

11.4 := Graphing in Polar Coordinates.

Identify the symmetry of the curves. Then sketch the curve.

1 $r = 1 + \cos \theta$.

a) check the symmetry about the x-axis.

$$\begin{aligned}(r, -\theta) &\Rightarrow r = 1 + \cos(-\theta) \\ &= 1 + \cos \theta \\ &= r \\ &\Rightarrow (r, \theta) \quad \checkmark\end{aligned}$$

Symmetric about the x-axis.

b) check the symmetry about the y-axis.

$$\begin{aligned}(-r, -\theta) &\Rightarrow -r \stackrel{?}{=} 1 + \cos(-\theta) \\ &-r \neq 1 + \cos(\theta)\end{aligned}$$

cannot tell.

$$\begin{aligned}(r, \pi - \theta) &\Rightarrow r \stackrel{?}{=} 1 + \cos(\pi - \theta) \\ &\stackrel{?}{=} 1 + \cos(\pi) \cos(\theta) + \sin(\pi) \sin \theta.\end{aligned}$$

$$r \neq 1 - \cos(\theta)$$

not symmetric about y-axis.

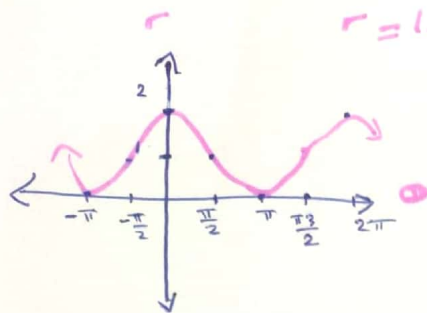
Therefore not symmetric about the origin.

(you can check)

c) check the symmetry about the origin.

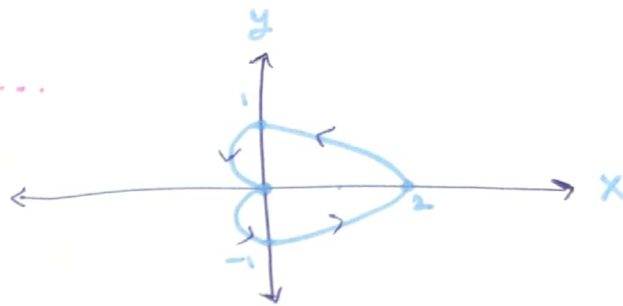
$$(-r, \theta) \Rightarrow -r \stackrel{?}{=} 1 + \cos \theta$$

$$\begin{aligned}(r, \theta + \pi) &\Rightarrow r \stackrel{?}{=} 1 + \cos(\theta + \pi) \\ &\stackrel{?}{=} 1 + \cos(\theta) \cos(\pi) - \sin(\theta) \sin(\pi) \\ &r \neq 1 - \cos(\theta).\end{aligned}$$



$$r = 1 + \cos \theta = 0$$

$$\Rightarrow \theta = \pi, -\pi, 3\pi, \dots$$



(r, θ)



(x, y)

$$\text{s.t. } x = r \cos \theta$$

$$y = r \sin \theta.$$

$$(2, 0) \Rightarrow (2, 0)$$

$$(1, \frac{\pi}{2}) \Rightarrow (0, 1)$$

$$(0, \pi) \Rightarrow (0, 0)$$

$$(1, \frac{3\pi}{2}) \Rightarrow (0, -1)$$

From the graph:

No symmetry about
y-axis or origin.

6 $r = 1 + 2 \sin \theta.$

a) x-axis.

$$(r, -\theta) \Rightarrow r \stackrel{?}{=} 1 + 2 \sin(-\theta).$$

$$\stackrel{?}{=} 1 - 2 \sin \theta.$$

$$\neq r$$

$$(-r, \pi - \theta) \Rightarrow -r \stackrel{?}{=} 1 + 2(\sin(\pi - \theta)).$$

$$-r \stackrel{?}{=} 1 + 2[\sin(\pi) \cos(\theta) - \cos(\pi) \sin(\theta)]$$

$$\stackrel{?}{=} 1 + 2[0 - -\sin \theta].$$

$$-r \neq 1 + 2 \sin \theta$$

Not symmetric about x-axis.

b) y-axis.

$$(-r, -\theta) \Rightarrow -r \stackrel{?}{=} 1 + 2 \sin(-\theta).$$

$$-r \neq 1 - 2 \sin \theta$$

$$(r, -\theta) \Rightarrow r \stackrel{?}{=} 1 + 2 \sin(\pi - \theta).$$

$$r \stackrel{?}{=} 1 + 2 [\sin(\pi) \cos(\theta) - \cos(\pi) \sin(\theta)].$$

$$r \stackrel{?}{=} 1 + 2 [0 + \sin \theta]$$

$$r = 1 + 2 \sin \theta.$$

Symmetric about the y-axis.

From a & b not symmetric about the origin.

c) about the origin.

$$(-r, \theta) \Rightarrow -r \neq 1 + 2 \sin \theta.$$

$$(r, \pi + \theta) \Rightarrow r \stackrel{?}{=} 1 + 2 \sin(\pi + \theta)$$

$$\stackrel{?}{=} 1 + 2 [\sin(\pi) \cos(\theta) + \cos(\pi) \sin(\theta)].$$

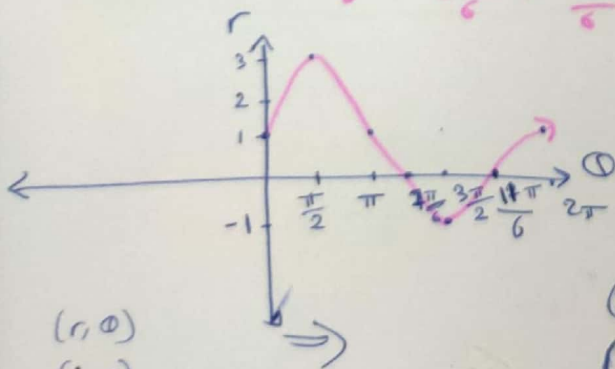
$$\neq 1 - 2 \sin \theta.$$

not symmetric about the origin.

$$\Rightarrow r = 1 + 2 \sin \theta = 0$$

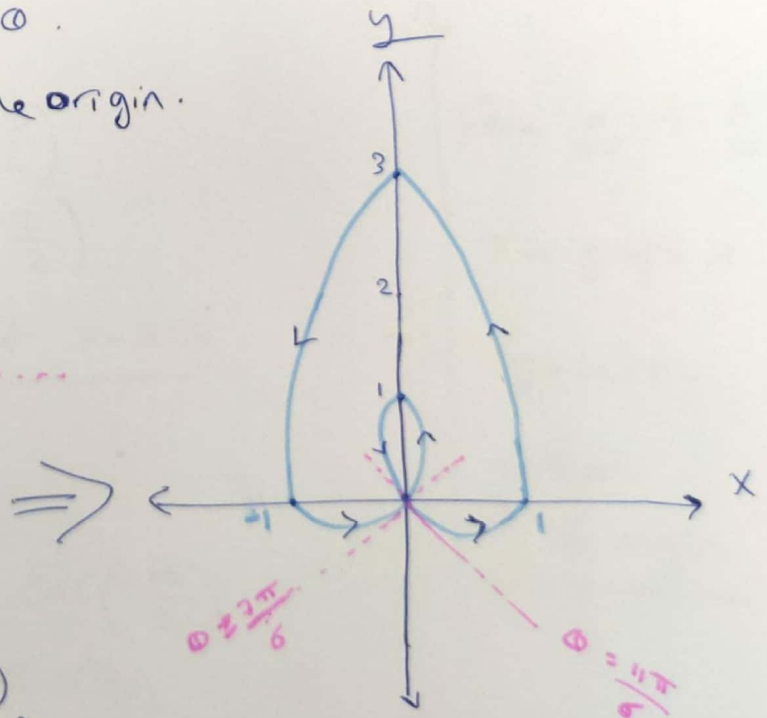
$$\sin \theta = -\frac{1}{2}$$

$$\theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$$

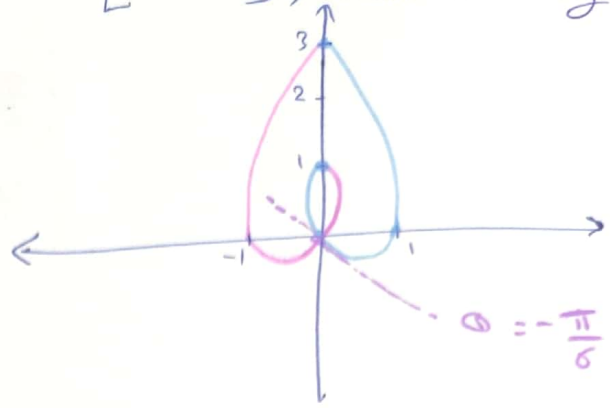
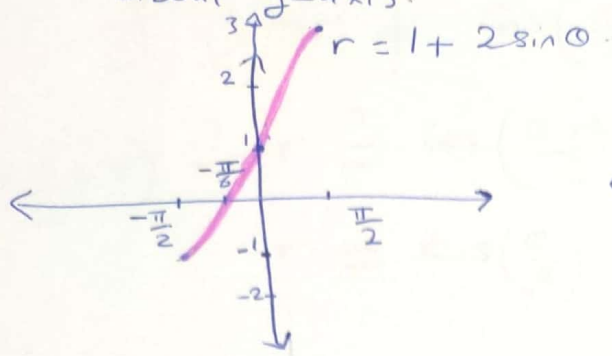


- (r, θ)
- $(1, 0)$
 - $(3, \frac{\pi}{2})$
 - $(1, \pi)$
 - $(0, \frac{7\pi}{6})$
 - $(-1, \frac{3\pi}{2})$

- (x, y)
- $(1, 0)$
 - $(0, 3)$
 - $(-1, 0)$
 - $(0, 0)$
 - $(0, -1)$



note You can have a graph on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then use symmetry about y-axis.



(r, θ)	$\Rightarrow$	(x, y)
$(-1, -\frac{\pi}{2})$	$\Rightarrow$	$(0, 1)$
$(0, -\frac{\pi}{6})$	$\Rightarrow$	$(0, 0)$
$(1, 0)$	$\Rightarrow$	$(1, 0)$
$(3, \frac{\pi}{2})$	$\Rightarrow$	$(0, 3)$

From the graph: no symmetry about x-axis or origin.

8 $r = \cos\left(\frac{\theta}{2}\right)$

a) x-axis:

$$(r, -\frac{\theta}{2}) \Rightarrow r \stackrel{?}{=} \cos\left(-\frac{\theta}{2}\right)$$

$$r = \cos\left(-\frac{\theta}{2}\right)$$

The graph is symmetric about x-axis.

b) y-axis:

$$(-r, -\frac{\theta}{2}) \Rightarrow -r \stackrel{?}{=} \cos\left(-\frac{\theta}{2}\right)$$

$$-r \neq \cos\left(\frac{\theta}{2}\right)$$

$$(r, \pi - \frac{\theta}{2}) \Rightarrow r \stackrel{?}{=} \cos\left(\frac{2\pi - \theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)$$

$$= r$$

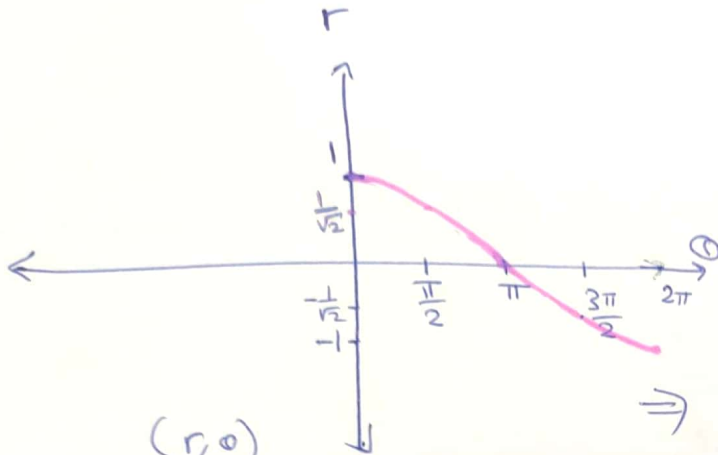
The graph is symmetric about y-axis.

From a & b

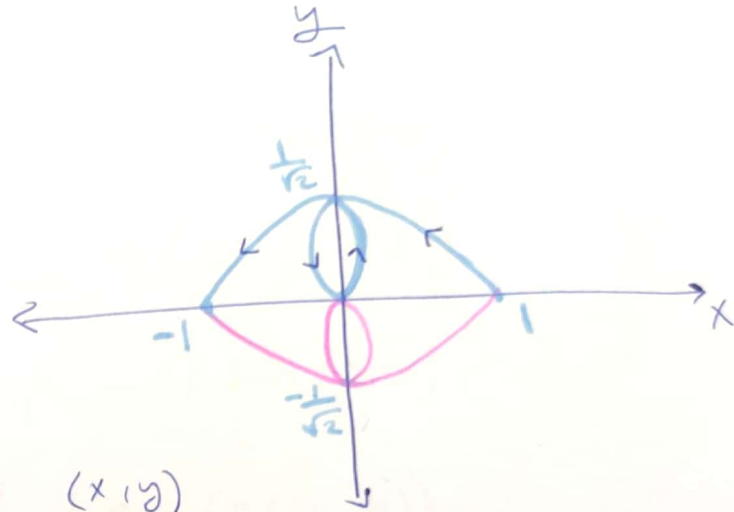
The graph is
symmetric
about
The origin.

$$r = \cos\left(\frac{\theta}{2}\right) = 0.$$

$$\frac{\theta}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots \Rightarrow \theta = \pm\pi, \pm3\pi, \dots$$



- (r, θ)
 $(1, 0)$
 $(\frac{1}{\sqrt{2}}, \frac{\pi}{2})$
 $(0, \pi)$
 $(-\frac{1}{\sqrt{2}}, \frac{3\pi}{2})$
 $(-1, 2\pi)$



- (x, y)
 $(1, 0)$
 $(0, \frac{1}{\sqrt{2}})$
 $(0, 0)$
 $(0, -\frac{1}{\sqrt{2}})$
 $(-1, 0)$

Note that:

From the graph $\Rightarrow$ The curve has all symmetries.

14 What symmetric does curve have?

$$r^2 = 4 \sin(2\theta).$$

x-axis $\Rightarrow (r, -\theta) \Rightarrow r^2 \stackrel{?}{=} 4(\sin(-2\theta))$
 $r^2 \neq -4 \sin(2\theta)$

$(-r, \pi - \theta) \Rightarrow (-r)^2 \stackrel{?}{=} 4 \sin(2(\pi - \theta))$
 $r^2 \stackrel{?}{=} 4 \sin(2\pi - 2\theta)$
 $\stackrel{?}{=} 4 [\sin(2\pi) \cos(2\theta) - \cos(2\pi) \sin(2\theta)]$
 $\stackrel{?}{=} 4 [0 - \sin(2\theta)]$
 $r^2 \neq -4 \sin(2\theta).$

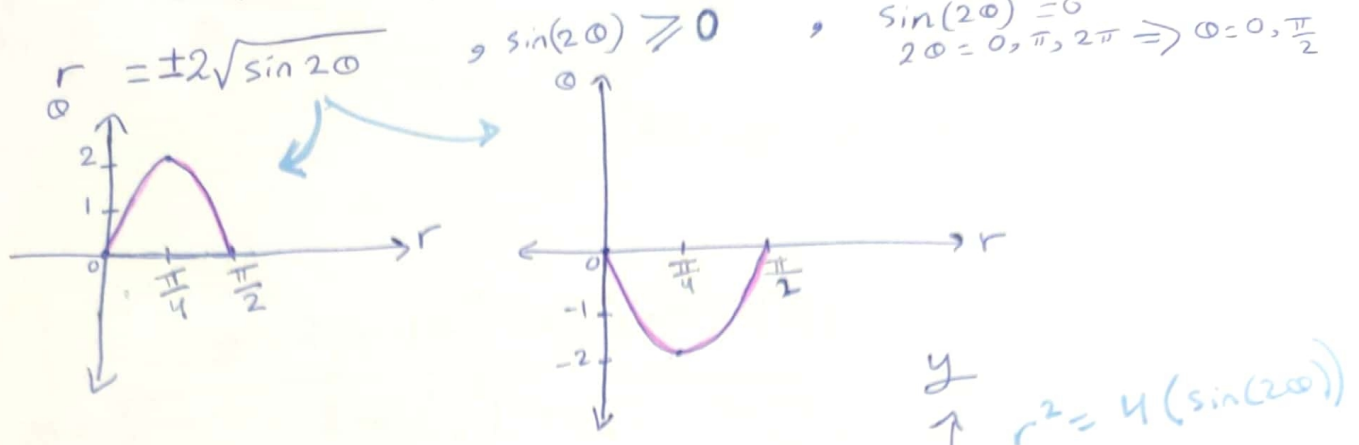
y-axis $\Rightarrow (-r, \theta) \Rightarrow (-r)^2 \stackrel{?}{=} 4(\sin(-2\theta))$
 $r^2 \neq -4 \sin(2\theta).$

$(r, \pi - \theta) \Rightarrow r^2 \stackrel{?}{=} 4 \sin(2(\pi - \theta))$
 $\stackrel{?}{=} 4 \sin(2\pi - 2\theta)$
 $\stackrel{?}{=} 4 [\sin(2\pi) \cos(2\theta) - \cos(2\pi) \sin(2\theta)]$
 $r^2 \neq -\sin(2\theta).$

origin $(-r, \theta) \Rightarrow (-r)^2 \stackrel{?}{=} 4 \sin(2\theta)$
 $r^2 = 4 \sin(2\theta)$

The graph is symmetric about the origin.

$$r^2 = 4 \sin(2\theta).$$



(r, θ)	$\Rightarrow (x, y)$
$(0, 0)$	$\Rightarrow (0, 0)$
$(2, \frac{\pi}{4})$	$\Rightarrow (\sqrt{2}, \sqrt{2})$
$(0, \frac{\pi}{2})$	$\Rightarrow (0, 0)$

Use the symmetry about the origin.

From the graph:

The graph only symmetric about the origin.

19 Find the slope of the curve.

$r = \sin(2\theta)$, at $\theta = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$.

$\theta = -\frac{\pi}{4}, r = -1(-1, -\frac{\pi}{4}), \theta = \frac{\pi}{4}, r = 1(1, \frac{\pi}{4}), \theta = -\frac{3\pi}{4}, r = 1(1, -\frac{3\pi}{4}), \theta = \frac{3\pi}{4}, r = -1(-1, \frac{3\pi}{4})$

Remember:

$$\frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

$$f'(\theta) \cos \theta - f(\theta) \sin \theta$$

$$r = f(\theta) = \sin(2\theta).$$

$$r' = f'(\theta) = 2 \cos(2\theta).$$

$$y' = \frac{2 \cos(2\theta) \sin(\theta) + \sin(2\theta) \cos(\theta)}{2 \cos(2\theta) \cos(\theta) - \sin(2\theta) \sin(\theta)}$$

$$\left. y' \right|_{\theta = -\frac{\pi}{4}} = \frac{2 \cos(-\frac{2\pi}{4}) \sin(-\frac{\pi}{4}) + \sin(-\frac{2\pi}{4}) \cos(-\frac{\pi}{4})}{2 \cos(-\frac{2\pi}{4}) \cos(-\frac{\pi}{4}) - \sin(-\frac{2\pi}{4}) \sin(-\frac{\pi}{4})}$$

$$= \frac{-2 \cos(\frac{\pi}{2}) \sin(\frac{\pi}{4}) - \sin(\frac{\pi}{2}) \cos(\frac{\pi}{4})}{2 \cos(\frac{\pi}{2}) \cos(\frac{\pi}{4}) - \sin(\frac{\pi}{2}) \sin(\frac{\pi}{4})}$$

$$y' \Big|_{\theta = -\frac{\pi}{4}} = \frac{0 - (1) \left(\frac{1}{\sqrt{2}}\right)}{0 - 1 \left(\frac{1}{\sqrt{2}}\right)} = \underline{\underline{1}}$$

= The slope at $\theta = -\frac{\pi}{4}$.

$$y' \Big|_{\theta = \frac{\pi}{4}} = \frac{2 \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{4}\right)}{2 \cos\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{0 + (1) \frac{1}{\sqrt{2}}}{0 - (1) \frac{1}{\sqrt{2}}} = \underline{\underline{-1}}$$

= the slope at $\theta = \frac{\pi}{4}$.

$$y' \Big|_{\theta = -\frac{3\pi}{4}} = \frac{2 \cos\left(-\frac{3\pi}{2}\right) \sin\left(-\frac{3\pi}{4}\right) + \sin\left(-\frac{3\pi}{2}\right) \cos\left(-\frac{3\pi}{4}\right)}{2 \cos\left(-\frac{3\pi}{2}\right) \cos\left(-\frac{3\pi}{4}\right) - \sin\left(-\frac{3\pi}{2}\right) \sin\left(-\frac{3\pi}{4}\right)}$$

$$= \frac{0 + -(-1) \left(-\frac{1}{\sqrt{2}}\right)}{0 - (-(-1)) \left(-\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \underline{\underline{-1}}$$

= The slope at $\theta = -\frac{3\pi}{4}$.

$$\begin{aligned}
 y' \Big|_{\theta = \frac{3\pi}{4}} &= \frac{2 \cos\left(\frac{3\pi}{2}\right) \sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{2}\right) \cos\left(\frac{3\pi}{4}\right)}{2 \cos\left(\frac{3\pi}{2}\right) \cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{2}\right) \sin\left(\frac{3\pi}{4}\right)} \\
 &= \frac{0 + (-1) \left(-\frac{1}{\sqrt{2}}\right)}{0 - (-1) \left(\frac{1}{\sqrt{2}}\right)} \\
 &= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \\
 &= \boxed{1} \\
 &= \text{The slope at } \theta = \frac{3\pi}{4}.
 \end{aligned}$$

Check the symmetry:

$$\begin{aligned}
 \underline{x\text{-axis}} \Rightarrow (r, \theta) &\Rightarrow r \stackrel{?}{=} \sin(-2\theta) \\
 &r \neq 2 \sin(2\theta)
 \end{aligned}$$

$$\begin{aligned}
 (-r, \pi - \theta) &\Rightarrow -r \stackrel{?}{=} \sin(2\pi - 2\theta) \\
 &\stackrel{?}{=} \sin(2\pi) \cos(2\theta) - \cos(2\pi) \sin(2\theta) \\
 &\stackrel{?}{=} 0 - (1) (\sin(2\theta)) \\
 &-r = -\sin(2\theta).
 \end{aligned}$$

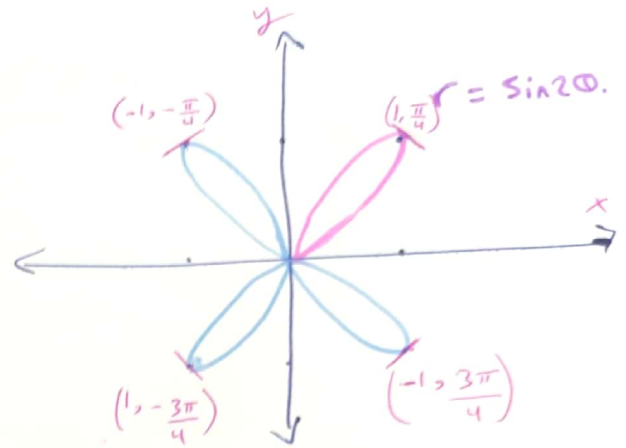
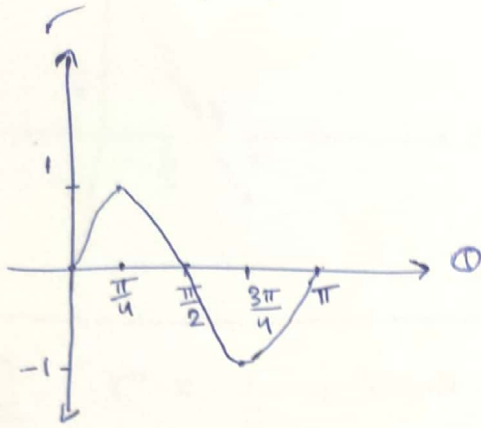
☞ The curve is symmetric about x-axis.

$$\begin{aligned}
 \underline{y\text{-axis}} \Rightarrow (-r, -\theta) &\Rightarrow -r \stackrel{?}{=} \sin(-2\theta) \\
 &-r = -\sin(2\theta).
 \end{aligned}$$

☞ The curve is symmetric about y-axis.

☞ So, the curve is symmetric about the origin.

$$r = \sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi, 2\pi \Rightarrow \theta = 0, \frac{\pi}{2}, \pi.$$



$$\begin{aligned} (r, \theta) &\Rightarrow (x, y) \\ (0, 0) &\Rightarrow (0, 0) \\ (1, \frac{\pi}{4}) &\Rightarrow (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \\ (0, \frac{\pi}{2}) &\Rightarrow (0, 0) \end{aligned}$$

Then use the symmetry

21 Graph:

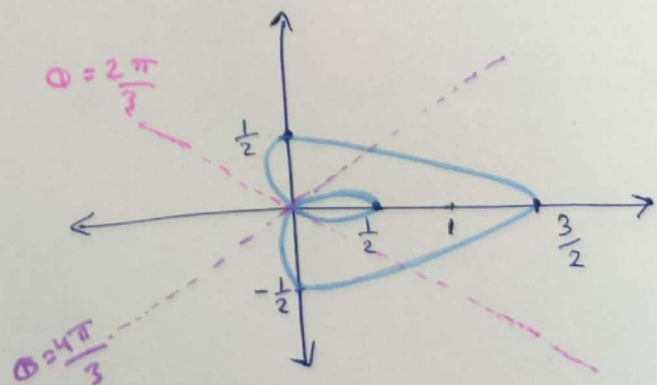
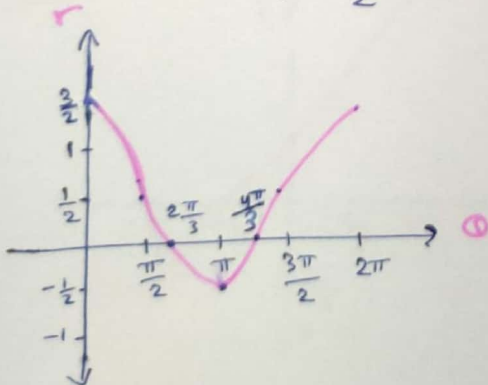
a $r = \frac{1}{2} + \cos \theta.$

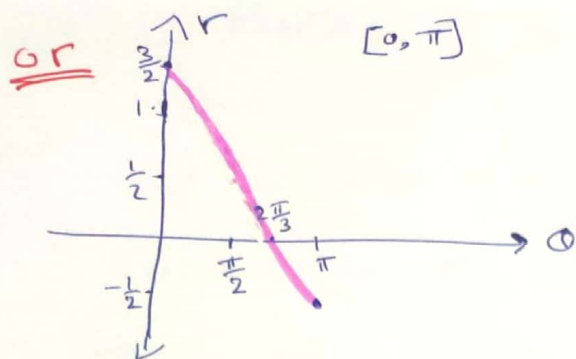
The curve is symmetric to the x-axis. check it

$$r = \frac{1}{2} + \cos \theta.$$

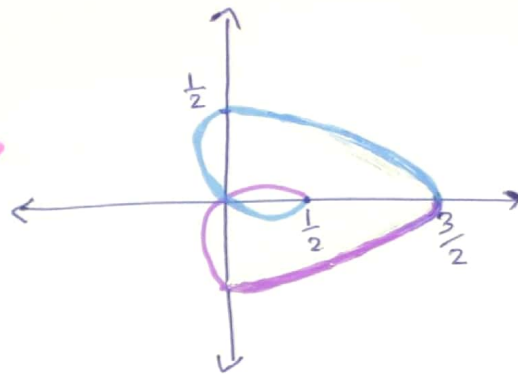
$$r = 0 \Rightarrow \frac{1}{2} + \cos \theta = 0 \Rightarrow \cos \theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}.$$





use symmetry
about x-axis

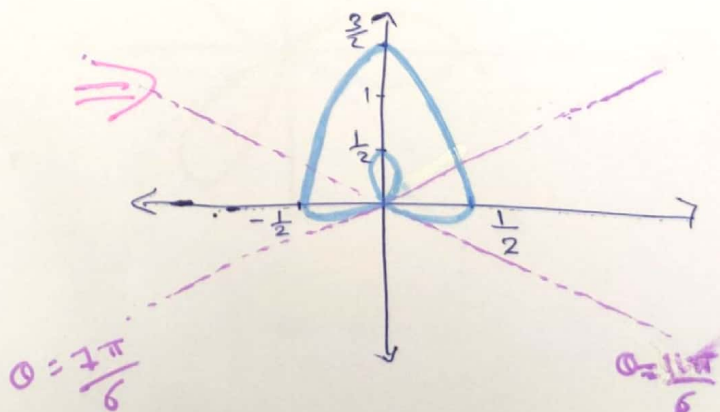
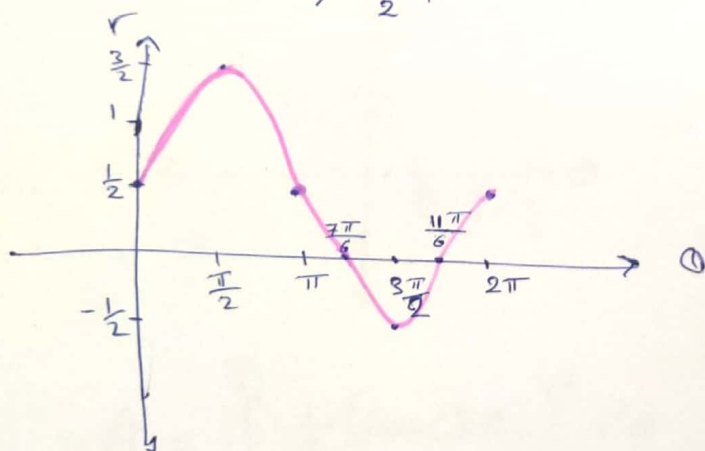


b $r = \frac{1}{2} + \sin \theta$

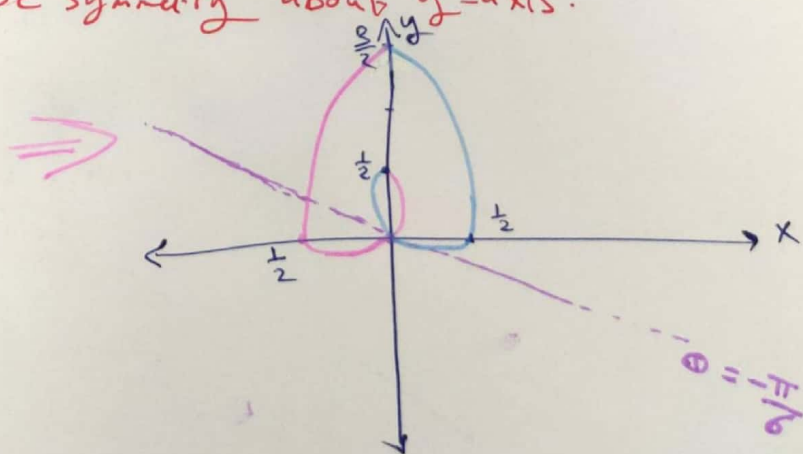
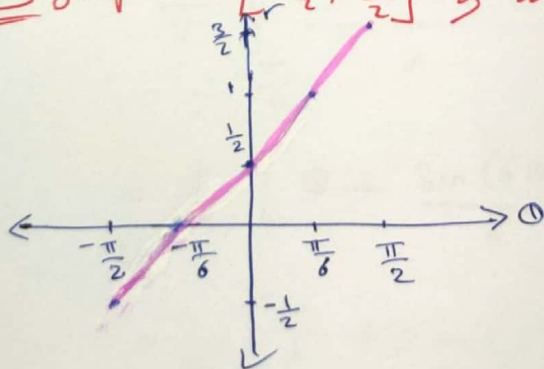
The curve is symmetric about y-axis check it

$$r = 0 \Rightarrow \frac{1}{2} + \sin \theta = 0 \Rightarrow \sin \theta = -\frac{1}{2}$$

$$\Rightarrow \theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$



or graph on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ & use symmetry about y-axis.



11.5 Areas & Lengths in Polar Coordinates.

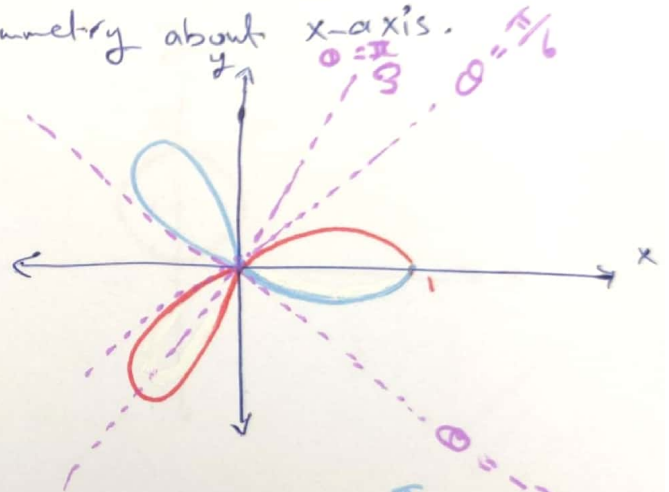
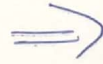
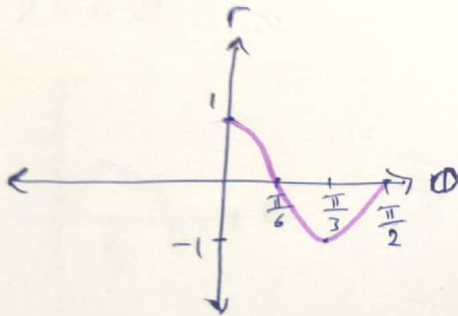
6 Find the area of region inside one leaf of the three-leaved rose $r = \cos(3\theta)$.

* Symmetric about x-axis (check it).

$$\cos(3\theta) = 0 \Rightarrow 3\theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$$

$$\Rightarrow \theta = \pm \frac{\pi}{6}, \pm \frac{\pi}{2}.$$

graph on $[0, \frac{\pi}{2}]$ then use symmetry about x-axis.



$$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2} (\cos(3\theta))^2 d\theta$$
$$= \frac{1}{2} \int_{-\pi/6}^{\pi/6} \frac{1 + \cos(6\theta)}{2} d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{\sin(6\theta)}{6} \right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4} \left[\frac{\pi}{6} + 0 - \left(-\frac{\pi}{6} + 0 \right) \right]$$

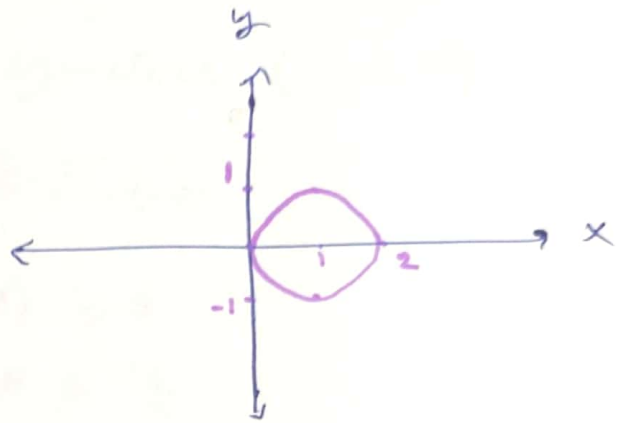
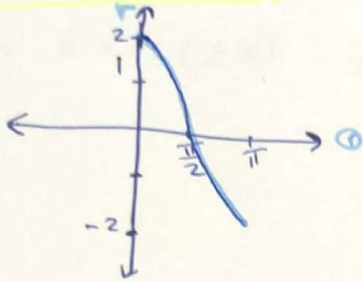
$$= \frac{1}{4} \left[\frac{\pi}{6} + \frac{\pi}{6} \right]$$

$$= \boxed{\frac{\pi}{12}}$$

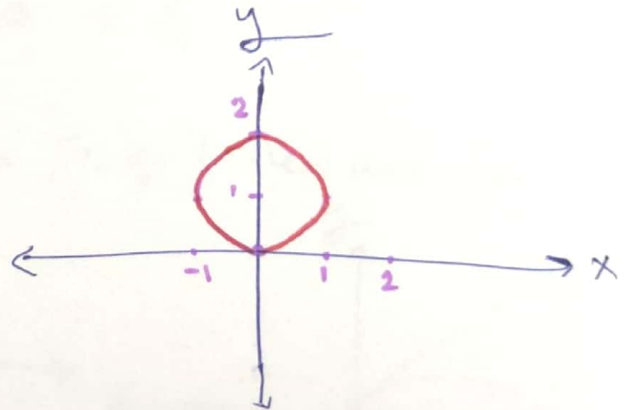
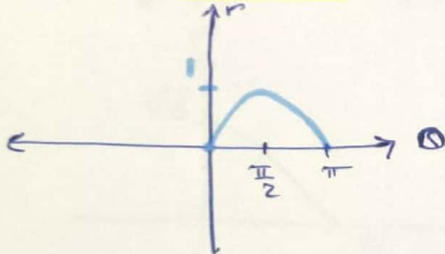
$$\text{or } A = 2 \int_0^{\pi/6} \frac{1}{2} \cos^2(3\theta) d\theta$$
$$\text{or } A = 2 \int_0^{\pi/6} \frac{1}{2} \cos^2(3\theta) d\theta$$
$$= \frac{\pi}{12}$$

9 Find the area of the region shared by the circles
 $r = 2 \cos \theta$ & $r = 2 \sin \theta$.

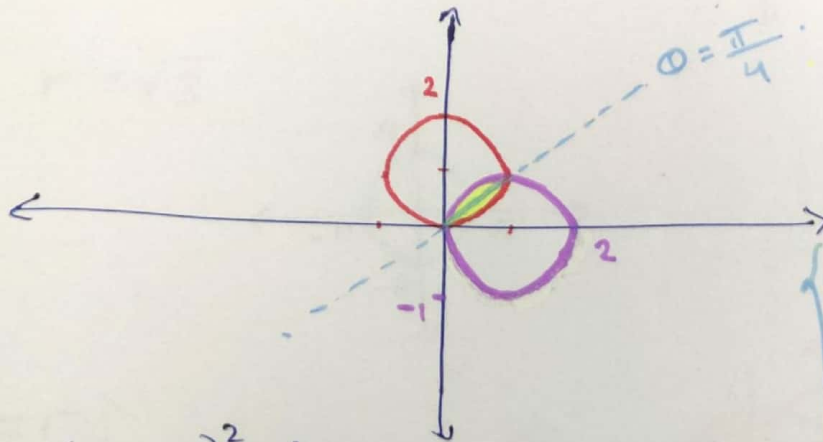
$r = 2 \cos \theta$



$r = 2 \sin \theta$



$$2 \cos \theta = 2 \sin \theta \Rightarrow \cos \theta = \sin \theta \Rightarrow \theta = \frac{\pi}{4}$$



$$A = 2 \int_0^{\pi/4} \frac{1}{2} (2 \sin \theta)^2 d\theta$$

$$= 4 \int_0^{\pi/4} \sin^2 \theta d\theta = 4 \int_0^{\pi/4} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= 2 \left[\theta - \frac{\sin(2\theta)}{2} \right]_0^{\pi/4} = 2 \left[\frac{\pi}{4} - \frac{\sin(\pi/2)}{2} \right] = \boxed{\frac{\pi}{2} - 1}$$

or
 $A = 2 \int_{\pi/4}^{\pi/2} \frac{1}{2} (2 \cos \theta)^2 d\theta$
 $= \frac{\pi}{2} - 1$

13 Find the area of the region inside the lemniscate $r^2 = 6 \cos(2\theta)$ & outside the circle $r = \sqrt{3}$.

$r^2 = 6 \cos(2\theta)$ has all symmetries (check it).

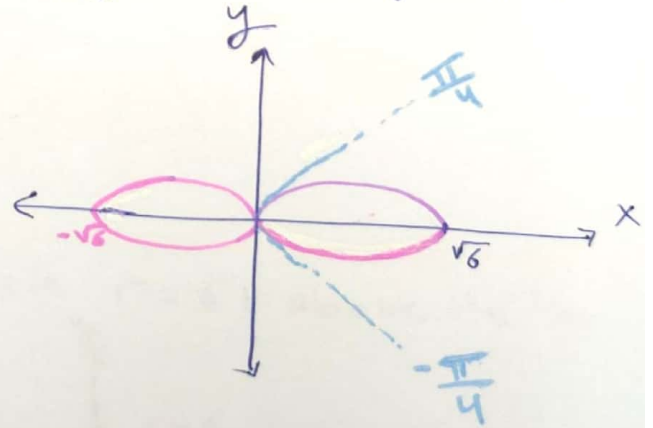
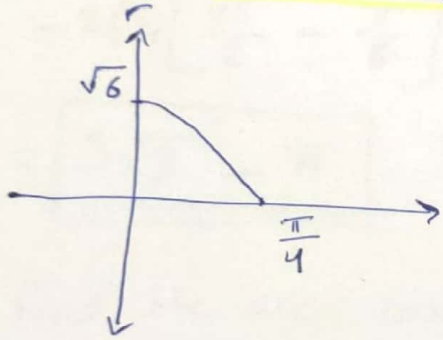
$$r^2 = 6 \cos(2\theta) \Rightarrow r = \sqrt{6} \sqrt{\cos(2\theta)}$$

$$\cos(2\theta) \geq 0$$

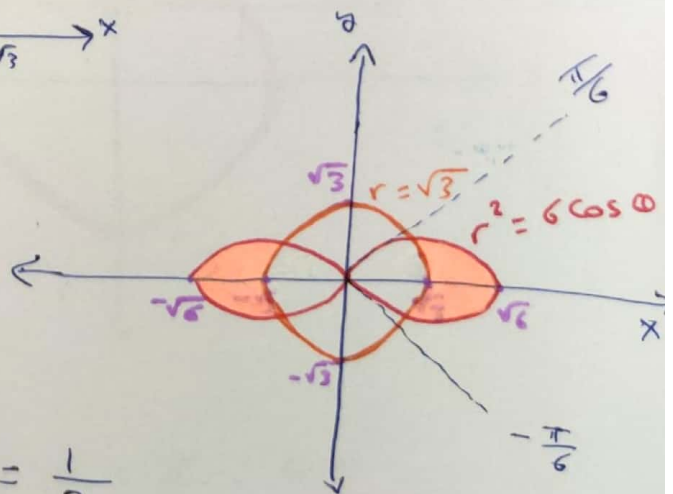
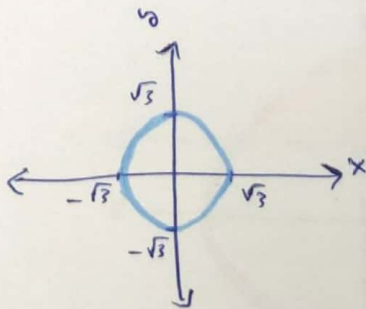
$$0 \leq 2\theta \leq \frac{\pi}{2}$$

$$0 \leq \theta \leq \frac{\pi}{4}$$

$\Rightarrow$ graph $r = \sqrt{6} \sqrt{\cos(2\theta)}$ on $[0, \frac{\pi}{4}]$ then use symmetries.



$\Rightarrow$ graph $r = \sqrt{3}$



$$r = r$$

$$\sqrt{6} \sqrt{\cos(2\theta)} = \sqrt{3}$$

$$\Rightarrow r^2 = r^2$$

$$6 \cos(2\theta) = 3 \Rightarrow \cos(2\theta) = \frac{1}{2}$$

$$\Rightarrow 2\theta = \pm \frac{\pi}{3}$$

$$\Rightarrow \theta = \pm \frac{\pi}{6}$$

$$\begin{aligned}
 A &= 4 \int_0^{\frac{\pi}{6}} \left[\frac{1}{2} (\sqrt{6} \cos(2\theta))^2 - \frac{1}{2} (\sqrt{3})^2 \right] d\theta \\
 &= 4 \int_0^{\frac{\pi}{6}} \left[\frac{1}{2} 6 \cos(2\theta) - \frac{3}{2} \right] d\theta \\
 &= 2 \left[\frac{6 \sin(2\theta)}{2} - 3\theta \right]_0^{\frac{\pi}{6}} \\
 &= 2(3) \left[\sin\left(2 \cdot \frac{\pi}{6}\right) - \frac{\pi}{6} - (0) \right] \\
 &= 2(3) \left[\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right] \\
 &= \boxed{3\sqrt{3} - \pi}
 \end{aligned}$$

16 Find the area inside the circle $r = 6$ above the line $r = 3 \csc \theta$.

$r = 6$

$r = 3 \csc \theta$

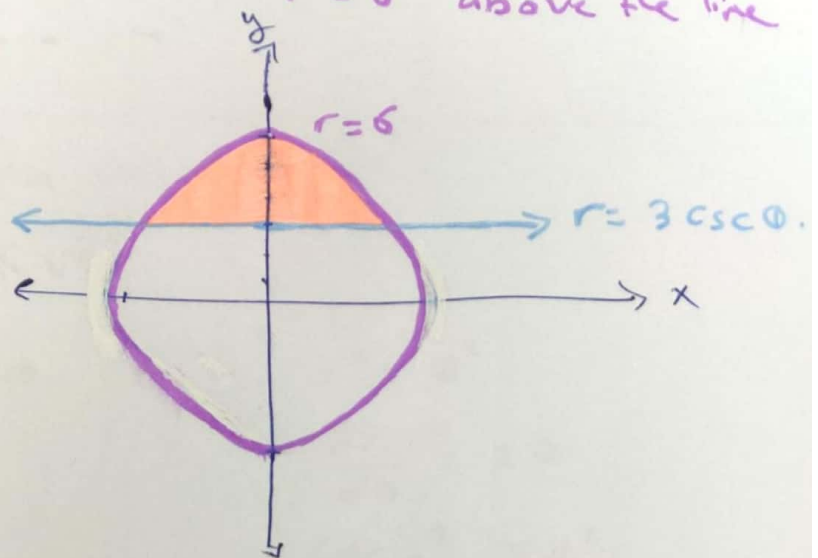
$r = 3 \frac{1}{\sin \theta}$

$r \sin \theta = 3$

$y = 3$

$3 \csc \theta = 6$

$\frac{3}{\sin \theta} = 6 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$



$$\begin{aligned}
 A &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{1}{2} (6)^2 - \frac{1}{2} (3 \csc \theta)^2 \right) d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(18 - \frac{9}{2} \csc^2 \theta \right) d\theta \\
 &= \left(18\theta + \frac{9}{2} \cot \theta \right) \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}
 \end{aligned}$$

$$= 18 \left(\frac{5\pi}{6} \right) + \frac{9}{2} \cot \left(\frac{5\pi}{6} \right) - 18 \left(\frac{\pi}{6} \right) - \frac{9}{2} \cot \left(\frac{\pi}{6} \right)$$

$$= 15\pi + \frac{9}{2} (-\sqrt{3}) - 3\pi - \frac{9}{2} \sqrt{3}$$

$$= \boxed{12\pi - 9\sqrt{3}}$$

21) Find the length of the curve

$$r = \theta^2, \quad 0 \leq \theta \leq \sqrt{5}.$$

$$L = \int_0^{\sqrt{5}} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta.$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2} d\theta.$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^2} \sqrt{\theta^2 + 4} d\theta = \int_0^{\sqrt{5}} \theta \sqrt{\theta^2 + 4} d\theta.$$

$$r = \theta^2$$

$$r^2 = \theta^4$$

$$\frac{dr}{d\theta} = 2\theta$$

$$\left(\frac{dr}{d\theta} \right)^2 = 4\theta^2.$$

but $|\theta| = \theta$ since $\theta \geq 0$.

$$\text{So, } \int_0^{\sqrt{5}} \theta \sqrt{\theta^2 + 4} d\theta.$$

$$= \frac{1}{2} \int_4^9 \sqrt{u} du.$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_4^9$$

$$= \frac{1}{3} \left[9^{\frac{3}{2}} - 4^{\frac{3}{2}} \right].$$

$$= \frac{1}{3} \left[\left(9^{\frac{1}{2}} \right)^3 - \left(4^{\frac{1}{2}} \right)^3 \right]$$

$$= \frac{1}{3} [27 - 8]$$

$$= \boxed{\frac{19}{3}}$$

$$\text{let } u = \theta^2 + 4.$$

$$du = 2\theta d\theta.$$

$$\frac{du}{2} = \theta d\theta$$

$$\text{when } \theta = 0 \Rightarrow u = 4.$$

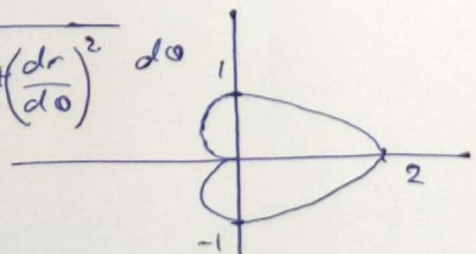
$$\theta = \sqrt{5} \Rightarrow u = 5 + 4 = 9.$$

23 Find the length of the cardioid $r = 1 + \cos \theta$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta = 2 \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta$$

$$= 2 \int_0^{\pi} \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_1} d\theta.$$

$$= 2 \int_0^{\pi} \sqrt{2 + 2\cos\theta} d\theta.$$



$$r = 1 + \cos \theta$$

$$r^2 = 1 + 2\cos \theta + \cos^2 \theta.$$

$$\frac{dr}{d\theta} = -\sin \theta$$

$$\left(\frac{dr}{d\theta} \right)^2 = \sin^2 \theta.$$

$$= 2 \int_0^{\pi} \sqrt{\frac{4}{2}(1 + \cos \theta)} d\theta.$$

$$= 2 \int_0^{\pi} \sqrt{4} \sqrt{\frac{1}{2} + \frac{\cos \theta}{2}} d\theta.$$

$$= 4 \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta.$$

$$= 4 \frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}} \Big|_0^{\pi}$$

$$= 8 [\sin(\pi) - \sin 0]$$

$$= 8 [1 - 0]$$

$$= \underline{\underline{8}}$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$\frac{\cos\theta + 1}{2} = \cos^2\left(\frac{\theta}{2}\right).$$