

Chapter 8: External Direct products.

Def : External Direct product.

Let G_1, G_2, \dots, G_n be a finite collection of groups. The external direct product of G_1, G_2, \dots, G_n written as $G_1 \oplus G_2 \oplus \dots \oplus G_n$, is the set of all n -tuples for which the i th component is an element of G_i and the operation is componentwise.

مُتَعَدِّل (متعدد)

Ex1 :

$$U(8) \oplus U(10) = U(8) = \{1, 3, 5, 7\}, \quad U(10) = \{1, 3, 7, 9\}$$

$$= \left\{ (1,1), (1,3), (1,7), (1,9), (3,1), (3,3), (3,7), (3,9) \right. \\ \left. (5,1), (5,3), (5,7), (5,9), (7,1), (7,3), (7,7), (7,9) \right\}$$

Ex2 :

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{0, 1\} \oplus \{0, 1, 2\}$$

$$= \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

(1,1) cyclic (ذري)

$\rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3$ the order equal 6, is cyclic pp? Yes

$$(1,1)^1 = (1,1)$$

$\rightarrow (\mathbb{Z}_6, \oplus)$ the order equal 6 and cyclic.

$$(1,1)^2 = ((1+1)_2 + (1+1)_3) = (0,2)$$

$$(1,1)^3 = (1,0)$$

\Rightarrow so $(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$ isomorphic to (\mathbb{Z}_6, \oplus) .

$$(1,1)^4 = (0,1)$$

$$(1,1)^5 = (1,2)$$

But since order of $(S_3, \otimes) = 6$

$$(1,1)^6 = (0,0) \text{ identity}$$

$(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$ Not isomorphic to (S_3, \otimes) .

so its cyclic.

→ The groups of order 6 up isomorphism = 2

$$2 : \textcircled{1} (\mathbb{Z}_6, +) = (\mathbb{Z}_2 \oplus \mathbb{Z}_3)$$

$$\textcircled{2} (S_3, \alpha).$$

exp3:

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$$

Not cyclic: $(0,1)^1 = (0,1)$

$$(0,1)^2 = (0,0)$$

$$|(0,1)| = 2$$

$(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong (\mathbb{Z}_4, \circ)$ isomorphic \Leftrightarrow 4 will order 4, 2, 1, 5

Thm 8.1: order of an element in a Direct product.

The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols: $|g_1 g_2 \dots g_n| = \text{LCM}(|g_1|, |g_2|, \dots, |g_n|)$.

exp: $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 = \{ \dots, 1, 4, 0, 3, \dots \}$

$$|G| = 4 \times 2 \times 5 = 40$$

→ let for exp $|(2,1,3)| = \text{LCM}(|2|_{\text{on } \mathbb{Z}_4}, |1|_{\text{on } \mathbb{Z}_2}, |3|_{\text{on } \mathbb{Z}_5})$

$$= \text{LCM}(2, 2, 5)$$

$$|(2,1,3)| = \underline{\underline{10}}$$

Ex 4: number of element of order 5. in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$

$$|(a,b)| = 5$$

case 1 : $|a|=1$ and $|b|=1$ or 5.

$|a|$ on \mathbb{Z}_{25} :

5 in order \mathbb{Z}_{25} is element φ^5

$$a: 5, 10, 15, 20$$

$$b: \underbrace{0, 1, 2, 3, 4}$$

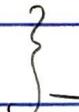
5 in 1 w order \mathbb{Z}_5 is element φ^{45}

5 element on a and 4 element on b $= 4 \times 5 = 20$.

case 2 : $|a|=1$ and $|b|=5$.

$$a: 0$$

$$b: 1, 2, 3, 4$$



$$1 \times 4 = 4$$

Thus, $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$ has 24 element of order 5.

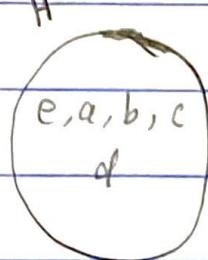
24 element of order 5 since $25 \times 5 = 125$ element

\Leftrightarrow 5 subgroups of order $(\mathbb{Z}_{25} \oplus \mathbb{Z}_5)$ is cyclic subgroup $\varphi(\mathbb{Z}_{25} \oplus \mathbb{Z}_5) \Leftarrow$

5 groups

each cyclic group of order 5 contains 4 elements of order 5. (24)

exp:



prime number
the group H have $\frac{5}{1}$ elements

$|H| = 5$ so H is cyclic since 5 is prime

on H any $x \neq e \Rightarrow |x| \mid |H|$.

$|x| \mid 5$ divisor of $\frac{5}{1}$

$$\underline{|x|=5}$$

$$\text{so } |a|=|b|=|c|=|d|=5.$$

$\frac{x \neq e}{\text{order} \neq 1}$

exp 5+6 .

on 5 : since $|H|=10$, $H=\langle a \rangle$

so the generators : a^1, a^3, a^7, a^9

$\rightarrow \text{all } 2^1 :$

Theorem 8.2 : criterion for $G \oplus H$ to be cyclic :

let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic iff $|G|$ and $|H|$ are relatively prime.

exp: $\mathbb{Z}_5 \oplus \mathbb{Z}_7$

$|\mathbb{Z}_5|=5$, $|\mathbb{Z}_7|=7$ and Both are cyclic.

$\Rightarrow |\mathbb{Z}_5 \oplus \mathbb{Z}_7| = 35 \Rightarrow (\mathbb{Z}_5 \oplus \mathbb{Z}_7)$ is cyclic of order 35.

$\Rightarrow \mathbb{Z}_5 \oplus \mathbb{Z}_7 = \langle (3,3) \rangle, \langle (1,1) \rangle, \langle (2,3) \rangle, \dots$

\rightarrow Relatively prime when there are no common factors other than 1.

$\rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$ and cyclic.

$\rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12}$ and cyclic.

Corollary 1: criterion for $G_1 \oplus G_2 \oplus \dots \oplus G_n$ to be cyclic.

An external direct product $G_1 \oplus G_2 \oplus G_3 \oplus \dots \oplus G_n$ of a finite number of finite cyclic groups is cyclic iff $|G_i|$ and $|G_j|$ are relatively prime when $i \neq j$.

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \xrightarrow{\text{isomorphism}} \mathbb{Z}_{30} \text{ and cyclic.}$$

(1,1,1) is the generator.

$$\begin{aligned} \emptyset : (1,1,1) &\rightarrow 1 && \text{the isomorphism.} \\ +(1,1,1) : (0,2,2) &\rightarrow 2 \\ +(1,1,1) : (1,0,3) &\rightarrow 3 \\ &\vdots && \end{aligned}$$

Corollary 2: criterion for $\mathbb{Z}_{n_1 n_2 \dots n_k} \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_k}$.

Let $m = n_1 n_2 \dots n_k$. Then \mathbb{Z}_m is isomorphic to $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_k}$ iff n_i and n_j are relatively prime when $i \neq j$.

$$\mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_{11} = \mathbb{Z}_{3,5,7,11}.$$

Date.