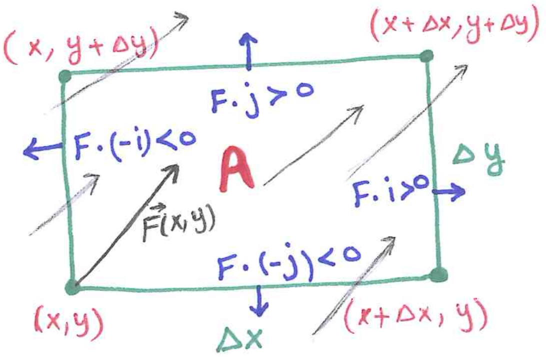
16.4 Green's Theorem in the Plane

- Recall that if $\vec{F}$ is a coservative vector field, then $\vec{F} = \nabla f$ for a diff function f. Thus, we calculate the line integral of $\vec{F}$ over any path C joining A to B as

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

- In this section we derive a method (Green's Theorem) to compute a work or flux integral over a closed curve C in the plane when the field $\vec{F}$ is not conservative.
- We will see Green's Theorem applies to any vector field, but we focus on velocity fields of fluid flows like a liquid flow (not compressible) and gas flow (compressible) as they are easy to visualize.

Divergence:



- Let $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a velocity field of a fluid flowing in the plane.
- Suppose the first partial derivatives of M and N are continuous at each point of a region R.
- Let (x,y) be a point in R and a corner point of a rectangle A as in this Figure, where A lies entirely in R.
- The small rectangle A has sides Δx and Δy parallel to the coordinate axes.
- Assume the components M and N do not change sign throughout a small region containing A.

- The rate at which fluid leaves the rectangle across the rectangle is approximately

$\approx$ top edge flow + bottom edge flow + right edge flow + left edge flow

$$\approx \vec{F}(x, y+\Delta y)\cdot\vec{j}\ \Delta x + \vec{F}(x,y)\cdot(-\vec{j})\ \Delta x + F(x+\Delta x, y)\cdot\vec{i}\ \Delta y + F(x,y)\cdot(-\vec{i})\ \Delta y$$

$$= N(x, y+\Delta y)\,\Delta x - N(x,y)\,\Delta x + M(x+\Delta x, y)\,\Delta y - M(x,y)\,\Delta y$$

$$= \left[N(x, y+\Delta y) - N(x,y)\right]\Delta x + \left[M(x+\Delta x, y) - M(x,y)\right]\Delta y$$

$$= \frac{N(x, y+\Delta y) - N(x,y)}{\Delta y}\,\Delta x\,\Delta y + \frac{M(x+\Delta x, y) - M(x,y)}{\Delta x}\,\Delta x\,\Delta y$$

$$\approx N_y\,\Delta x\,\Delta y + M_x\,\Delta x\,\Delta y$$

Hence, the net effect of the flow rates or the

$$\text{Flux across rectangle boundary} \approx (M_x + N_y)\,\Delta x\,\Delta y$$

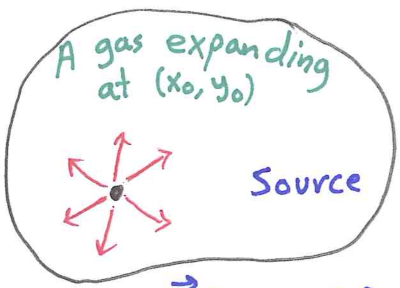
- To estimate the total flux per unit area (or flux density) for the rectangle, we divide by $\Delta x\,\Delta y$ to get:

$$\frac{\text{Flux across rectangle boundary}}{\text{Rectangle area}} \approx M_x + N_y$$

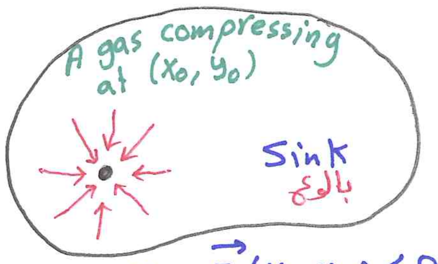
- Finally, let $\Delta x \to 0$ and $\Delta y \to 0$ to define the flux density of $\vec{F}$ at the point (x,y) or the divergence of $\vec{F}$ at (x,y):

$$\boxed{\operatorname{div}\vec{F} = M_x + N_y}$$

Remark:
- A gas is compressible and liquid is not.
- If gas is expanding at the point (x_0, y_0), then lines of flow would diverge there and gas would be flowing out of a small rectangle about (x_0, y_0) which makes $\text{div}\,\vec{F}(x_0, y_0) > 0$.
- If gas is compressing at the point (x_0, y_0), then lines of flow would be flowing in about (x_0, y_0) which makes $\text{div}\,\vec{F}(x_0, y_0) < 0$.



$\text{div}\,\vec{F}(x_0, y_0) > 0$

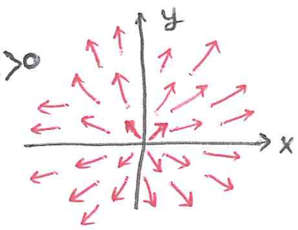


$\text{div}\,\vec{F}(x_0, y_0) < 0$

Exp The following vector fields represent the velocity of a gas flowing in the xy-plane. Find the divergence of each vector field and interpret its physical meaning.

a Uniform expansion $\vec{F}(x,y) = cx\,\vec{i} + cy\,\vec{j}$, $c > 0$

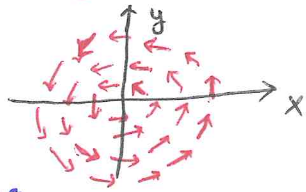
$\text{div}\,\vec{F} = M_x + N_y = c + c = 2c > 0$



b Uniform rotation $\vec{F}(x,y) = -cy\,\vec{i} + cx\,\vec{j}$

$\text{div}\,\vec{F} = M_x + N_y = 0 + 0 = 0$

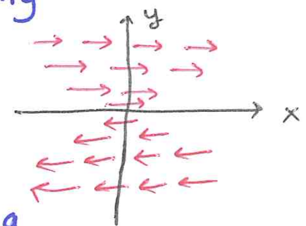
The gas is neither expanding nor compressing



c Shearing flow $\vec{F}(x,y) = y\,\vec{i}$

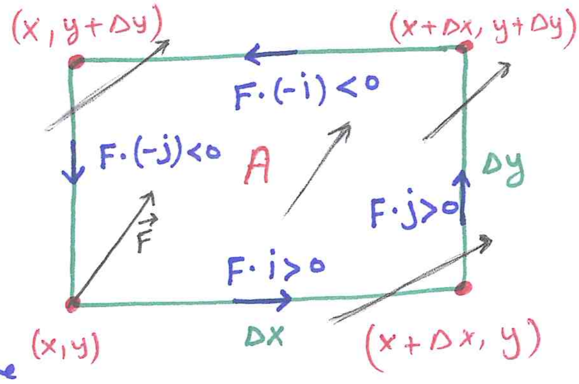
$\text{div}\,\vec{F} = M_x + N_y = 0 + 0 = 0$

The gas is neither expanding nor compressing



Spin around an axis: The k-component of Curl

- After divergence, we need a second idea for Green's Theorem which is the circulation density of a vector field $\vec{F} = M\vec{i} + N\vec{j}$ at point (x, y).



- Same as before, but we approximate the rate of flow along each edge in the directions of blue arrows

$$\approx \vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x + \vec{F}(x,y)\cdot \vec{i}\,\Delta x + \vec{F}(x+\Delta x, y)\cdot\vec{j}\,\Delta y + \vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$$

$$= -M(x, y+\Delta y)\,\Delta x + M(x,y)\,\Delta x + N(x+\Delta x, y)\,\Delta y - N(x,y)\,\Delta y$$

$$= -\left[\frac{M(x, y+\Delta y) - M(x,y)}{\Delta y}\right]\Delta x\,\Delta y + \frac{\left[N(x+\Delta x, y) - N(x,y)\right]}{\Delta x}\Delta x\,\Delta y$$

$$\approx -M_y\,\Delta x\,\Delta y + N_x\,\Delta x\,\Delta y$$

Hence, the net circulation counterclockwise orientation is

$$\text{Circulation around rectangle} \approx (N_x - M_y)\,\Delta x\,\Delta y$$

- To estimate the circulation density of $\vec{F}$ at the point (x, y), we divide by $\Delta x\,\Delta y$ to get

$$\frac{\text{Circulation around rectangle}}{\text{Rectangle area}} \approx N_x - M_y$$

- Finally, Let $\Delta x \to 0$ and $\Delta y \to 0$ to define the circulation density (or the k-component of the curl) of a vector field $\vec{F}$ at (x, y) is

$\vec{k} \perp$ بدون

$$\boxed{(\text{curl } \vec{F})\cdot\vec{k} = N_x - M_y}$$

Exp Find the circulation density (with interpretation) for the vector fields described in Exp':

[a] Uniform expansion $\vec{F}(x,y) = cx\,\vec{i} + cy\,\vec{j}$ $M = cx$, $N = cy$

$(\text{curl}\,\vec{F})\cdot\vec{k} = N_x - M_y = 0 - 0 = 0$

- The gas is not circulating at very small scales.

[b] Uniform rotation $\vec{F}(x,y) = -cy\,\vec{i} + cx\,\vec{j}$ $M = -cy$, $N = cx$

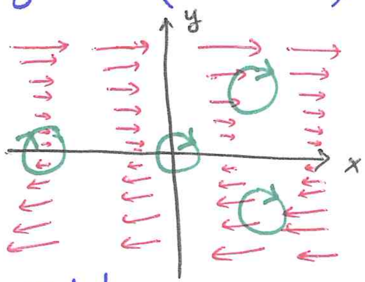
$(\text{curl}\,\vec{F})\cdot\vec{k} = N_x - M_y = c + c = 2c$

- The circulation density is constant and there is a rotation at every point
- If $c > 0$, then the rotation is counterclockwise and if $c < 0$, then the rotation is clockwise.

[c] Shearing flow $\vec{F}(x,y) = y\,\vec{i}$ $M = y$, $N = 0$

$(\text{curl}\,\vec{F})\cdot\vec{k} = N_x - M_y = 0 - 1 = -1$

- The circulation density is constant and negative (clockwise)
- The average effect of the fluid flow is to push fluid clockwise around each of the small circles



- The rate of rotation is the same at each point.

Recall that
- the curve C is simple if it does not cross itself. (147 page)
- the outward flux of $\vec{F}$ across C is (see also page 147)

$$\oint_C M\,dy - N\,dx = \int_C \vec{F}\cdot\vec{n}\,ds$$

- the circulation around the curve C that starts and ends at same point is the flow integral $\int_C \vec{F}\cdot\vec{T}\,ds$ (see page 145)

Th (Green's Theorem)

Let C be a piecewise smooth and simple closed curve enclosing a region R in the plane.

Let $\vec{F} = M\vec{i} + N\vec{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R.

Then • the outward flux of $\vec{F}$ across C equals the double integral of div $\vec{F}$ over the region R enclosed by C:

Normal form

$$\underbrace{\oint_C \vec{F}\cdot\vec{n}\, ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R (M_x + N_y)\, dx\, dy}_{\text{Divergence integral}} \quad *^1$$

• the counterclockwise circulation of $\vec{F}$ around C equals the double integral of $(\text{curl } \vec{F})\cdot\vec{k}$ over R:

Tangent form

$$\underbrace{\oint_C \vec{F}\cdot\vec{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C M\,dx + N\,dy = \underbrace{\iint_R (N_x - M_y)\, dx\, dy}_{\text{Curl integral}} \quad *^2$$

Notes: • $*^1$ and $*^2$ are equivalent. That is, applying $*^1$ to the field $G^1 = N\vec{i} - M\vec{j}$ gives $*^2$ and applying $*^2$ to the field $G^2 = -N\vec{i} + M\vec{j}$ gives $*^1$.

• The simple closed curve C can be traversed in two directions:

1. The curve C is traversed counterclockwise (and called positively oriented) if the region C encloses is always to the left of an object as it moves along the path.
2. Otherwise, C is traversed clockwise (and called negatively oriented).

Exp Verify both forms of Green's Theorem for the vector field $\vec{F} = -x^2 y\,\vec{i} + xy^2\,\vec{j}$ and the region R is the disk $x^2+y^2 \leq 4$ bounded by the circle $C: \vec{r} = (2\cos t)\vec{i} + (2\sin t)\vec{j}$, $0 \leq t \leq 2\pi$. 164

- $x = 2\cos t$ and $y = 2\sin t$ $\Rightarrow$ $dx = -2\sin t\, dt$, $dy = 2\cos t\, dt$
- $M = -x^2 y = -8\cos^2 t \sin t$
  $N = xy^2 = 8\cos t \sin^2 t$
- $M_x = -2xy$, $M_y = -x^2$, $N_x = y^2$, $N_y = 2xy$
- Flux (form *1): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} (-16\cos^3 t \sin t + 16\cos t \sin^3 t)\,dt$

$$= \Big[4\cos^4 t + 4\sin^4 t\Big]_0^{2\pi} = 0$$

$$\iint_R (M_x + N_y)\,dx\,dy = \iint_R (-2xy + 2xy)\,dx\,dy = 0$$

- Circulation (form *2):

$$\oint_C M\,dx + N\,dy = \int_0^{2\pi} (16\cos^2 t \sin^2 t + 16\cos^2 t \sin^2 t)\,dt$$

$$= \int_0^{2\pi} 32\cos^2 t \sin^2 t\,dt = \int_0^{2\pi} 16\sin^2 2t\,dt$$

$$= 4\int_0^{4\pi} \sin^2 u\,du = 4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = 8\pi$$

$$\iint_R (N_x - M_y)\,dx\,dy = \iint_R (y^2 + x^2)\,dx\,dy = \int_0^{2\pi}\int_0^2 r^2(r)\,dr\,d\theta$$

$$= \int_0^{2\pi} 4\,d\theta = 8\pi$$

Exp Use Green's Theorem to find the counterclockwise circulation and outward flux for the vector field $\vec{F} = (x^2+4y)\,\vec{i} + (x+y^2)\,\vec{j}$ and the curve

C : the squar bounded by $x=0$, $x=1$, $y=0$, $y=1$.

- $M = x^2 + 4y \Rightarrow M_x = 2x$ and $M_y = 4$

  $N = x + y^2 \Rightarrow N_x = 1$ and $N_y = 2y$

- Flux $= \iint_R (M_x + N_y)\,dx\,dy = \iint_R (2x+2y)\,dx\,dy$

  $= \int_0^1\int_0^1 (2x+2y)\,dx\,dy = \int_0^1 (x^2+2xy)\Big|_0^1 dy = \int_0^1 (1+2y)\,dy = 2$

- Circulation $= \iint_R (N_x - M_y)\,dx\,dy = \iint_R (1-4)\,dx\,dy$

  $= \int_0^1\int_0^1 -3\,dx\,dy = -3$

Exp Evaluate the integral $\oint_C (3y\,dx + 2x\,dy)$

- Apply Green's Theorem form $*^2$
- $M = 3y \Rightarrow M_y = 3$

  $N = 2x \Rightarrow N_x = 2$

- $\oint_C (3y\,dx + 2x\,dy) = \iint_R M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$

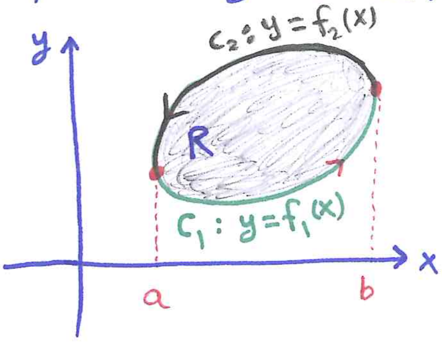
  $= \iint_R (2-3)\,dx\,dy$

  $= \int_0^{\pi}\int_0^{\sin x} (-1)\,dx\,dy$

  $= -\int_0^{\pi} \sin x\,dx = -2$

Proof of Green's Theorem for special Regions

- Let C be smooth simple closed curve in xy-plane as given here:
- Let R be the region enclosed by C.
- Given $\vec{F} = M\vec{i} + N\vec{j}$ with M, N and their first partial derivatives are continuous at every point in some open region containing C and R.



- We need to prove the circulation-curl form $*^2$:

$$\oint M\,dx + N\,dy = \iint_R \left(N_x - M_y\right) dx\,dy$$

- Note that the curve C consists of two directed parts:

$$C_1 : y = f_1(x) \;,\quad a \overset{\rightarrow}{\leq} x \leq b \quad \text{and}$$

$$C_2 : y = f_2(x) \;,\quad a \leq x \overset{\leftarrow}{\leq} b$$

- $\forall\, x \in [a,b] \Rightarrow \displaystyle\int_{f_1(x)}^{f_2(x)} M_y\,dy = M(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)} = M(x,f_2(x)) - M(x,f_1(x))$

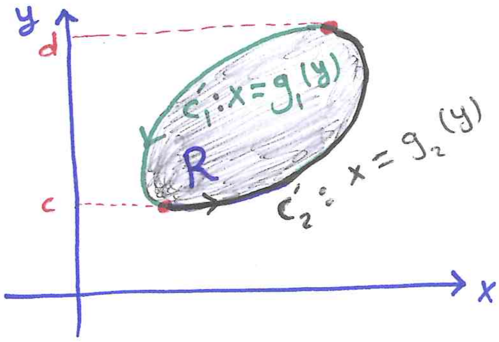
$$\Rightarrow \int_a^b \int_{f_1(x)}^{f_2(x)} M_y\,dy\,dx = \int_a^b \left[M(x,f_2(x)) - M(x,f_1(x))\right] dx$$

$$= -\int_b^a M(x,f_2(x))\,dx - \int_a^b M(x,f_1(x))\,dx$$

$$= -\int_{C_2} M\,dx - \int_{C_1} M\,dx = -\oint_C M\,dx$$

- Hence, $\displaystyle\oint_C M\,dx = \iint_R -M_y\,dx\,dy \quad \cdots \quad (1)$

- Now we derive the other half by integrating N_x w.r.t x then w.r.t y.



- Note that the curve C consists of two directed parts:

$C_1' : x = g_1(y)$, $c \leq y \leq d$ (←)

$C_2' : x = g_2(y)$, $c \leq y \leq d$ (→)

- $\forall\, y \in [c,d] \Rightarrow \displaystyle\int_{g_1(y)}^{g_2(y)} N_x \, dx = N(x,y)\Big|_{x=g_1(y)}^{x=g_2(y)} = N(g_2(y),y) - N(g_1(y),y)$

$$\Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} N_x \, dx \, dy = \int_c^d \left[N(g_2(y),y) - N(g_1(y),y)\right] dy$$

$$= \int_c^d N(g_2(y),y)\, dy + \int_d^c N(g_1(y),y)\, dy$$

$$= \int_{C_2'} N \, dy + \int_{C_1'} N \, dy = \oint_C N \, dy$$

- Hence, $\displaystyle\oint_C N \, dy = \iint_R N_x \, dx \, dy$ ---- ②

- Thus, adding ① + ② gives:

$$\oint M \, dx + \oint N \, dy = \oint M \, dx + N \, dy$$

$$= \iint_R (N_x - M_y) \, dx \, dy$$