$\boxed{5.6}$ Substitution and Area Between Curves

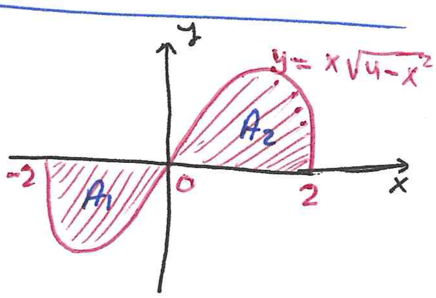
Example: Evaluate the definite integral $\int_0^1 (t^5+2t+1)^{-\frac{1}{2}}(5t^4+2)\,dt$

$$\int_0^1 (t^5+2t+1)^{-\frac{1}{2}}(5t^4+2)\,dt = \int_1^4 u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}}\Big|_1^4 = 2\sqrt{u}\Big|_1^4 = 2(2)-2(1) = 4-2 = 2$$

$u = t^5+2t+1$

$du = (5t^4+2)\,dt$

Example: Consider the graph of $y = x\sqrt{4-x^2}$



[a] Find $\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$ since y is odd

[b] Find the total area of the shaded regions

$$A = |A_1| + |A_2| = 2\int_0^2 x\sqrt{4-x^2}\,dx$$

$u = 4-x^2$

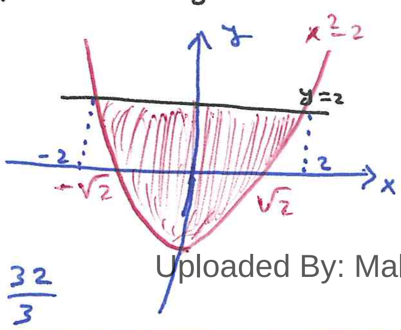
$du = -2x\,dx$

$$= -\int_4^0 u^{\frac{1}{2}}\,du = -\frac{2}{3}u^{\frac{3}{2}}\Big|_4^0 = -\frac{2}{3}\left[\sqrt{0^3}-\sqrt{4^3}\right] = -\frac{2}{3}(-8) = +\frac{16}{3}$$

Theorem Let f be continuous on the symmetric interval $[-a,a]$.

- If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example: Find the area of the region enclosed by the curve $y = x^2-2$ and $y = 2$. $x^2-2 = 2 \Leftrightarrow x^2 = 4 \Leftrightarrow x = \pm 2$

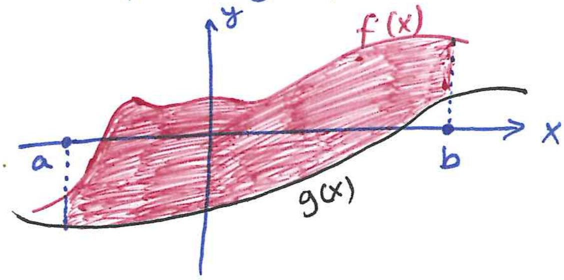


$$\text{Area} = 2\int_0^2 (2-x^2+2)\,dx = 2\int_0^2 (4-x^2)\,dx$$

$$= 2\left[4x - \frac{x^3}{3}\Big|_0^2\right] = 2\left[8-\frac{8}{3}\right] = 2\left(\frac{16}{3}\right) = \frac{32}{3}$$

Definition: If f and g are continuous on $[a,b]$ with $f(x) \geq g(x)$, then the area of the region between f and g from a to b is

$$A = \int_a^b \left[f(x) - g(x)\right]dx$$



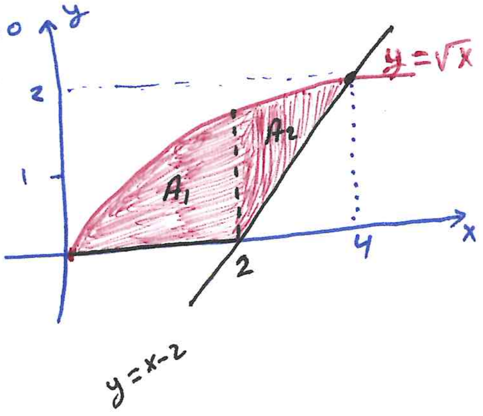
Example*: Find the area of the region in the 1^{st} quadrant that is bounded above by $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$ 108

$\sqrt{x} = x-2 \Leftrightarrow x = x^2-4x+4 \Leftrightarrow x^2-5x+4=0$

$\Leftrightarrow (x-4)(x-1)=0 \Leftrightarrow \boxed{x=4}$

Does not satisfy $\leftarrow$ $\cancel{x=1}$

$\sqrt{x} = x-2$



The total area $= A_1 + A_2$

$$= \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]\Big|_2^4 = \frac{10}{3}$$

Example: Find the area of the region in Example* by integrating w.r.t. y.

$x = y+2$ upper curve

$x = y^2$ lower curve

$y+2 = y^2 \Leftrightarrow$

$y^2 - y - 2 = 0 \Leftrightarrow$

$(y-2)(y+1) = 0 \Leftrightarrow$

$\boxed{y=2}$ and $\cancel{y=-1}$ below x-axis

$$A = \int_0^2 (y+2-y^2)\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2$$

$$= 2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{10}{3}$$
