

Chapter 3

linear systems:-

Iterative methods:-

1. Fixed point iteration
2. Gauss - Seidel method
3. Newton Method.

Direct methods:- (A is nonsingular).

1. Gaussian Elimination $[A:b] \rightarrow [U|c]$ + Back substitution:
2. Gauss - Jordan $[A|b] \rightarrow [I|X]$.
3. Inverse method $x = A^{-1}b$
4. Cramer's $x_i = \frac{|A_i|}{|A|}$
5. L - U Factorization.

Section 3.3

back substitution

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$4x_2 + 6x_3 = 10$$

$$10x_3 = 10$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 9 \\ 10 \\ 10 \end{bmatrix}$$

$$10x_3 = 10 \rightarrow \boxed{x_3 = 1}$$

$$4x_2 + 6x_3 = 10$$

$$4x_2 = 10 - 6$$

$$4x_2 = 4 \rightarrow \boxed{x_2 = 1}$$

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$3x_1 = 9 - 4 - 2$$

$$3x_1 = 3 \rightarrow \boxed{x_1 = 1}$$

$$P_n = 10^{-2^n} \rightarrow 0 \text{ quadratically}$$

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n \rightarrow \infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1$$

$$P_n = 10^{-n^2} \not\rightarrow 0 \text{ quadratically.}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^2} &= \lim_{n \rightarrow \infty} \frac{10^{-(n+1)^2}}{(10^{-n^2})^2} = \lim_{n \rightarrow \infty} \frac{10^{-(n+1)^2}}{10^{-2(n^2)}} \\ &= \lim_{n \rightarrow \infty} \frac{10^{2(n^2)}}{10^{(n+1)^2}} = \lim_{n \rightarrow \infty} \frac{10^{n^2} \cdot 10^{n^2}}{10^{(n+1)^2}} \rightarrow \infty \end{aligned}$$

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2.3

$$1,564,000 = 1,000,000 e^2 + \frac{435,000}{2} (e^2 - 1)$$

$$1564 = 1000 e^2 + \frac{435}{2} (e^2 - 1)$$

$$f(2) = 1000 e^2 + \frac{435}{2} (e^2 - 1) - 1564 = 0$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ \vdots & & & & & \\ 0 & & & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & & a_{n,n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix}$$

$a_{n,n}x_n = b_n$
 $x_n = \frac{b_n}{a_{n,n}}$
 $a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$
 $x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}$$

$$x_k = b_k$$

$$x_k = \frac{b_k - a_{k,k+1}x_{k+1} - a_{k,k+2}x_{k+2} - \dots - a_{k,n}x_n}{a_{k,k}}$$

$$x_k = b_k - \sum_{j=k+1}^n a_{k,j}x_j \quad k = n, n-1, \dots, 1$$

33 Cost

Steps	+/-	x/÷
1	0	1
2	1	2
3	2	3
k	k-1	k
n	n-1	n
Total	$\frac{(n-1)(n)}{2}$	$\frac{n(n+1)}{2}$

$$\text{Total cost} = \frac{n^2 - n}{2} + \frac{n^2 + n}{2} = n^2$$

3.4 Gaussian Elimination

$$AX=b$$

$$[A|b] \rightarrow [0|C] + \text{back sub.}$$

Row operations:

1. multiply any row by a nonzero constant
2. switch any two rows
3. Replace any row by adding to it a nonzero multiple of another row

$$\text{row } r := \text{row } r + C \text{ row } p$$

$$= \text{row } r - m_{rp} \text{ row } p$$

$$m_{rp} = \frac{a_{rp}}{a_{pp}} \quad r > p$$

Example

Solve:-

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$

$$2x_1 + 4x_3 + 3x_4 = 28$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$$

Pivot element

1	2	1	4	13
2	0	4	3	28
4	2	2	1	20
-3	1	3	2	6

Pivot Row

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$$

$$m_{41} = \frac{a_{41}}{a_{11}} = \frac{-3}{1} = -3$$

$R_2 - 2R_1$
 $R_3 - 4R_1$
 $R_4 + 3R_1$

1	2	1	4	13
0	-4	2	-5	2
0	-6	-2	-15	-32
0	7	6	14	45

Pivot Row

$$m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}} = \frac{6}{-4} = -1.5$$

$$m_{42} = \frac{a_{42}^{(2)}}{a_{22}^{(2)}} = \frac{7}{-4} = -1.75$$

$R_3 - 1.5R_2$
 $R_4 + 1.75R_2$

1	2	1	4	13
0	-4	2	-5	2
0	0	-5	-7.5	-35
0	0	9.5	15.25	48.5

Pivot Row

$$m_{43} = \frac{a_{43}^{(3)}}{a_{33}^{(3)}} = \frac{9.5}{-5} = -1.9$$

$R_4 - 1.9R_3$

1	2	1	4	13
0	-4	2	-5	2
0	0	-5	-7.5	-35
0	0	0	-9	-18

$$x_4 = 2$$

$$x_3 = 4$$

$$x_2 = -1$$

$$x_1 = 3$$

Coast

Step	+/-	X / ÷
1	4x3	3 + 4x3
2	3x2	2 + 3x2
3	2x1	1 + 2x1
total	20	26

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in general for nxn matrix

Step	+/-	X / ÷
1	$(n-1)n$	$(n-1)n + n - 1$
2	$(n-2)(n-1)$	$(n-2)(n-1) + n - 2$
3	$(n-3)(n-2)$	$(n-3)(n-2) + n - 3$
⋮		
P	$(n-P)(n-P+1)$	$(n-P)(n-P+1) + n - P$
last step →	$(n-1)$	

$$\text{total +/- : } \sum_{P=1}^{n-1} (n-P)(n-P+1)$$

$$\text{X / ÷ : } \sum_{P=1}^{n-1} (n-P)(n-P+1) + (n-P)$$

$$\sum_{P=1}^{n-1} (n-P)(n-P+1) = \sum_{P=1}^{n-1} (n-P)^2 + (n-P)$$

Let $k = n - p$

if $p = 1 \rightarrow k = n - 1$

$p = n - 1 \rightarrow k = 1$

$$\therefore \sum_{k=1}^{n-1} k^2 + k$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k = \frac{(n-1)n}{2}$$

$$\therefore \sum_{k=1}^{n-1} k^2 + k = \frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n}{2}$$

total $x/\%$ = $\frac{(n-1)n(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$

Grand Total = $2 \left[\frac{(n^2-n)(2n-1)}{6} + \frac{n(n-1)}{2} \right] + \frac{n(n-1)}{2}$

$$= 2 \left[\frac{2n^3 - 3n^2 + n}{6} + \frac{3n^2 - 3n}{6} \right] + \frac{n^2 - n}{2}$$

$$= \frac{2n^3 - 2n}{3} + \frac{n^2 - n}{2}$$

$$= \frac{4n^3 - 4n + 3n^2 - 3n}{6} = \frac{4n^3 + 3n^2 - 7n}{6}$$

$$\approx \frac{2}{3} n^3$$

• Coast for Gaussian

Coast = $\frac{4n^3 + 3n^2 - 7n}{6} + (n^2)$ ← Coast For back substitution.

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

$$\approx \frac{2}{3} n^3$$

Algorithm

will store the Augmented matrix in $n+1$ column.

$$\begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \dots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & \dots & a_{2,n}^{(1)} & a_{2,n+1}^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n,1}^{(1)} & a_{n,2}^{(1)} & \dots & a_{n,n}^{(1)} & a_{n,n+1}^{(1)} \end{bmatrix}$$

and will construct an equivalent upper triangular, U

$$\begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \dots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{2,2}^{(2)} & \dots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & 0 & a_{3,3}^{(3)} & \dots & a_{3,n}^{(3)} & a_{3,n+1}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n}^{(n)} & a_{n,n+1}^{(n)} \end{bmatrix}$$

Step 1 store the coefficient in array

Step 2 switch rows if necessary so that $a_{1,1}^{(1)} \neq 0$

find $m_{n,1} = \frac{a_{n,1}^{(1)}}{a_{1,1}^{(1)}}$ for $n=2$ to n .

for c From 2 to $n+1$.

set $a_{n,c}^{(2)} = a_{n,c}^{(1)} - m_{n,1} a_{1,c}^{(1)}$

we get

$$\begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \dots & a_{1,n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{2,2}^{(2)} & \dots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & 0 & a_{3,3}^{(3)} & \dots & a_{3,n}^{(3)} & a_{3,n+1}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n}^{(n)} & a_{n,n+1}^{(n)} \end{bmatrix}$$

in general

$P+1$ step find $a_{p,p}^{(p)} \neq 0$ From $r = p+1$ to N

$m_{r,p} = \frac{a_{r,p}^{(p)}}{a_{p,p}^{(p)}}$ and $a_{r,p}^{(p+1)} = 0$

For $c = p+1$ to $n+1$

$a_{r,c}^{(p+1)} = a_{r,c}^{(p)} - m_{r,p} a_{p,c}^{(p)}$

we have 3 loop

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Error

$$0.37205 * (7) = 2.60435 \approx 2.6044$$

$$0.12345 * (7) = 0.86415 \approx 0.86415$$

نقرب بعد صغر امن

من الضرب بعد كبير

- Gaussian elimination with pivoting:-

To avoid propagation of error we use the pivot element to be the largest in the remaining of the column in $|a_{k-p}| = \max[|a_{pp}|, |a_{p+1,p}|, \dots, |a_{n-1,p}|, |a_{n,p}|]$ and switch row p with row k if

$$k > p$$

Example:-

$$(1.000, 1.000) \text{ is a solution to } \begin{aligned} 1.133x_1 + 5.281x_2 &= 6.414 \\ 24.14x_1 - 1.210x_2 &= 22.93 \end{aligned}$$

Solve the above by Gaussian with pivoting and without pivoting

• without pivoting

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array} \right] \quad m_{21} = \frac{24.14}{1.133} = 21.31$$

$$\rightarrow \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array} \right] \quad \begin{aligned} x_2 &= 1.001 \\ x_1 &= 0.9956 \end{aligned}$$

with pivoting

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array} \right] \quad m_{21} = \frac{1.133}{24.14} = 0.0464$$

$$\rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array} \right] \quad \begin{aligned} x_1 &= 1.000 \\ x_2 &= 1.000 \end{aligned}$$

$$Ax=b$$

Gaussian $[A|b] \rightarrow [U|c] + \text{backsubstitution}$

2. Gauss - Jordan Elimination

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array} \right] \xrightarrow{(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array} \right] \xrightarrow{(3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right]$$

Step	+/-	* / ÷
1	3×2	$3 + 3 \times 2$
2	2×2	$2 + 2 \times 2$
3	1×2	$1 + 1 \times 2$
4	trans	

Solve

$$3x_1 + 2x_2 + 4x_3 = 9$$

$$x_1 - 2x_2 + 3x_3 = 2$$

$$3x_1 + 4x_2 - x_3 = 6$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & -8/3 & 5/3 & -1 \\ 0 & 2 & -5 & -3 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & 1 & -5/8 & 3/8 \\ 0 & 2 & -5 & -3 \end{array} \right]$$

Exercise

Find the total cost for Gauss Jordan elimination

3. Inverse method.

$$[A \mid I] \rightarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array} \mid A^{-1} \right] \rightarrow \left[\begin{array}{ccc|ccc} * & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & 1 \end{array} \right]$$

$$Ax = b$$

$$x = A^{-1}b$$

$$\text{multiplication cost} = 2n^2 - n$$

Cost

Step	+/-	* / ÷
1	5×2	$5 + 5 \times 2$
2	4×2	$4 + 4 \times 2$
3	3×2	$3 + 3 \times 2$
	$\vdots$	$(2n-p)(n-1) + (2n-p)$
		$(2n-p)(n-1)$

$$(2n-1)(n-1) \rightarrow$$

Step	+/-	* / ÷
1	$(2n-1) \times (n-1)$	$(2n-1) + (2n-1)(n-1)$
2	$(2n-2) \times (n-1)$	$(2n-2) + (2n-2)(n-1)$
$\vdots$	$(2n-p) \times (n-1)$	$(2n-p) + (2n-p)(n-1)$
n	$n(n-1)$	$n + n(n-1)$

4. Cramer's method

$$x_i = \frac{|A_i|}{|A|}$$

$$\text{cost} = \frac{16n^3 - 9n^2 - n}{6} \approx 2\frac{2}{3}n^3$$

Find the cost of Cramer's method for 3×3 matrix.

$$x_1 = \frac{|A_1|}{|A|}$$

$$x_2 = \frac{|A_2|}{|A|}$$

$$x_3 = \frac{|A_3|}{|A|}$$

$$4 \mid 3 \times 3 + 3$$

↓ determinant ↓

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\text{Cost} = 4 \times (14) + 3 = 59$$

$$4 \times 3 + 2 = 14$$

↓

C.P.

3.6 L-U Factorization

$$Ax = b$$

$$LUX = b$$

1. $LY = b \rightarrow$ Forward Substitution

2. $UX = Y \rightarrow$ backward substitution (cost = n^2)

$$[A] \rightarrow \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{(n)} \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & \dots & 0 \\ m_{21} & 1 & \dots & 0 \\ m_{31} & m_{32} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Ex:-

Solve using L-U Factorization.

$$4x_1 + 3x_2 - x_3 = -2$$

$$-2x_1 + 4x_2 + 5x_3 = 20$$

$$x_1 + 2x_2 + 6x_3 = 7$$

no switch in Row

$$\begin{bmatrix} 4 & 3 & -1 \\ -2 & 4 & 5 \\ 1 & 2 & 6 \end{bmatrix} \begin{matrix} -2 \\ 20 \\ 7 \end{matrix}$$
$$m_{21} = \frac{-2}{4} = -\frac{1}{2}$$
$$m_{31} = \frac{1}{4} = 0.25$$

$$\begin{matrix} R_2 + 0.5R_1 \\ R_3 - 0.25R_1 \end{matrix} \rightarrow \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix} \begin{matrix} -2 \\ 19 \\ 6.75 \end{matrix}$$
$$m_{32} = \frac{1.25}{-2.5} = -0.5$$

$$\begin{matrix} R_3 + 0.5R_2 \end{matrix} \rightarrow \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} = U \quad \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix} = L$$

$$LY = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \\ 7 \end{bmatrix}$$

$$y_1 = -2$$

$$y_2 = 19$$

$$y_3 = 17$$

backward substitution

$$0x = y$$

$$\begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 19 \\ 17 \end{bmatrix}$$

$$x_3 = \frac{17}{8.5} = 2$$

$$x_2 = -4$$

$$x_1 = 3$$

total cost = backsubstitution

Row operation (addition)

$$\frac{n^3 - n}{3} + n^2 + \frac{n^2 - n}{6} + \frac{2n^3 - 3n^2 + n}{6}$$

Forward substitution

$$= \frac{2n^3 - 2n}{6} + \frac{6n^2}{6} + \frac{6n^2 - 6n}{6} + \frac{2n^3 - 3n^2 + n}{6}$$

$$= \frac{4n^3 + 9n^2 - 7n}{6}$$

Step	+ / -	* / ÷
1	$(n-1)(n-1)$	$(n-1) + (n-1)(n-1)$
2	$(n-2)(n-2)$	$(n-2) + (n-2)(n-2)$
P	$(n-p)^2$	$(n-p) + (n-p)^2$

$$\text{total} = \{(n-p)^2 + \{(n-p) + (n-p)^2\}$$

$$= 2\{(n-p)^2 + \{(n-p)$$

$$= 2 \frac{(n-1)(n)(2n-1)}{6} + \frac{(n-1)(n)}{2}$$

$$= \frac{2(2n^3 - 3n^2 + n)}{6} + \frac{n^2 - n}{2}$$

$$= \frac{4n^3 - 6n^2 + 2n + 3n^2 - 3n}{6}$$

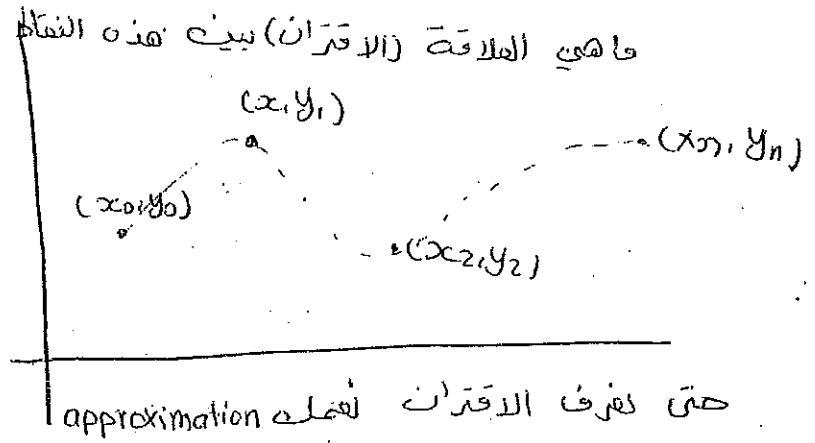
$$= \frac{4n^3 - 3n^2 - n}{6}$$

Chapter 4

Interpolation by polynomials:-

given $(x_0, y_0) (x_1, y_1) (x_2, y_2) \dots (x_n, y_n)$

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n



Curve Fitting
أخذت خط يمر بين النقاط حتى
تكون العلاقة بين النقاط مفيدة

Interpolation :- is estimation of the unknown function by poly which passes through all given points

$$P_n(x_i) = f(x_i)$$

P_n is the approximation polynomial
 f is the unknown function

n Polonomial من الدرجة

إذا كانت $(n+1)$ Points

Example

$(1, 2), (3, 5), (7, 10)$

$$P_2(x) = Ax^2 + Bx + C$$

$$P_2(1) = A + B + C = 2$$

$$P_2(3) = 9A + 3B + C = 5$$

$$P_2(7) = 49A + 7B + C = 10$$

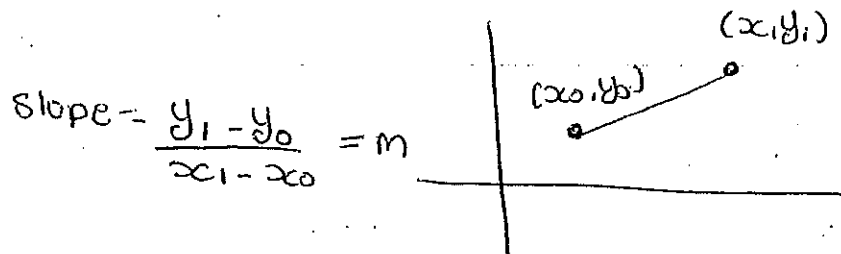
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• given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.

we need to find the polynomial $P_n(x)$ which satisfies

$$P_n(x_i) = y_i, \quad i = 0, \dots, n$$

• given $(x_0, y_0), (x_1, y_1)$.



$$\text{slope} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

$$y - y_0 = m(x - x_0)$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0} (x - x_0) \rightarrow y = y_0 + \frac{y_1 - y_0}{x_1 - x_0} (x - x_0)$$

$$= \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

~~$$P_1(x) = \frac{x - (x_1 + x_2)}{x_0 - (x_1 + x_2)} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 + \frac{x - x_0}{x_2 - x_0} y_2$$~~

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$P_n(x) = \underbrace{\frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}}_{L_{n,0}} y_0 + \underbrace{\frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}}_{L_{n,1}} y_1 + \dots + \underbrace{\frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}}_{L_{n,n}} y_n$$

Lagrange Coefficient Polynomial

$$P_1(x) = \sum_{k=0}^1 L_{1,k}(x) y_k$$

$$P_n(x) = \sum_{k=0}^n L_{n,k}(x) y_k \quad \leftarrow \text{Lagrange Polynomial}$$

Proof $\rightarrow$ $P_n(x_i) = y_i$

$$n=1 \rightarrow P_1(x_0) \stackrel{??}{=} y_0$$

$$P_1(x_0) = y_0$$

$$P_1(x_1) = y_1$$

$$n=2 \rightarrow$$

$$P_2(x_0) = y_0$$

$$P_2(x_1) = y_1$$

$$P_2(x_2) = y_2$$

228
85

$$n=k \rightarrow P_n(x_k) = y_k$$

Example:

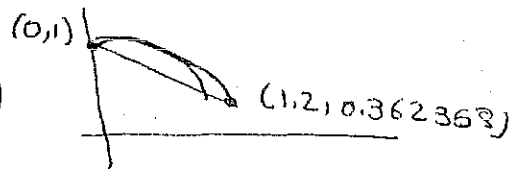
given $f(x) = \cos x$ on $[0, 1.2]$

Find $P_1(x)$, $P_2(x)$, $P_3(x)$ and compare the answers $P_1(0.35)$, $P_2(0.35)$, $P_3(0.35)$ to the exact.

to Find $A(x)$.

$$(x_0, y_0) = (0, \cos 0) = (0, 1)$$

$$(x_1, y_1) = (1.2, \cos 1.2) = (1.2, 0.362358)$$



$$P_1(x) = \frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1$$

$$= \frac{x-1.2}{0-1.2} (1) + \frac{x-0}{1.2-0} (0.362358)$$

$$P_1(x) = -0.833333(x-1.2) + 0.301965x$$

$$P_1(0.35) = -0.833333(0.35-1.2) + 0.301965(0.35)$$

$$= 0.8140208$$

$$\text{exact} = \cos(0.35) = 0.9393727$$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$(x_0, y_0), (x_1, y_1), (x_2, y_2)$$

$$(0, \cos 0), (\cancel{0.6, \cos 0.6}), (1.2, \cos(1.2))$$

$$(0.6, \cos 0.6)$$

$$(0, 1), (0.6, 0.825336), (1.2, 0.362258)$$

$$P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)} (1) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)} (0.825336) + \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)} (0.362258)$$

$$= 0.38889(x-0.6)(x-1.2) - 2.292599x(x-1.2) + 0.503275x(x-0.6)$$

$$P_2(0.35) = 0.9233150528$$

$$h = \frac{1.2-0}{3} = 0.4$$

$$(0, 1), (0.4, 0.921061), (0.8, 0.696707), (1.2, 0.362258)$$

$$P_3(x) = \frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)} (1) + \frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)} (0.921061) \\ + \frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)} (0.696707) + \frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)} (0.362258)$$

$$= -2.60417(x-0.4)(x-0.8)(x-1.2) \dots$$

$$= 0.439607167$$

4.3. Lagrange interpolating polynomial.

given $(x_0, y_0) (x_1, y_1) \dots (x_n, y_n)$

$$P_n(x) = \sum_{k=0}^n L_{n,k}(x) \cdot y_k = L_{n,0}(x) y_0 + L_{n,1}(x) y_1 + \dots + L_{n,n}(x) y_n$$

$$L_{n,i}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

Theory:-

if

$$f(x) = P_n(x) + E_n(x)$$

$(n+1)$

$$E_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi)$$

for the
previous
example

$$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(\xi)$$

$$(0, 1), (0.6, 0.825336), (1.2, 0.362326)$$

$$E_2(x) = \frac{x(x-0.6)(x-1.2)}{6} f^{(3)}(\xi)$$

$$E_2(0.35) = \frac{(0.35)(0.35-0.6)(0.35-1.2)}{6} f^{(3)}(\xi)$$

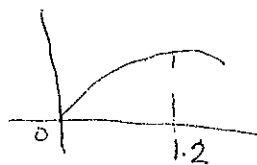
$$|E_2(x)| \leq \frac{|x(x-0.6)(x-1.2)|}{6} \max_{x_0 \leq x \leq x_n} |f^{(3)}(\xi)|$$

$$f(x) = \cos x \quad [0, 1.2]$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$



$$\max |f'''(x)| = \sin(1.2)$$

$$0 \leq x \leq 1.2 \quad \approx 0.9320$$

$$|E_2(x)| \leq \frac{|x(x-0.6)(x-1.2)|}{6} \quad (0.9320).$$

$$|E_2(0.35)| \leq \frac{0.35(0.25)(0.85)}{6} \quad (0.9320)$$

$$= 0.01155$$

$$P_2(0.35) = 0.93315.$$

$$\text{Exact} = 0.9393727.$$

$$\text{Error} = 0.0062$$

Find an upperbound for $E_2(x)$ for all x .

Find the maximum value of the function $E_2(x)$ for all x in the interval $[0, 1.2]$.
 $x_{\max}$ ← maximum value of the function.
 max upper bound. ←

$g'(x) = 0 \rightarrow$ uniform bound.

Theorem:- Uniform bound

For uniform partition $h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$ $n \in \text{integer}$

$$\rightarrow x_k = x_0 + k h$$

$$x_n = x_0 + n h$$

$$\text{Let } M_n = \max_{a \leq x \leq b} |f^{(n)}(x)|$$

$$\text{then } 1. |E_1(x)| \leq \frac{h^2 M_2}{8} \quad \text{For all } x \in [x_0, x_1]$$

$$2. |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} \quad \text{for all } x \in [x_0, x_2]$$

$$3. |E_3(x)| \leq \frac{h^4 M_4}{24} \quad \text{for all } x \in [x_0, x_3]$$

Ex 2-

Using the theorem for the previous example.

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.6)^3 (0.9320)}{9\sqrt{3}} = 0.03587$$

↓
Upper bounded
for all x

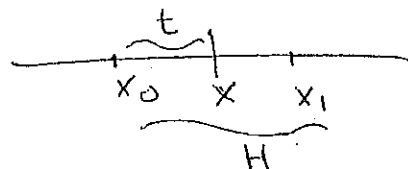
Proof

$$|E_1(x)| \leq \frac{h^2 M_2}{8} \text{ for all } x \in [x_0, x_1]$$

to show $|E_1(x)| \leq \frac{h^2 M_2}{8}$

$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2!} M_2$$

$$h(x) = (x-x_0)(x-x_1)$$



let $t = x - x_0$

$$h(x) = t(x-x_1) = t(t-h)$$

$$x - x_1 \neq \\ = (x_0 + t) - (x_0 + h)$$

$$h(t) = t(t-h)$$

$$h(t) = t^2 - th$$

$$h'(t) = 2t - h = 0$$

$$t = \frac{h}{2} \rightarrow \text{critical point}$$

$$H\left(\frac{h}{2}\right) = \frac{h}{2} \left(\frac{h}{2} - h\right)$$

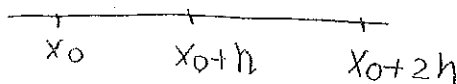
$$= \frac{h}{2} \left(-\frac{h}{2}\right) = -\frac{h^2}{4}$$

$$\max |H(x)| = \frac{h^2}{4}$$

$$E_1(x) \leq \frac{h^2}{4} \cdot \frac{M_2}{2} = \frac{h^2 M_2}{8}$$

for
proof

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$$



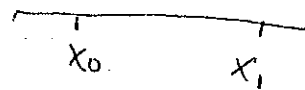
let $x = x_0 + th$
 $0 < t < 2$
so -

Theorem

$$f(x) = P_n(x) + E_n(x).$$

then

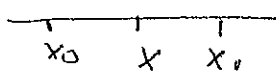
$$E_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{(n+1)}(c).$$



proof: For $n=1$.

To show that the error.

$$E_1(x) = f(x) - P_1(x) \text{ is equal to } \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c).$$



$$\text{Let } h(t) = f(t) - P_1(t) - E_1(x) \frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$$

$h(x)$ are continuous and differentiable.

$$h(x_0) = f(x_0) - P_1(x_0) - E_1(x) \frac{(x_0-x_0)(x_0-x_1)}{(x-x_0)(x-x_1)} = 0$$

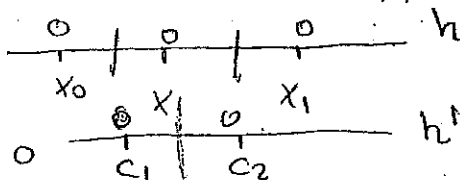
$$h(x_1) = f(x_1) - P_1(x_1) - 0 = 0$$

$$h(x) = f(x) - P_1(x) - E_1(x) \frac{(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)} = f(x) - P_1(x) - E_1(x) = 0$$

$$h(x) = 0.$$

Using MVT on (x_0, x) , $\exists c \in (x_0, x)$.

such that



$$h'(c_1) = \frac{h(x) - h(x_0)}{x - x_0} = 0$$

Similarly $\exists c_2 \in (x_1, x)$ such that

$$h'(c_2) = \frac{h(x) - h(x_1)}{x - x_1} = 0$$

Similarly $\exists c \in (c_1, c_2)$ such that

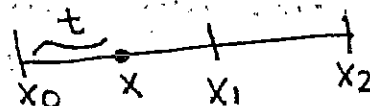
$$h''(c) = \frac{h'(c_2) - h'(c_1)}{c_2 - c_1} = \frac{0 - 0}{c_2 - c_1} = 0$$

• proof $|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$

$$x = x_0 + t$$

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$



$$|E_2(x)| = \frac{(x-x_0)(x-x_1)(x-x_2) F'''(c)}{3!}$$

$$|E_2(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2) M_3}{6}$$

$$x - x_0 = t$$

$$x - x_1 = t - h$$

$$x - x_2 = t - 2h$$

$$|E_2(x)| \leq \frac{(t)(t-h)(t-2h) M_3}{6}$$

$$\phi(t) = t(t-h)(t-2h)$$

$$= (t^2 - th)(t-2h)$$

$$= t^3 - 2ht^2 - ht^2 + 2h^2t = t^3 - 3ht^2 + 2h^2t$$

$$\phi'(t) = 3t^2 - 6ht + 2h^2$$

$$\phi'(t) = 0$$

$$t = \frac{6 \pm \sqrt{36 - 4 \times 3 \times 2}}{6} h$$

$$t = 0.42264973 h$$

$$t = 1.577350269 h$$

$$\phi(t) = 0.384900179 h^3$$

$$|E_2(x)| \leq \frac{0.384900179 h^3 M_3}{6}$$

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$$

• proof $|E_3(x)| \leq \frac{h^4 M_4}{24}$

$$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(\xi)$$

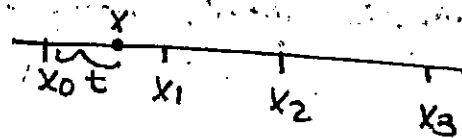
$$|E_3(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{24} M_4$$

$$x = x_0 + t$$

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$x_3 = x_0 + 3h$$



$$x - x_0 = t$$

$$x - x_1 = t - h$$

$$x - x_2 = t - 2h$$

$$x - x_3 = t - 3h$$

$$|E_3(x)| \leq \frac{(t)(t-h)(t-2h)(t-3h)}{24} M_4$$

$$\begin{aligned} \text{Let } \phi(t) &= t(t-h)(t-2h)(t-3h) \\ &= (t^2 - th)(t-2h)(t-3h) \\ &= (t^3 - 2t^2h - t^2h + 2th^2)(t-3h) \\ &= (t^3 - 3t^2h + 2th^2)(t-3h) \\ &= t^4 - 6ht^3 + 11h^2t^2 - 6h^3t \end{aligned}$$

$$\phi'(t) = 4t^3 - 18ht^2 + 22h^2t - 6h^3$$

$$\phi'(t) = 0$$

$$t = 2.618033989h$$

$$= 0.381966011h$$

$$= 0.05h$$

$$\phi(t) = 1 - 1 = 0$$

For $\phi(t)$ the max = h^4 @ $t = 2.618033989h$

$$|E_3(x)| \leq \frac{h^4 M_4}{24}$$

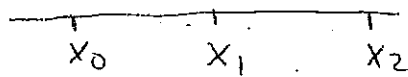
$$h'(t) = f'(t) - P_1'(t) - E_1(x) \left(\frac{(t-x_0) + (t-x_1)}{(x_0-x_1)(x-x_1)} \right)$$

$$h''(t) = f''(t) - \underset{\substack{\downarrow \\ P_1''(t) \text{ because} \\ \text{the function} \\ \text{is linear}}}{0} - \frac{E_1(x)(2)}{(x_0-x_1)(x-x_1)}$$

$$\rightarrow h''(c) = f''(c) - \frac{2E_1(x)}{(x-x_0)(x-x_1)} = 0$$

$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2} f''(c)$$

for $n=2$



Exercise

$$h(t) = f(t) - P_2(t) - E_2(x) \frac{(t-x_0)(t-x_1)(t-x_2)}{(x-x_0)(x-x_1)(x-x_2)}$$

$$h(x_0) = f(x_0) - P_2(x_0) - 0$$

$$h(x_1) = f(x_1) - P_2(x_1) - 0$$

$$h(x_2) = f(x_2) - P_2(x_2) - 0$$

$$h(x) = f(x) - P_2(x) - E_2(x) \cdot (1) \\ = 0$$

5.4 Newton interpolation polynomial

given $x_0, x_1, x_2, \dots, x_n$.

$(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

$$P_n(x) = f(x)$$

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$$

$$P_3(x) = P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$P_1(x_0) = a_0 + a_1(x_0 - x_0)$$

$$P_1(x_0) = a_0 = f(x_0) = y_0 \rightarrow \boxed{a_0 = y_0}$$

$$P_1(x_1) = \underset{a_0}{f(x_0)} + a_1(x_1 - x_0) = f(x_1)$$

$$\boxed{a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}} = F[x_0, x_1] \quad \text{First divided difference.}$$

$$f(x_2) = P_2(x_2) = f(x_0) + F[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \frac{f(x_2) - f(x_0) - F[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)} = F[x_0, x_1, x_2]$$

Then

$$a_{12} = F[x_0, x_1, x_2, \dots, x_{12}] \quad \text{1st } k^{\text{th}} \text{ divided difference.}$$

Definition

$$F[x_k] = f(x_k) \quad \text{Zero}^{\text{th}} \text{ divided difference.}$$

$$F[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad \text{1st divided difference.}$$

2nd divided difference.

$$F[x_{k-2}, x_{k-1}, x_k] = \frac{F[x_{k-1}, x_k] - F[x_{k-2}, x_{k-1}]}{x_k - x_{k-2}}$$

3rd divided difference

$$F[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = \frac{F[x_{k-2}, x_{k-1}, x_k] - F[x_{k-3}, x_{k-2}]}{x_k - x_{k-3}}$$

example:-

$$(1, 3), (2, 5), (4, 7), (8, 11), (9, 15)$$

$$\begin{aligned} F[2, 4, 8] &= \frac{F[4, 8] - F[2, 4]}{8 - 2} = \frac{\frac{f(8) - f(4)}{8 - 4} - \frac{f(4) - f(2)}{4 - 2}}{6} \\ &= \frac{\frac{11 - 7}{4} - \frac{7 - 5}{2}}{6} = 0 \end{aligned}$$

$$\begin{aligned} F[1, 2, 4, 8] &= \frac{F[2, 4, 8] - F[1, 2, 4]}{8 - 1} \\ &= \frac{\frac{f(4, 8) - f(2, 4)}{8 - 2} - \left[\frac{f(2, 4) - f(1, 2)}{4 - 1} \right]}{8 - 1} \end{aligned}$$

another way

x_k	$f(x_k)$	1 st divided $f[x_{k-1}, x_k]$	2 nd divided $f[x_{k-2}, x_{k-1}, x_k]$	3 rd divided $f[x_{k-3}, x_{k-2}, x_{k-1}, x_k]$
x_0	$f(x_0)$ a_0			
x_1	$f(x_1)$	$f[x_0, x_1]$ a_1		
x_2	$f(x_2)$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$ a_2	
x_3	$f(x_3)$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$ a_3
x_4	$f(x_4)$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$
x_5	$f(x_5)$	$f[x_4, x_5]$	$f[x_3, x_4, x_5]$	$f[x_2, x_3, x_4, x_5]$
x_6	$f(x_6)$	$f[x_5, x_6]$	$f[x_4, x_5, x_6]$	$f[x_3, x_4, x_5, x_6]$

example

Find Newton interp $P_1, P_2, P_3, P_4, \dots$ for the following table

x_k	$f(x_k)$	1 st	2 nd	3 rd	4 th
1	-3 a_0				
2	0	3 a_1			
3	15	15	6 a_2		
4	48	33	9	1 a_3	
5	105	67	12	1	0
6	192	87	15	1	0

$$P_1(x) = a_0 + a_1(x - x_0) = -3 + 3(x - 1)$$

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1) = -3 + 3(x - 1) + 6(x - 1)(x - 2)$$

$$P_3(x) = P_2(x) + 1(x - 1)(x - 2)(x - 3)$$

$$P_4 = P_3$$

$$P_5 = P_4 = P_3$$

Note:-

Error for newton interpolation polynomial equal to the error for lagrange because they uses the same Polynomial.

Example:-

Estimate $f(5.5)$ using Newton int polynomial P_1, P_2, P_3 for the following table.

x_i	$f(x_i)$	1 st	2 nd	3 rd	4 th
1	3	/	/	/	/
3	4.5	0.75	/	/	/
4.25	6		0.138462	/	/
5.75	7.25			0.05722	/
6	8				0.10287

$$P_1(x) = a_0 + a_1(x-x_0)$$

$$= 3 + 0.75(x-1)$$

$$P_1(5.5) \approx P_1(5.5)$$

$$= 3 + 0.75(5.5-1)$$

$$= 6.375$$

$$P_2(x) = P_1(x) + a_2(x-1)(x-3)$$

$$= 0.13846(x-1)(x-3) + P_1(x)$$

$$P_2(5.5) \approx P_2(5.5)$$

$$P_3(x) = P_2(x) + 0.05722(x-1)(x-3)(x-4.25)$$

$$\approx 6.375 + 0.13846(5.5-1)(5.5-3)$$

$$\approx 7.93267$$

$$P_3(5.5) \approx P_3(5.5)$$

$$\approx 7.93267 + 0.05722(5.5-1)(5.5-3)(5.5-4.25)$$

$$\approx 8.7373$$

$$P_4(x) = P_3 + 0.10287(x-1)(x-3)(x-4.5)(x-5.75)$$

$$f(5.5) \approx P_4(5.5)$$

$$\approx 8.7373 + 0.10287(4.5)(2.5)(1.25)(-0.25)$$

$$\approx 8.375.$$