بسم الله الرحمن الرحيم

MATH 1321

Summary

Aseel hamed

8.2 Trigonometric Integrals

* $\sin^2 x + \cos^2 x = 1$
* $\cos 2x = 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$
* $\sin^2 x = \frac{1-\cos 2x}{2}$
* $\cos^2 x = \frac{1+\cos 2x}{2}$

Exp: $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$

$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$

Exp: $\int \cos^{\overset{\text{odd}}{3}} x \sin^2 x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int \cos^2 x \sin^2 x \cos x \, dx$

$= \int (1-\sin^2 x) u^2 \, du$

$= \int (1-u^2) u^2 \, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C$

Exp: $\int \sin^5 x \cos^2 x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \int \sin^4 x \cos^2 x \sin x \, dx$

$= \int \sin^2 x \sin^2 x \, u^2 (-du)$

$= \int (1-\cos^2 x)(1-\cos^2 x) u^2 (-du)$

$= -\int u^2 (1-u^2)^2 \, du$

$= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$

Exp: $\int \cos x \sin^3 x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int u^3 \, du$

$= \frac{u^4}{4} + C$

$\overbrace{}^{\text{not odd}}$

Exp: $4\int \sin^2 x \cos^2 x \, dx$

$= 4\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$

$= \int (1-\cos^2 2x)\, dx$

$= x - \int \cos^2 2x \, dx$

$= x - \int \frac{1+\cos 4x}{2} \, dx$

$= x - \frac{x}{2} - \int \frac{\cos 4x}{2} \, dx$

$= \frac{x}{2} - \frac{1}{2} \int \cos 4x \, dx$

$= \frac{x}{2} - \frac{\sin 4x}{8} + C$

Exp: $\int \cos 3x \sin 5x \, dx$

$n = 3$ ، $m = 5$

$= \frac{1}{2} \int \sin 2x + \sin 8x \quad dx$

$= \frac{1}{2}\left[\frac{-\cos 2x}{2} - \frac{\cos 8x}{8}\right] + C$

حفظ

$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

Exp: $\int \cos 4x \cos 7x \, dx$

$\cos -x = \cos x$

$\sin -x = -\sin x$

$= \frac{1}{2} \int [\cos(-3x) + \cos(11x)] \, dx$

$= \frac{1}{2} \int \cos 3x + \cos 11x \, dx$

$= \frac{1}{2} \left[\frac{\sin 3x}{3} + \frac{\sin 11x}{11} \right] + C$

Exp: $\int_0^{\pi} \sqrt{1 - \sin^2 x} \, dx$

$\int_0^{\pi} \sqrt{\cos^2 x} \, dx$

$\int_0^{\pi} |\cos x| \, dx$

$\int_0^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x \, dx$

$= \sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\frac{\pi}{2}}^{\pi}$

$= 2 \left(\sin x \Big|_0^{\frac{\pi}{2}} \right)$

$= 2(1-0) = 2$

Exp: $\int \sec^4 x \, dx$

$= \int \sec^2 x \sec^2 x \, dx$

$= \int (\tan^2 x + 1) \sec^2 x \, dx$ $\qquad u = \tan x$

$du = \sec^2 dx$

$= \int (u^2 + 1) \, du$

$= \frac{u^3}{3} + u + C$

8.2: outline

5

$$\int \sin^3 x \, dx$$

$$= \int \sin^2 x \sin x \, dx$$

$$= \int (1-\cos^2 x) \sin x \, dx$$

$u = \cos x$

$du = -\sin x$

$$= -\int (1-u^2) \, du$$

$$= \frac{u^3}{3} - u + c$$

11

$$\int \sin^3 x \cos^3 x \, dx$$

$$= \int \sin^3 x \cos^2 x \cos x \, dx$$

$$= \int \sin^3 x (1-\sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^3 (1-u^2) du$$

$$= \int u^3 - u^5 \, du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + C$$

18

$$\int 8 \cos^4(2\pi x) \, dx$$

$$= 8 \int (\cos^2(2\pi x))^2 \, dx$$

$$= 8 \int \left(\frac{1+\cos 4\pi x}{2}\right)^2$$

$$= 2 \int 1 + 2\cos 4\pi x + \frac{1+\cos 8\pi x}{2} \, dx$$

$$= 3x + \frac{\sin 4\pi x}{\pi} + \frac{\sin 8\pi x}{8\pi} + C$$

22

$$\int_0^{\frac{\pi}{2}} \sin^2(2\phi) \cos^3(2\phi) \, d\phi$$

$\sin 2\phi = u$

$2\cos 2\phi = \frac{du}{dx}$

$$= \int \sin^2(2\phi) (1-\sin^2(2\phi)) \cos(2\phi)$$

$$= \int u^2 (1-u^2) \frac{du}{2}$$

$$= \frac{1}{2} \int u^2 - u^4 \, du$$

$$\frac{u^3}{3} - \frac{u^5}{5} \Big|_0^0 = 0$$

28

$$\int_0^{\frac{\pi}{6}} \sqrt{1+\sin x} \, dx$$

$$\int_0^{\frac{\pi}{6}} \sqrt{1+\sin x} \cdot \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$$

$$\int_0^{\frac{\pi}{6}} \frac{\sqrt{(1+\sin x)(1-\sin x)}}{\sqrt{1-\sin x}} \, dx$$

$$\int_0^{\frac{\pi}{6}} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}} \, dx$$

$$\int_0^{\frac{\pi}{6}} \frac{|\cos x|}{\sqrt{1-\sin x}} \, dx$$

موجب في هاي الفترة

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{\sqrt{1-\sin x}} \, dx$$

$u = \sqrt{1-\sin x}$

$du = \frac{-\cos x}{2\sqrt{1-\sin x}}$

$$\int_1^{\frac{1}{\sqrt{2}}} -2 \, du = -2\left(\frac{1}{\sqrt{2}} - 1\right)$$

38 $\int \sec^4 x \tan^2 x \, dx$

$= \int \sec^2 x \sec^2 x \tan^2 x \, dx$

$= \int \sec^2 x \, (1+\tan^2 x) \tan^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \int (1+u^2) u^2 \, du$

$= \frac{u^3}{3} + \frac{u^5}{5} + C$

47 $\int \tan^5 x$

$= \int \tan^3 x \tan^2 x \, dx$

$= \int \tan^3 x \, (\sec^2 - 1) \, dx$

$= \int \tan^3 x \sec^2 x - \tan^3 x \, dx$

$= \int \tan^3 \sec^2 x - \tan^2 x \tan x \, dx$

$= \int \tan^3 x \sec^2 x - \sec^2 x \tan x + \tan x \, dx$

$u = \tan x$ $\quad$ $u = \tan x$ $\quad$ $\tan = \frac{\sin x}{\cos x}$

51 $\int \sin 3x \cos 2x \, dx$

$\frac{1}{2} \int \left(\sin 5x + \sin x \right) dx$

$= \frac{1}{2} \left(\frac{-\cos 5x}{5} + -\cos x \right) + C$

20 $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$

$\int_0^{\pi} 8 \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right)$

$\int_0^{\pi} (1-\cos 2y)^2 \cdot (1+\cos 2y)$

$\int_0^{\pi} (1 - 2\cos 2y + \cos^2 2y) \cdot (1+\cos 2y)$

$\int_0^{\pi} 1 - 2\cos 2y + \cos^2 2y + \cos 2y - 2\cos^2 2y + \cos^3 2y$

$\int_0^{\pi} 1 - \cos 2y - \cos^2 2y + \cos^3 2y$

$\sin 2y = u$

$2\cos 2y = \frac{du}{dx}$

$\int_0^{\pi} 1 - \cos 2y - \frac{1}{2} - \frac{\cos 4y}{2} + \int_0^{\pi} \cos^3 2y$

$\int_0^{\pi} \frac{1}{2} - \cos 2y - \frac{\cos 4y}{2} + \int_0^{\pi} \cos^2 2y \cos 2y$

$\int_0^{\pi} \frac{1}{2} - \cos 2y - \frac{\cos 4y}{2} + \int \frac{1}{2} (1-u^2) \, du$

$\frac{y}{2} - \frac{\sin 2y}{2} - \frac{\sin 4y}{8} + \frac{u}{2} - \frac{u^3}{3}$

$\frac{y}{2} - \frac{\sin 2y}{2} - \frac{\sin 4y}{8} + \frac{\sin 2y}{2} - \frac{\sin 2y^3}{3}$

$\frac{y}{2} - \frac{\sin 4y}{8} - \frac{\sin 2y^3}{3} \Big|_0^{\pi}$

$\frac{\pi}{2} - 0 = \frac{\pi}{2}$

33 $\int \sec^2 x \tan x \, dx$

$= \int u \, du$

$\boxed{\begin{array}{l} \tan x = u \\ \sec^2 x \, dx = du \end{array}}$

$= \frac{u^2}{2} + C$

$= \frac{\tan^2}{2} + C$

36 $\int \sec^3 x \tan^3 x \, dx$

$\int \sec^3 x \, (\sec^2 - 1) \tan x \, dx$

$\int u^2 \, (u^2 - 1) \, du$

$\boxed{\begin{array}{l} \sec x = u \\ \sec x \tan x \, dx = du \end{array}}$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

42 $\int 3 \sec^4 3x \, dx$

$\int 3 \sec^2 3x \sec^2 3x \, dx$

$\int 3 \, (1 + \tan^2 3x) \sec^2 3x \, dx$

$\int (1 + u^2) \, du$

$\boxed{\begin{array}{l} \tan 3x = u \\ 3\sec^2 3x \, dx = du \end{array}}$

$= u + \frac{u^3}{3} + C$

$= \tan 3x + \frac{\tan^3 3x}{3} + C$

45 $\int 4 \tan^3 x \, dx$

$\int 4 \tan^2 x \tan x \, dx$

$\int 4 \, (\sec^2 x - 1) \tan x \, dx$

$\boxed{\begin{array}{l} \sec x = u \\ \sec x \tan x \, dx = du \end{array}}$

$\int 4 \frac{(u^2 - 1)}{u}$

$\int 4 \left(u - \frac{1}{u}\right)$

$4\left(\frac{u^2}{2} - \ln u\right) + C$

$4\left(\frac{\sec^2 x}{2} - \ln \sec x\right) + C$

64 $\int \frac{\sin 3}{\cos^4 x} \, dx$

$\int \frac{(1 - \cos^2 x) \sin x}{\cos^4 x} \, dx$

$\boxed{\begin{array}{l} \cos x = u \\ -\sin x \, dx = du \end{array}}$

$\int \frac{u^2 - 1}{u^4} \, du$

$\int \frac{1}{u^2} - \frac{1}{u^4} \, du$

$= \frac{-1}{u} - \frac{1}{3u^3} + C$

67 $\int x \sin^2 x \, dx$

$\int x \, \frac{(1 - \cos 2x)}{2} \, dx$

$\boxed{\begin{array}{ll} u = x & dv = \cos 2x \\ du = 1 & v = \frac{\sin 2x}{2} \end{array}}$

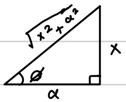
$\int \frac{x}{2} - \frac{1}{2} \int x \cos 2x$

$\frac{x^2}{4} - \frac{1}{2}\left(\frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2}\right)$

$\frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} +$

8.3 : Trigonometric subtitution:

(1)

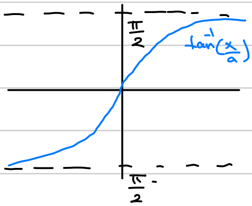


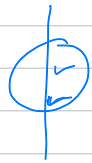
$x = a\tan\phi$

$\tan\phi = \frac{x}{a}$

$dx = a\sec^2\phi \, d\phi$

$\phi = \tan^{-1}\left(\frac{x}{a}\right)$

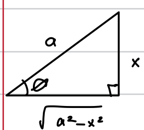




$-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\phi + a^2} = \sqrt{a^2(\tan^2\phi+1)} = |a|\sqrt{\sec^2\phi} = a\sec\phi$

(2)



$x = a\sin\phi \Rightarrow \sin\phi = \frac{x}{a}$

$dx = a\cos\phi \, d\phi$

$\phi = \sin^{-1}\left(\frac{x}{a}\right)$

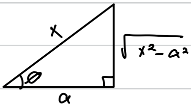
$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$

$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\phi}$

$= \sqrt{a^2(1-\sin^2\phi)}$

$= |a|\sqrt{\cos^2\phi} = a|\cos\phi| = a\cos\phi$

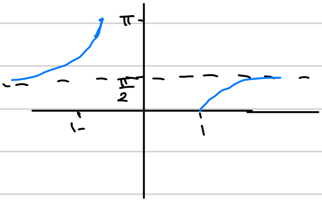
3



$x = a\sec\phi \implies \sec\phi = \frac{x}{a}$

$dx = a\sec\phi\tan\phi\, d\phi$

$\phi = \sec^{-1}\left(\frac{x}{a}\right)$



$0 \leq \phi < \frac{\pi}{2}$ if $\frac{x}{a} \geq 1$

$\frac{\pi}{2} \leq \phi < \pi$ if $\frac{x}{a} \leq -1$

$\sqrt{x^2 - a^2}$

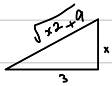
$= \sqrt{a^2\sec^2\phi - a^2}$

$= \sqrt{a^2(\sec^2\phi - 1)}$

$= |a|\sqrt{\tan^2\phi}$

$= a\tan\phi$

(1) $$\int \frac{1}{\sqrt{x^2+9}}\,dx$$



$$\int \frac{3\sec^2\phi}{3\sec\phi}$$

$x = 3\tan\phi$

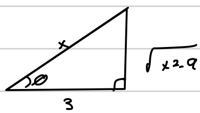
$dx = 3\sec^2\phi\, d\phi$

$$= \int \sec\phi$$

$$= \ln(\sec\phi + \tan\phi) + C$$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

(2) $$\int \frac{dx}{\sqrt{x^2-9}}$$



$x = 3\sec\phi$

$$= \int \frac{3\sec\phi\tan\phi\, d\phi}{3\tan\phi}$$

$dx = 3\sec\phi\tan\phi$

$$= \int \sec\phi\, d\phi$$

$$= \ln|\sec\phi + \tan\phi| + C$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

(3) $$\int \frac{dx}{\sqrt{9-x^2}}$$

$$= \int \frac{3\cos\phi}{3\cos\phi}\, d\phi$$

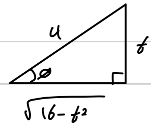
$$= \int d\phi$$

$$= \phi + C$$

$$= \sin^{-1}\left(\frac{x}{3}\right) + C$$

Exp:

$$\int \sqrt{16-t^2}\, dt$$



$$= \int 4\cos\emptyset \cdot 4\cos\emptyset \, d\emptyset \qquad t = 4\sin\emptyset$$

$$dt = 4\cos\emptyset \, d\emptyset$$

$$= 16\int \cos^2\emptyset \, d\emptyset$$

$$= 16\int \frac{1+\cos 2\emptyset}{2}$$

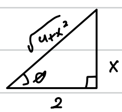
$$= 8\left(\emptyset + \frac{\sin 2\emptyset}{2}\right) + C$$

$$= 8\left(\emptyset + \frac{\cancel{2}\sin\emptyset\cos\emptyset}{\cancel{2}}\right) + C$$

$$= 8\left(\emptyset + \sin\emptyset\cos\emptyset\right) + C$$

$$= 8\left[\sin^{-1}\left(\frac{t}{4}\right) + \frac{t}{4}\left(\frac{\sqrt{16-t^2}}{4}\right)\right] + C$$

Exp: $$\int_{-2}^{2} \frac{dx}{4+x^2}$$



$$x = 2\tan\emptyset$$

$$= \int \frac{2\cancel{\sec^2}\emptyset}{4\cancel{\sec^2}\emptyset} \, d\emptyset \qquad dx = 2\sec^2\emptyset$$

$$= \int \frac{1}{2} \, d\emptyset$$

$$= \frac{\emptyset}{2} \Big|_{\square}^{\square}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) \Big|_{-2}^{2}$$

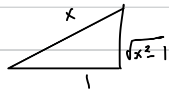
$$= \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} -1\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4}$$

Exp:

$$\int \frac{x^2}{(x^2-1)^{\frac{5}{2}}}\,dx \qquad , x>1 \qquad x = \sec\emptyset$$



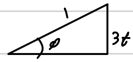
$$\int \frac{\sec^2\emptyset \cdot \sec\emptyset\tan\emptyset}{\tan^5\emptyset}\,d\emptyset$$

$$(x^2-1)^{\frac{1}{2}} = \sqrt{x^2-1} = \tan\emptyset$$

$$= \frac{\sec^3\emptyset}{\tan^4\emptyset}\,d\emptyset$$

8.3:

8) $\int \sqrt{1-9t^2}\,dt$



$\sin\varnothing = \frac{3t}{1}$

$\cos\varnothing\, d\varnothing = 3\,dt$

$\varnothing = \sin^{-1}(3t)$

$\frac{1}{3}\int \cos\varnothing \cos\varnothing\, d\varnothing$

$\frac{1}{3}\int \cos^2\varnothing\, d\varnothing$

$\frac{1}{3}\int \frac{1+\cos 2\varnothing}{2}$

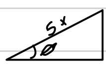
$\frac{1}{3}\left(\frac{\varnothing}{2} + \frac{\sin 2\varnothing}{4}\right)$

$\frac{1}{3}\left(\frac{\varnothing}{2} + \frac{\sin\varnothing\cos\varnothing}{2}\right)$

$= \frac{1}{3}\left(\frac{\sin^{-1}(3t)}{2} + \frac{3t}{2}\left(\sqrt{1-9t^2}\right)\right) + C$

---

10) $\int \frac{5}{\sqrt{25x^2-9}}\,dx$



$\sec\varnothing = \frac{5x}{3}$

$\sec\varnothing\tan\varnothing\, d\varnothing = \frac{5}{3}\,dx$

$\varnothing = \sec^{-1}\left(\frac{5x}{3}\right)$

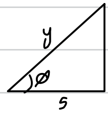
$\int \frac{\cancel{5}}{\cancel{3}\cancel{\tan\varnothing}} \cdot \frac{\cancel{3}}{\cancel{5}} \sec\varnothing \cancel{\tan\varnothing}\, d\varnothing$

$= \int \sec\varnothing\, d\varnothing$

$= \ln\left|\sec\varnothing + \tan\varnothing\right| + C$

$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

(12) $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy \quad , y>5$



$\sec\theta = \frac{y}{5}$

$\sec\theta\tan\theta\,d\theta = \frac{dy}{5}$

$\theta = \sec^{-1}\left(\frac{y}{5}\right)$

$$\int \frac{\cancel{5}\tan\theta \cdot \cancel{5}\,\cancel{\sec\theta}\tan\theta\,d\theta}{5^{\cancel{3}}\sec^{\cancel{3}\,2}\theta}$$

$$\int \frac{\tan^2\theta}{5\sec^2\theta}\,d\theta$$

$$\frac{1}{5}\int \frac{\sec^2\theta - 1}{\sec^2\theta}$$

$$\frac{1}{5}\int 1-\cos^2\theta$$

$$\frac{1}{5}\int 1-\frac{1}{2}-\frac{\cos 2\theta}{2}$$

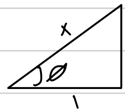
$$\frac{1}{5}\left(\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)$$

$$\frac{1}{5}\left(\frac{\sec^{-1}\left(\frac{y}{5}\right)}{2}-\frac{\sin\theta\cos\theta}{2}\right)$$

$$\frac{1}{5}\left(\frac{\sec^{-1}\left(\frac{y}{5}\right)}{2}-\frac{1}{2}\left(\frac{5}{y}\cdot\frac{\sqrt{y^2-25}}{y}\right)\right)$$

---

(14) $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}} \quad , x>1$



$\sec\theta = x$

$\sec\theta\tan\theta\,d\theta = dx$

$\theta = \sec^{-1}(x)$

$$\int \frac{2\cdot\cancel{\sec\theta}\cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}\cdot\sec^{\cancel{3}\,2}\theta}$$

$$\int \frac{2}{\sec^2\theta}\,d\theta$$

$$\int 2\cos^2\theta\,d\theta$$

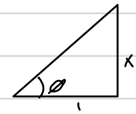
$$\int 1+\cos 2\theta\,d\theta$$

$$= \theta + \frac{\sin 2\theta}{2}$$

$$= \theta + \sin\theta\cos\theta$$

$$= \sec^{-1}x + \frac{1}{x}\cdot\sqrt{x^2-1} + C$$

18 $\displaystyle \int \frac{dx}{x^2\sqrt{x^2+1}}$



$\tan\varnothing = \frac{x}{1}$

$\sec^2\varnothing\, d\varnothing = dx$

$\varnothing = \tan^{-1}(x)$

$$\int \frac{\sec^{\cancel{2}}\varnothing \; d\varnothing}{\tan^2\varnothing \cdot \cancel{\sec}\varnothing}$$

$$\int \frac{\sec\varnothing}{\tan^2\varnothing}\, d\varnothing$$

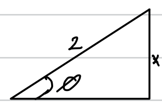
$$\int \frac{1}{\cancel{\cos}\varnothing} \cdot \frac{\cos^{\cancel{2}}\varnothing}{\sin^2\varnothing}$$

$$\int \frac{\cos\varnothing}{\sin^2\varnothing}$$

$$= -\frac{1}{\sin\varnothing} + C$$

$$= -\frac{1}{\sqrt{x^2+1}} + C$$

24 $\displaystyle \int_0^1 \frac{dx}{(4-x^2)^{\frac{3}{2}}}$



$\sin\varnothing = \frac{x}{2}$

$\cos\varnothing\, d\varnothing = \frac{dx}{2}$

$\varnothing = \sin^{-1}\left(\frac{x}{2}\right)$

$$\int \frac{2\cos\varnothing \; d\varnothing}{(2\cos\varnothing)^3}$$

$$\int \frac{1}{4\cos^2\varnothing}\, d\varnothing$$

$$\frac{1}{4}\int \sec^2\varnothing \; d\varnothing$$

$$\frac{1}{4}\tan\varnothing \Big|_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4\sqrt{3}}$$

* 26 $\int \frac{x^2}{(x^2-1)^{\frac{5}{2}}}dx$, $x>1$

$\int \frac{\sec^2\phi \cdot \sec\phi \cancel{\tan\phi}\, d\phi}{\tan^{\cancel{5}\,4}\phi}$

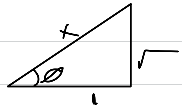
$\int \frac{\sec^3\phi}{\tan^4\phi}\, d\phi$

$\int \frac{1}{\cos^3\phi} \cdot \frac{\cos^4\phi}{\sin\phi^4}$

$= \int \frac{\cos}{\sin\phi^4}$

$= \frac{-1}{3\sin^3\phi} + C$

$= \frac{-x}{3\sqrt{x^2-1}}$



$\sec\phi = \frac{x}{1}$

$\sec\phi\tan\phi\, d\phi = dx$

$\sec^{-1}x = \phi$

29 $\int \frac{8\, dx}{(4x^2+1)^2}$

$\int \frac{8 \cdot \sec^{\cancel{2}}\phi\, d\phi}{\sec^{\cancel{4}\,2}x}$

$\int \frac{8}{\sec^2\phi}\, d\phi$

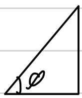
$\int 4\cos^2\phi$

$\int 2(1+\cos 2\phi)\, d\phi$

$= 2(\phi + \sin 2\phi)\, d\phi$

$= 2(\tan^{-1}(2x) + 2\cos\phi\sin\phi)\, d\phi$

$= 2(\tan^{-1}(2x) + 2\sqrt{x}$



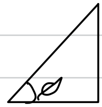
$\tan\phi = \frac{2x}{1}$

$\sec^2\phi\, d\phi = 2dx$

$\tan^{-1}(2x) = \phi$

33

$$\int \frac{v^2}{(1-v^2)^{\frac{5}{2}}}\, dv$$



$\sin\phi = \frac{v}{1}$

$\cos\phi \, d\phi = dv$

$\sin^{-1} v = \phi$

$$\int \frac{\sin^2\phi \cancel{\cos\phi}}{\cos^{\cancel{5}\,4}\phi}\, d\phi$$

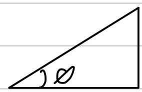
$$\int \frac{\sin^2\phi}{\cos^4\phi}\, d\phi$$

$$= \int \tan^2\phi \sec^2\phi \, d\phi$$

$$= = \frac{\tan^3\phi}{3} + C$$

38

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$$



$\tan\phi = \frac{\ln y}{1}$

$\sec^2\phi \, d\phi = \frac{1}{y}\, dy$

$\tan^{-1}(\ln y) = \phi$

$$\int \frac{y \sec^2\phi \, d\phi}{y \sec\phi}$$

$$\int \sec\phi \, d\phi$$

$$\ln|\sec\phi + \tan\phi| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln(1+\sqrt{2})$$

$y=1$	$y=e$
$\ln 1 = \tan\phi$	$\ln e = \tan\phi$
$\phi = 0$	$1 = \tan\phi$
	$\phi = \frac{\pi}{4}$

45 $\int \sqrt{\frac{4-x}{x}}\, dx$

$\sqrt{x} = u$

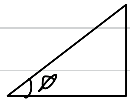
$\frac{1}{2u} = \frac{du}{dx}$

$x = u^2$

$\int \frac{\sqrt{4-x}}{\sqrt{x}}$

$\int \frac{\sqrt{4-u^2}}{\cancel{u}} \cdot 2\cancel{u}\, du$

$2\int \sqrt{4-u^2}$



$\sin Ø = \frac{u}{2}$

$\cos Ø \, dØ = \frac{du}{2}$

$Ø = \sin^{-1}\left(\frac{u}{2}\right)$

$2\int \frac{2\cos Ø}{2\cos Ø}\, dØ$

$= 2\int dØ$

$= 2Ø + C$

$= 2\sin^{-1}\left(\frac{u}{2}\right) + C$

46 $\int \sqrt{\frac{x}{1-x^3}}\, dx$

8.4 : Integration by partial fraction.

Exp: $\displaystyle\int \frac{8+x}{x^2-x-2}\,dx$

$$\int \frac{8+x}{(x-2)(x+1)}$$

$$\frac{8+x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

بنحط اشي اقل منه درجة

درجة 1

$$\frac{8+x}{(x+2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

توحيد مقامات

$$8+x = A(x+1) + B(x-2)$$

1 $x=2 \Rightarrow 8+2 = A(2+1)+0$

$$A = \frac{10}{3}$$

2 $x=-1 \Rightarrow 8-1 = 0 + B(-3)$

$$B = -\frac{7}{3}$$

$$\Rightarrow \int \frac{8+x}{(x+1)(x-2)}\,dx = \int \left(\frac{\frac{10}{3}}{x-2} + \frac{\frac{-7}{3}}{x+1} \right) dx$$

$$= \frac{10}{3}\ln|x-2| - \frac{7}{3}\ln|x+1| + C$$

طريقة سريعة اذا كان المقام خطيات 0

Distincte linear factor we use covermethod

$A = \dfrac{8+2}{2+1} = \dfrac{10}{3}$ بنغطي $(x-2)$ ونعوض 2

$B = \dfrac{8-1}{-1-2} = \dfrac{-7}{3}$ بنغطي $(x+1)$ ونعوض -1

طريقة 3.

مقارنة معاملات

$1 = A + B$

$8 = A - 2B$

طريقة ثانية لاخراج B, A

1 نشتق الطرفين

$1 = A + B$

2 نعوض اي رقم بالمعادلة الاساسية

$x = -1 \Rightarrow B = -\frac{7}{3}$, $A = \frac{10}{3}$

Exp: $\displaystyle\int \frac{dx}{x^3+x^2-2x}$

$$= \frac{dx}{x(x^2+x-2)}$$

$$= \frac{dx}{x(x+2)(x-1)}$$

$$= \int \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}\, dx$$

we use cover method

$A = \dfrac{1}{(0+2)(0-1)} = \dfrac{-1}{2}$ نعوض صفر ونغطي جزء A

$$B = \frac{1}{-2(-2-1)} = \frac{1}{6}$$

$$C = \frac{1}{1(1+2)} = \frac{1}{3}$$

$$= \frac{-1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

Exp: $\displaystyle\int \frac{3}{(x+1)(x^2-5)}\,dx = \int \frac{A}{x+1} + \frac{Bx+C}{x^2-5}$

اذا ما كان حلي التمام بنحط الصورة العامة الاقل بدرجة

Exp: $\displaystyle\int \frac{dx}{(x-1)\,x^3} = \int \frac{A}{x-1} + \frac{Bx^2+Cx+D}{x^3}\,dx$

Exp: $\displaystyle\int \frac{dx}{(x-1)^3(x^2+3)^2}$

Repeted

$$= \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+3} + \frac{Fx+G}{(x^2+3)^2}$$

Exp: $\int \frac{3x+2}{(x+1)^2}\,dx$

$= \int \frac{A}{x+1} + \frac{B}{(x+1)^2}\,dx$

$\frac{3x+2}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$

$3x+2 = A(x+1)+B$

$3 = A$ نشتق *

$x=-1 \Rightarrow -3+2 = 0 + B$ نعوض اي رقم *

$-1 = B$ $x=-1$

$= \int \left(\frac{3}{x+1} - \frac{1}{(x+1)^2} \right) dx$

$\boxed{\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}}$

$= 3\ln|x+1| - \int \frac{du}{u^2}$

$= 3\ln|x+1| - \int u^{-2}\,du$

$= 3\ln|x+1| + \int \frac{1}{x+1} + C$

Exp: $\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$

اذا كانت درجة البسط اكبر او تساوي درجة المقام
لازم نعمل قسمة طويلة

$$\begin{array}{r|l} & x - 2 \\ \hline x^2+2x+1 & x^3 \\ & -(x^3+2x^2+x) \\ \hline & -2x^2 - x \\ & -(-2x^2-4x-2) \\ \hline & 3x+2 \end{array}$$

stop.

$= \int x - 2 + \frac{3x+2}{x^2+2x+1}\,dx$

$= \int \frac{x^2}{2} - 2x + \underbrace{3\ln|x+1| + \frac{1}{x+1}}_{\text{من المثال السابق}} \Big|_0^1$

$= -2 + 3\ln 2$

Exp: $\displaystyle\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx$

$$= \int \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}\,dx$$

$$= \frac{(Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$4-2x = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$

$$x=1 \Rightarrow 4-2 = 0+0+2D \Longrightarrow D=1$$

نشتق

$$-2 = (Ax+B)(2)(x-1) + (x-1)^2 A + C(x-1)2x + C(x^2+1) + 2xD$$

$$x=1 \Rightarrow -2 = 0+0+0+2C+2 \Longrightarrow C=-2$$

نشتق مرة ثانية : $0 = (Ax+B)(2) + 2(x-1)A + 2A(x-1) + C(x-1)(2) + 2Cx + 2Cx + 2D$

او نعوض رقم اخر

$D=1$

$B=1$

$A=2$

$$= \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}$$

$$= \ln x^2+1 + \tan^{-1}x - 2\ln|x-1| \frac{-1}{x-1} + C$$

8.7 Improper integral :

Type I

$$\int_a^\infty f \ , \int_{-\infty}^b f \ , \int_{-\infty}^\infty f$$

و فيه عنا ∞ او ∞- بحدود التكامل

Type II

$$\int_0^1 \frac{dx}{x} \ , \int_0^3 \frac{dx}{x-3}$$

و في عنا نقطة الفنكشن غيرمتصل عندها

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

$$\int_0^\infty \frac{dx}{x}$$

I

II

$$* \int_{-\infty}^4 \frac{dx}{x^2} \ x\neq 0$$

Type I + II

Type I :

*How to find the improper Integral ?

f cont $[a,\infty]$

$$* \int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$* \int_{-\infty}^a f(x)\,dx = \lim_{c\to-\infty} \int_b^a f(x)\,dx \ ; f(x) \text{ cont } [-\infty, a]$$

$$* \int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

$$= \lim_{c\to-\infty} \int_c^a f(x)\,dx + \lim_{b\to\infty} \int_a^b f(x)\,dx$$

; $f(x)$ cont on $[-\infty, \infty]$

Remark: If limit exists (finite) (الجواب رقم)
Then we say Improper integral [converge] to this number

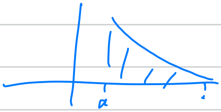
• // // " " $\{^{\infty}_{-\infty}$ //

// // // Diverges

Remark 2 : If $f(x) \geq 0$ and $\int_a^\infty f(x)\,dx$

converges to the number $L \geq 0$

The L represents the area under f



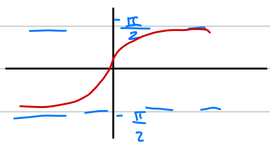
Exp 1:

$$\int_0^{\infty} \frac{dx}{x^2+1}$$

$$\lim_{b \to \infty} \int_0^{b} \frac{dx}{x^2+1}$$

$$\lim_{b \to \infty} \tan^{-1} x \Big|_0^{b}$$

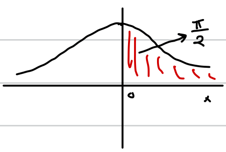
$$\lim_{b \to \infty} \left[\tan^{-1} b - \tan^{-1} 0 \right]$$



يساوي $\frac{\pi}{2}$

$\Longrightarrow \int_0^{\infty} \frac{dx}{x^2+1}$ converges to $\frac{\pi}{2}$

$$f(x) = \frac{1}{x^2+1}$$



* Exp $\int_{-\infty}^{0} \frac{dx}{x^2+1} = \frac{\pi}{2}$

* Exp $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{\boxed{0}} \frac{dx}{x^2+1} + \int_{\boxed{0}}^{\infty} \frac{dx}{x^2+1}$

ممكن نختار اي رقم

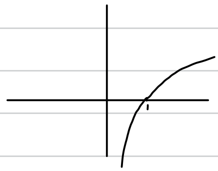
$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Exp:

$$\int_1^{\infty} \frac{dx}{x}$$

$$\lim_{b\to\infty} \int_1^{b} \frac{dx}{x}$$

$$= \lim_{b\to\infty} \left[\ln x \right]\Big|_1^{b}$$



$$= \lim_{b\to\infty} \left[\ln b - \ln 1 \right]$$

$\lim_{x\to\infty} \ln b = \infty$; it is Diverge.

Exp: $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx$

بنتأكد اذا هو Type II

$x^2 - 1 = 0$

$x = \pm 1 \in [-\infty, -2]$

only Type I

$$\lim_{b\to-\infty} \int_b^{-2} \frac{2}{x^2-1}\,dx$$

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$A = \frac{2}{1+1} = 1$$

$$B = \frac{2}{-1-1} = -1$$

$$= \lim_{b\to-\infty} \int_b^{-2} \frac{1}{x-1} - \frac{1}{x+1}$$

$$= \lim_{b\to\infty} \left[\ln|x-1| - \ln|x+1| \right]\Big|_b^{-2}$$

$$\lim_{b\to-\infty} \ln\left|\frac{x-1}{x+1}\right| \Bigg|_b^{-2}$$

$$\lim_{b\to-\infty} \left[\ln 3 - \ln\left|\frac{b-1}{b+1}\right| \right]$$

$$\ln 3 - \lim_{b\to-\infty} \overset{\text{cont}}{\ln} \frac{b-1}{b+1}$$

$$\ln 3 - \ln\left(\lim_{b\to-\infty} \frac{b-1}{b+1} \right) \quad \frac{\infty}{\infty} \text{ lopetal}$$

$\ln 3 - \ln 1 = \ln 3$, the integral is converge to $\ln 3$

Exp*

$$\int_{1}^{\infty} \frac{dx}{x^{p}} = \begin{cases} \frac{1}{p-1} , & p>1 \\ \infty , & p \le 1 \end{cases}$$

لازم حدود التكامل ١ ، ∞

Exp**

$$\int_{0}^{1} \frac{dx}{x^{p}} = \begin{cases} \frac{1}{1-p} , & p<1 \\ \infty , & p \ge 1 \end{cases}$$

حدود التكامل لازم ٠ ، ١

Exp: $\int_{1}^{\infty} \frac{dx}{x^{3}} = \frac{1}{3-1} = \frac{1}{2}$, I , converges

Exp: $\int_{1}^{\infty} \frac{dx}{x^{\frac{2}{3}}} = \infty$; diverge. , I

Exp: $\int_{1}^{\infty} \frac{dx}{\sqrt{x}} = \infty$, diverge , I

Exp: $\int_{0}^{1} \frac{dx}{\sqrt{x}} = \frac{1}{1-\frac{1}{2}} = 2$, converge , II

✱ Type II :

Improper Integrals of Type II are integrals of functions that become infinite at a point within the integral of integration (virtical Asymptots)

$\boxed{1}$ If f is discontinous at a, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$

$\boxed{2}$ If f is discontinouse at b then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$

$\boxed{3}$ If f is discontinous at c, wher $a<c<b$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area" otherwise the improper integral dirverge "infinit area"

Exp:

$$\int_0^4 \frac{dx}{\sqrt{4-x}}$$

, not I

, II ✓ , $x=4$ discont.

$$\lim_{c\to 4^-} \int_0^c \frac{dx}{\sqrt{4-x}}$$

$$\lim_{c\to 4^-} -2\sqrt{4-x}\,\Big|_0^c$$

$$\lim_{c\to 4^-} -2\left[\sqrt{4-c}-\sqrt{4}\right]$$

$$\lim_{c\to 4^-} \left[-2\sqrt{4-c}+(2)(2)\right]$$

$$\lim_{c\to 4^-} \left[-2\sqrt{4-4}+4\right] = 4 \quad \text{converge.}$$

Exp: $\int_0^1 \frac{dx}{\sqrt{x}}$, II

$\lim_{a\to 0^+} 2\int_a^1 \frac{dx}{2\sqrt{x}}$

$\lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1$

$\lim_{a\to 0^+} \left[2\sqrt{1} - 2\sqrt{a}\right]$

$2 - \lim_{x\to 0} 2\sqrt{a} = 2$ converge

Exp: $\int_0^1 \frac{\phi+1}{\sqrt{\phi^2+2\phi}}\, d\phi$, $\phi^2+2\phi=0$

$\phi = 0, -2$

$\lim_{a\to 0^+} \int_a^1 \frac{\phi+1}{\phi^2+2\phi}$

$u = \phi^2+2\phi$

$\frac{du}{2} = (\phi+1)\,d\phi$

$\lim_{a\to 0^+} \int \frac{du}{2\sqrt{u}}$

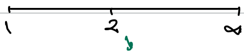
$\lim_{a\to 0^+} \sqrt{u}\,\Big|$

$\lim_{a\to 0^+} \sqrt{\phi^2+2\phi}\,\Big|_a^1$

$\lim_{a\to 0^+} \left[\sqrt{1+2} - \sqrt{a^2+2a}\right]$

$\sqrt{3} - \lim_{a\to 0^+} \sqrt{a^2+2a} = \sqrt{3}$

Exp: $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$; I ✓ II ✓ $[1,\infty)$



بنفصل التكامل عند اي قيمة بدي ياها ما فيها مشاكل

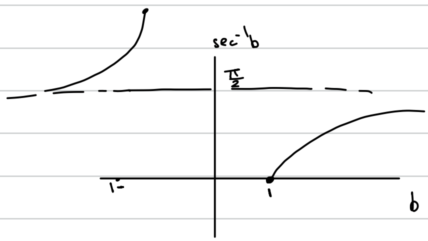
$$\int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}}$$

$$\lim_{c\to 1^+} \int_c^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$$

$$\lim_{c\to 1^+} \sec^{-1}|x| \Big|_c^2 + \lim_{b\to\infty} \sec^{-1} \Big|_2^b$$

$$\lim_{c\to 1^+} [\cancel{\sec^{-1} 2} - \sec^{-1}(c)] + \lim_{b\to\infty} [\sec^{-1}[b] - \cancel{\sec^{-1}[2]}]$$

$$\lim_{c\to 1^+} \sec^{-1} c + \lim_{b\to\infty} \sec^{-1}[b]$$



$$0 + \frac{\pi}{2} = \frac{\pi}{2}$$, converge to $\frac{\pi}{2}$

Exp: $\displaystyle\int_0^{\infty} \frac{16\tan^{-1}x}{1+x^2}\, dx$

$$\lim_{b\to\infty} \int_0^b \frac{16\tan^{-1}x}{1+x^2}$$

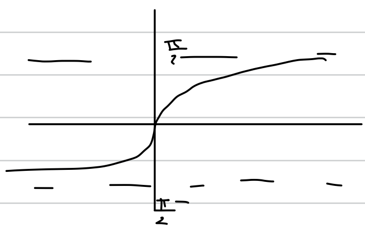
$u = \tan^{-1}x$

$16\, du = 16\, \frac{1}{1+x^2}\, dx$

$$\lim_{b\to\infty} \int 16\, u\, du$$

$$\lim_{b\to\infty} 8u^2 \Big|$$

$$\lim_{b\to\infty} 8(\tan^{-1}x)^2 \Big|_0^b$$



$$8\left(\lim_{b\to\infty} (\tan^{-1}b)^2 - (\tan^{-1}0)^2\right)$$

$$8\left(\left(\frac{\pi}{2}\right)^2 - 0\right) = 2\pi^2$$

* Two Tests to check convergnece / Divergence.

① Direct Comparsion Test (DCT)

f, g are cont on $[a,\infty]$ s.t $0 \leq f(x) \leq g(x)$, $\forall x \in [a,\infty]$ Then:

* If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converges

الكبير   بدوج لرقم   الصغير

* If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges

الصغير ∞ الكبير

Exp: check the conv, Dive :-

① $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

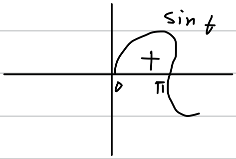
ممكن نختار اي واحد يزبط معنا

$\bigcirc < \dfrac{\sin^2 x}{x^2} < \dfrac{1}{x^2}$

$\displaystyle\int_1^\infty \frac{1}{x^2} = 1$ by Exp*

so $\displaystyle\int_0^\pi \frac{\sin^2 x}{x^2}$ converges by DCT → يكون جوابه اقل او يساوي 1

② **Exp:** $\displaystyle\int_0^\pi \frac{dx}{\sqrt{t} + \sin t}$



Div = 2.2   Conv = 2.2

$\bigcirc \leq \dfrac{1}{\sqrt{t} + \sin t} \leq \dfrac{1}{\sqrt{t}}$

$\displaystyle\int_0^\pi \frac{1}{\sqrt{t}} = \lim_{b\to 0^+} \int_b^\pi \frac{2\,dt}{2\sqrt{t}} = \lim_{b\to 0^+} 2\sqrt{t}\,\Big|_0^\pi$

$= \lim_{b\to 0^+}(2\sqrt{\pi} - 2\sqrt{b}) = 2\sqrt{\pi}$

$\displaystyle\int_0^\pi \frac{dx}{\sqrt{t} + \sin x}$ converges by DCT

③ $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}}$, type I

Div $\qquad \dfrac{1}{\sqrt{x^2}} \leq \dfrac{1}{\sqrt{x^2-0.1}} \leq \qquad$ conv

$$\int_1^\infty \frac{1}{\sqrt{x^2}} = \int_1^\infty \frac{1}{|x|} = \int_1^\infty \frac{1}{x} = \infty \quad \text{Div by Exp}^*$$

so $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^2-0.1}}$ by DCT

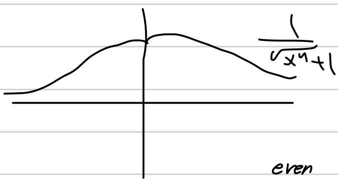
④ $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$

Div $\qquad \leq \dfrac{1}{x^3+1} \leq \dfrac{1}{x^3}$ conv

$\displaystyle\int_1^\infty \frac{1}{x^3}$ conv by Exp*

so $\displaystyle\int_1^\infty \frac{1}{x^3+1}$ conv DCT

⑤ $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}} = \int_{-\infty}^{\boxed{0}} \frac{dx}{\sqrt{x^4+1}} + \int_{\boxed{0}}^\infty \frac{dx}{\sqrt{x^4+1}}$



even

متماثل حول y

$$\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{\sqrt{x^4}}$$

مشكلة عند 0 , ∞ تقسم التكاملين

Conv by Exp*

$$\underbrace{\int_0^1 \frac{1}{\sqrt{x^4+1}}}_{\text{رقم مش ضروري تحسبه}} + \int_1^\infty \frac{dx}{\sqrt{x^4+1}}$$

رقم + رقم = رقم , converge by DCT

② Limit comparsion Test

f, g are $+$, cont on $[a, \infty]$ and

$\lim_{x\to\infty} \frac{f}{g} = L$ where $0 < L < \infty$ Then

$\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both are diverge

or " " converge.

Exp : check conv, Div.

① $\int_1^\infty \frac{dx}{1+x^2}$

??

$f = \frac{1}{1+x^2}$, $g = \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1 \leftarrow L$

so $\int \frac{dx}{1+x^2}$ conv. by LCT

Exp. $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

$f(x) = \frac{1}{\sqrt{x-1}}$, $g = \frac{1}{\sqrt{x}}$

$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} 2\int_2^b \frac{dx}{2\sqrt{x}} = \lim_{b\to\infty} 2\sqrt{x}\Big|_2^b = \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{2} = \infty$

$\lim_{x\to\infty} \frac{f}{g} = \lim \frac{\frac{1}{\sqrt{x-1}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x-1}} = 1$

$\int_2^\infty \frac{dx}{\sqrt{x-1}}$ div by LCT

Def: let $f(x)$ and $g(x)$ be positive for x sufficiently large

① f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \iff \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$

② f and g grow at the same rate as $x \to \infty$ if

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where L is finite & positive

$e^x \succ x^a \succ \ln / \log.$

10.1 :-

A sequence : is a list of numbers , $a_1, a_2, \ldots\ldots, a_n, \ldots$

- where a_i is a number with index i "order"
- it can be finite or infinite $\begin{cases}\to \text{Diverge} \\ \to \text{Converge}\end{cases}$

it is a function that sends 1 to a_1
2 to a_2
⋮
n to a_n " the n^{th} term "

Exp:

$a_n = \sqrt{n}$, $n = 1, 2, 3 \ldots$

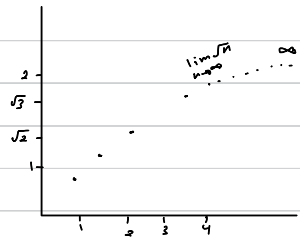
$n=1 \Rightarrow a_1 = \sqrt{1}$ 1st term

$n=2 \Rightarrow a_2 = \sqrt{2}$ 2nd term

$n=3 \Rightarrow a_3 = \sqrt{3}$ 3rd term

$\Rightarrow a_n = \sqrt{n}$ n^{th} term

$\lim\limits_{n\to\infty} \sqrt{n} = \infty \Rightarrow a_n = \sqrt{n}$ div



Exp:

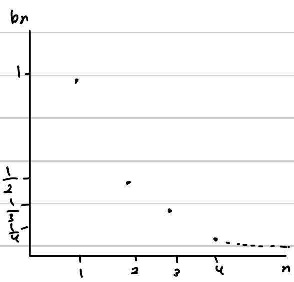
$b_n = \frac{1}{n}$, $n = 1, 2, 3 \ldots$

$n=1 \Rightarrow b_1 = \frac{1}{1}$

$n=2 \Rightarrow b_2 = \frac{1}{2}$

$n=3 \Rightarrow b_3 = \frac{1}{3}$

$b_n \xrightarrow[\text{تقترب}]{} 0$



$\lim\limits_{n\to\infty} \frac{1}{n} = 0$, converge.

$b_n \to 0$

as $n \to \infty$

Exp:

$C_n = (-1)^n \frac{1}{n}$; Alternating sequence

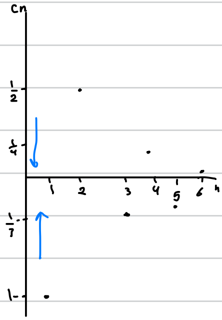
$n = 1, 2, 3 \ldots$

$n=1 \Rightarrow C_1 = (-1)^1 \frac{1}{1} = -1$

$n=2 \Rightarrow C_2 = (-1)^2 \frac{1}{2} = \frac{1}{2}$

$n=3 \Rightarrow C_3 = -\frac{1}{3}$

$n=4 \Rightarrow C_4 = \frac{1}{4}$



$\lim_{n \to \infty} (-1)^n \frac{1}{n} = 0$

$C_n \to 0$ as $n \to \infty$

Th

$\lim_{n \to \infty} a_n = A$; A number , $a_n \to A$ as $n \to \infty$

$\lim_{n \to \infty} b_n = B$; B number , $b_n \to B$ as $n \to \infty$

$a_n, b_n \to$ conv

1. $\lim_{n \to \infty} (a_n + b_n) = A + B$

2. $\lim_{n \to \infty} (a_n - b_n) = A - B$

3. $\lim_{n \to \infty} K a_n = KA$, K constant.

4. $\lim_{n \to \infty} (a_n b_n) = A\,B$

5. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B}$, $B \neq 0$

Exp:

① $\lim\limits_{n\to\infty} \frac{-\sqrt{5}}{n} = -\sqrt{5} \lim\limits_{n\to\infty} \frac{1}{n} = -\sqrt{5}(0) = 0$

② $\lim\limits_{n\to\infty} \frac{7n^1 - 3}{5 + 14n^1} = \frac{7}{14} = \frac{1}{2}$

المقام والبسط نفس الدرجة

أو $\lim\limits_{n\to\infty} \frac{7 - \frac{3}{n}}{\frac{5}{n} + 14} = \frac{7-0}{0+14} = \frac{7}{14} = \frac{1}{2}$

③ $\lim\limits_{n\to\infty} \frac{7n-3}{5+14n^2} = 0$

درجة المقام اعلى

أو $\lim \frac{\frac{7}{n} - \frac{3}{n^2}}{\frac{5}{n^2} + 14} = \frac{0-0}{0+14} = 0$

④ $\lim\limits_{n\to\infty} \frac{7n^3 - 3}{5 + 14n} = \infty$

درجة البسط اعلى

أو $\lim\limits_{n\to\infty} \frac{7n^2 - \frac{3}{n}}{\frac{5}{n} + 14} = \frac{\infty - 0}{0 + 14} = \frac{\infty}{14} = \infty$

Sandwich Th

I need to know $\lim\limits_{n\to\infty} b_n$??

If $a_n \leq b_n \leq c_n$ for all n

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$; L finite

So : $\lim\limits_{n\to\infty} b_n = L$

Exp:

Check conv/Div for :-

① $\lim\limits_{n\to\infty} \frac{\sin n}{n}$

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad ; \forall n$$

$$\underbrace{\lim_{n\to\infty} \frac{-1}{n}}_{0} \leq \underbrace{\lim_{n\to\infty} \frac{\sin n}{n}}_{0} \leq \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{0}$$

by S.T

② $\lim\limits_{n\to\infty} \frac{\sin^2 n}{2^n}$

$$\frac{0}{2^n} \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$$

$$\underbrace{\lim_{n\to\infty} 0}_{0} \leq \underbrace{\lim_{n\to\infty} \frac{\sin^2 n}{2^n}}_{0} \leq \underbrace{\lim_{n\to\infty} \frac{1}{2^n}}_{0} \rightarrow$$

conv. to 0
by S.T

$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \xrightarrow{x\in(-1,1)} \Rightarrow \lim\limits_{n\to\infty} x^n = 0 \;, |x|<1$

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{4}$

$a_3 = \frac{1}{9}$

$\vdots$

$a_n \underset{n\to\infty}{\longrightarrow} 0$

③ $\lim (-1)^n \frac{1}{n}$; Alternating

$$\frac{-1}{n} \leq \underbrace{(-1)^n}_{\text{بين } -1 \text{ ، } 1} \frac{1}{n} \leq \frac{1}{n}$$

$$\underbrace{\lim_{n\to\infty} \frac{-1}{n}}_{0} \leq \underbrace{\lim_{n\to\infty} (-1)^n \frac{1}{n}}_{\text{converge to } 0 \text{ by S.T}} \leq \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{0}$$

Th 5 :

(1) $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

(2) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

(3) $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$, $x > 0$

(4) If $|x| < 1 \overset{\text{then}}{\Longrightarrow} \lim\limits_{n\to\infty} x^n = 0$

$-1 < x < 1$

(5) $\lim\limits_{n\to\pm\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

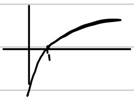
(6) $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$; for any x "Taylor series"

Exp: $\lim\limits_{n\to\infty} \frac{\ln n^3}{3n} = \lim\limits_{n\to\infty} \frac{\cancel{3}\ln n}{\cancel{3}n} = 0$

Exp $\lim\limits_{n\to\infty} e^{\ln \sqrt[n]{n}} = \lim\limits_{n\to\infty} \ln n^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\frac{\ln n}{n}} = \lim\limits_{n\to\infty} e^0 = 1$

Exp $\lim\limits_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} = \frac{\lim\limits_{n\to\infty} \ln n}{\lim\limits_{n\to\infty} \sqrt[n]{n}} = \frac{\infty}{1} = \infty$

متزايدة



Exp: $\lim\limits_{n\to\infty} \sqrt[n]{n^3} = \lim\limits_{n\to\infty} (n^3)^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3 = 1^3 = 1$

Exp: $\lim\limits_{n\to\infty} \sqrt[n]{\pi n} = \lim (\pi)^{\frac{1}{n}}\left(n^{\frac{1}{n}}\right) = \lim\limits_{n\to\infty} \pi^{\frac{1}{n}} \cdot \lim\limits_{n\to\infty} \sqrt[n]{n} = 1 \cdot 1 = 1$

Exp: $\lim \left(\frac{1}{2}\right)^n = 0$

Exp : $\lim \left(\frac{-2}{3}\right)^n = 0$

Exp : $\lim\limits_{n\to\infty} 2^n = \infty \Rightarrow \left(\frac{1}{2}\right)$

Exp: $\lim\limits_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$

Exp: $\lim\limits_{n\to\infty} \frac{\pi^{-n}}{e^{-n}} = \lim\limits_{n\to\infty} \left(\frac{\pi}{e}\right)^{-n} = \lim\limits_{n\to\infty} \left(\frac{e}{\pi}\right)^n = 0$

$e \approx 2.718$

$\pi \approx 3.14$

* $\lim \left(\frac{\pi}{e}\right)^n = \infty$

$a_n = \dfrac{4^n}{1+9^n}$

$\lim\limits_{n\to\infty} \dfrac{4^n}{1+9^n}$

$\dfrac{\lim\limits_{n\to\infty} 4^n}{\lim\limits_{n\to\infty} 1+9^n} \overset{\text{تعويض}}{=} \dfrac{\infty}{\infty}$

لوبيتال $\dfrac{\cancel{n}4^{n-1}}{\cancel{n}9^{n-1}}$

$\lim\limits_{n\to\infty} \left(\dfrac{4}{9}\right)^{n-1}$

$= 0$ حسب القاعدة

4. If $|x| < 1 \overset{\text{then}}{\Longrightarrow} \lim\limits_{n\to\infty} x^n = 0$

$-1 < x < 1$

Exp: $\lim\limits_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^3$

Exp: $\lim\limits_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}$

Exp: $\lim\limits_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \ln e = 1$

Exp: $\lim\limits_{n\to\infty} \sqrt[n]{n^2+n} = \lim\limits_{n\to\infty} e^{\ln(n^2+n)^{\frac{1}{n}}} = \lim\limits_{n\to\infty} e^{\frac{\ln(n^2+n)}{n}} = e^{\lim \frac{\ln(n^2+n)}{n}}$ $\frac{\infty}{\infty}$ lopetal

$= e^{\lim\limits_{n\to\infty} \frac{\frac{2n+1}{n^2+n}}{1}} = e^{\lim\limits_{n\to\infty} \frac{2n+1}{n^2+n}} = e^0 = 1$

Exp: $\lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$

Exp: $\lim\limits_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = 0$

Exp: $\lim\limits_{n\to\infty} \frac{\pi^n}{n!} = 0$

Exp: $\lim\limits_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = \lim\limits_{n\to\infty} \left(\frac{n-1+2}{n-1}\right) = \lim\limits_{n\to\infty} \left(1+\frac{2}{n-1}\right)^n$

assum: $u = n-1$; $n\to\infty$

$u+1 = n$; $u\to\infty$

$= \lim\limits_{u\to\infty} \left(1+\frac{2}{u}\right)^{u+1} = \lim\limits_{u\to\infty} \left(1+\frac{2}{u}\right)\left(1+\frac{2}{u}\right)^u$

$= \lim\limits_{u\to\infty} \left(1+\frac{2}{u}\right) \cdot \lim\limits_{u\to\infty} \left(1+\frac{2}{u}\right)^u = (1+0)(e^2) = e^2$

my way: $\lim\limits_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n$

$$\frac{\lim\limits_{n\to\infty} (n+1)^n}{\lim\limits_{n\to\infty} (n-1)^n}$$

ذ n عامل مشترك

$$\frac{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n}{\lim\limits_{n\to\infty} \left(1-\frac{1}{n}\right)^n}$$

$$= \frac{e^1}{e^{-1}} = e^2$$

Exp :- Find the n^{th} term of the following sequences: $\{a_n\}$

① $1, -4, 9, -16, 25, \ldots$

$+ \quad - \quad + \quad - \quad \ldots$

$(-1)^n$

$n^{th} = (-1)^{n+1} n^2 \quad , n = 1, 2, 3 \ldots$

② $0, 3, 8, 15, 24, \ldots \; ??$

a_n

$n^{th} = n^2 - 1$

③ $-3, -2, -1, 0, 1, \ldots$

$a_n = n - 4$

Recursive sequence:

$a_{100} \xrightarrow{\text{لازم نعرف}} a_{99} \longrightarrow a_{98}$

Exp: $a_1 = 1 \quad , \quad a_{n+1} = \frac{1}{2} a_n$

assume this sequence converges, find it's limit Find $\lim_{n\to\infty} a_n$

$a_1 = 1$

$a_2 = a_{1+1} = \frac{1}{2} a_1 = \frac{1}{2} a_1 = \frac{1}{2}$

$a_3 = a_{2+1} = \frac{1}{2} a_2 = \frac{1}{2} a_2 = \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^2$

$a_4 = \frac{a}{3+1} = \frac{1}{2} a_3 = \frac{1}{2} a = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^3$

$a_5 = \left(\frac{1}{2}\right)^4$

$a_n = \left(\frac{1}{2}\right)^{n-1}$

$* \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{-1} \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 2 \cdot 0 = 0$

upper bounde

Def:
- A sequence $\{a_n\}$ is bounded from above if $\exists$ a number M s.t $a_n \leq M$ for all n
- A sequence $\{a_n\}$ is bounded from below if $\exists$ a number m s.t $a_n \geq m$ for all n
- A sequence $\{a_n\}$ is bounded if it's bounded from above and is bounded from below.
- A sequence $\{a_n\}$ is not bounded if it's not bounded from above and is bounded from below.

Exp: $1, 2, 3, 4, 5, \cdots n$

① $a_n = n$, $n = 1, 2, 3 \ldots$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} n = \infty$

M ?? → a_n is not bounded from above
$m = 1, 0, -1, -2 \cdots$ (Lower bounde)
↳ greatest Lower bounde.

} only bounded from below (not bounded)

② $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots b_n \left(\frac{1}{2}\right)^n$, $n = 0, 1, 2, 3 \ldots$

$\lim_{n \to \infty} b_n = 0$

b_n converge to 0

$M = 1, 2, 3, 4 \cdots$ (upper bounde)
↳ least upper bounde
$m = 0, -1, -2, -3$
↳ greatest Lower bounde

} b_n is bounded from below and above (bounded)

③ $3, 3, 3, 3$

c_n , $n = 1, 2, 3, \ldots$

$\lim_{n \to \infty} c_n = \lim_{n \to \infty} 3 = 3$

c_n converges to 3

④ $\ldots -2, -1, 0, 1, 2 \ldots$ (not bounded sequence)

$M = 3, 4.5, 3.1, 9 \cdots$ (upper boundes)
↳ least upper bounde
$m = 3, 2, 1, 0, -1$
↳ greatest Lower bounde.

} c_n is bounded from below and above (bounded)

Defs -

- A sequence $\{a_n\}$ is nondecreasing if $a_n \leq a_{n+1}$ $\forall n$
  $a_1 \leq a_2 \leq a_3 \leq \cdots$

- A sequence $\{a_n\}$ is non increasing if $a_n \geq a_{n+1}$ $\forall n$
  $a_1 \geq a_2 \geq a_3 \geq \cdots$

- A sequence $\{a_n\}$ is monotonic if it is either nondecreasing or non increacing

Exp:

① $1, 2, 3, 4, 5, \cdots n$ $\Rightarrow$ non decreasing, monotonic

② $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots b_n \left(\frac{1}{2}\right)^n, n = 0, 1, 2, 3.$ $\Rightarrow$ non increasing, monotonic $\Rightarrow$ converge

③ $3, 3, 3, 3$ $\Rightarrow$ nondecreasing, nondecreasing, monotonic $\Rightarrow$ converge

Exp:

$-1, 1, -1, 1, -1, 1$ is not monotonic

Th: If a sequnce $\{a_n\}$ is both
1) bounded
2) monotonic

Then $\{a_n\}$ Converge.

Exp: $a_n = \frac{1}{n}$, $n = 1, 2, 3 \ldots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

① find M, m:

$m = 0, -1, \ldots$

$M = 1, 2, 3 \ldots$

② is a_n monotonic?

a_n is nonincreasing

yes it's monotonic.

③ is a_n bounded?

yes since we found m, M

④ Does a_n converge?

yes since $\{a_n\}$ monotonic and bounded

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

10.2 :- Infinite series

* An infinite series is the sum of an infinite sequence of numbers $a_1 + a_2 + a_3 \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$

where
- a_n is the n^{th} term of the series
- $s_1 = a_1$ is the 1^{st} portial sum of the series
- $s_2 = a_1 + a_2$ is the 2^{nd} portial sum of the series
- $s_n = a_1 + a_2 + \cdots a_n = \sum_{k=1}^{n} a_k$ is the n^{th} portial sum of the series.
- If the sequence of partial sums converges to a limit L then we say the series converge.

$$\sum_{n=1}^{\infty} a_n \longrightarrow \text{conv / Div ??}$$

Tests ⟹ Th's

Conv only

Div only

Con and Div

Test 1: (n^{th} partial sum Test) ; $\sum_{n=1}^{\infty} a_n = a_1 + a_2 \cdots$ (con, Div)

Find $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots a_n$

- If $\lim_{n \to \infty} s_n = L$ then $\sum_{n=1}^{\infty} a_n$ converges to L
- If $\lim_{n \to \infty} s_n$ div then $\sum_{n=1}^{\infty} a_n$ div

الحدود بتشطب مع بعض

Exp: (Telescoping series) ; check for conv / Div ?

① $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

نعوض

$$S_n = \left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{2}}} \right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}} \right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}} \right) + \cdots \cancel{} \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}} \right)$$

$$= 1 - \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}} \right) = 1 - \underbrace{\lim_{n\to\infty} \frac{1}{\sqrt{n+1}}}_{0} = 1 = L$$

$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ converges to 1 by n^{th} partial sum Test.

② Use n^{th} partial sum Test to check conv/Div

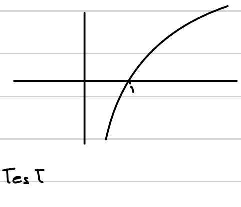
$$\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n} \right)$$

$$S_n = \left(\cancel{\ln\sqrt{2}} - \ln\sqrt{1} \right) + \left(\cancel{\ln\sqrt{3}} - \cancel{\ln\sqrt{2}} \right) + \left(\ln\sqrt{4} - \cancel{\ln\sqrt{3}} \right) \cdots + \left(\ln\sqrt{n+1} - \cancel{\ln\sqrt{n}} \right)$$

$$= -\ln 1 + \ln\sqrt{n+1}$$

$$= 0 + \ln\sqrt{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\sqrt{n+1} = \infty$$



so, $\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n} \right)$ div by sum partial Test we cant use test 2

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ نغير شكله عشان يشطب مع بعض

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$A = 1$; $B = -1$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \Rightarrow \text{Telescoping} \checkmark$$

$$S_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right) \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ Converges to 1 by the n^{th} partial sum TesT.

Test 2 : n^{th} term test for div $\displaystyle\sum_{n=1}^{\infty} a_n$

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ div

If $\displaystyle\lim_{n\to\infty} a_n$ fails to exist then $\displaystyle\sum_{n=1}^{\infty} a_n$ div
$(\infty, -\infty, DNE)$

Exp: cKeck for conv/Div??

① $\sum_{n=1}^{\infty} \frac{n+1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{1} = 1 \neq 0$

$\Longrightarrow \sum_{n=1}^{\infty} \frac{n+1}{n}$ div by n^{th} term test

② $\sum_{n=1}^{\infty} \sqrt{n} \Longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \infty$ so

$\sum \sqrt{n}$ div by n^{th} term test

③ $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

$\lim_{n\to\infty} \frac{2^n + 4^n}{3^n + 4^n}$

$\lim_{n\to\infty} \frac{\frac{2^n}{4^n} + 1}{\frac{3^n}{4^n} + 1} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n + 1}{\left(\frac{3}{4}\right)^n + 1} = 1$

So the infinite series div by n^{th} term test.

③ $\sum_{n=1}^{\infty} (-1)^{n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}$

$s_n = 1 - 1 + 1 - 1 + 1 - 1 \dots (-1)^{n+1}$

$\lim s_n$ DNE

so $\sum_{n=1}^{\infty} (-1)^{n+1}$ div

* $\sum_{n=1}^{\infty} (-1)^{n+1} = (1-1) + (1-1) + 1 \dots = 0$

* $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 + (-1+1) + (-1+1) + (-1+1) = 1$

Div

ممكن تفكر فيها بطريقتين

Exp:

$$\sum_{n=1}^{\infty} \frac{2n+1}{5n}$$

$$\lim_{n\to\infty} \frac{2n+1}{5n} = \frac{2}{5} \neq 0$$

so $\sum a_n$ div

② $$\sum_{n=1}^{\infty} \sqrt{5n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{5n} = \infty$$

so $\sum \sqrt{5n}$ div by n^{th} term test

* Harmonic Series div (دايمًا Div) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \frac{1}{9} \cdots \frac{1}{16} + \cdots \longrightarrow \text{أكبر}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} \cdots\right)$$

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots \Rightarrow \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \infty$$

$\hookrightarrow$ Div.

Th:

If $\lim_{n \to \infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ may div, may con

If $\sum_{n=1}^{\infty} a_n$ conv then $\lim_{n \to \infty} a_n = 0$

converse is not true $\Rightarrow$ means if $\lim_{n \to \infty} a_n$ this does not mean $\sum_{n=1}^{\infty} a_n$ conv

Exp: $\sum_{n=1}^{\infty} \frac{1}{n}$ div but $\lim_{n \to \infty} a_n \; \frac{1}{n} = 0$

Remarks

assume:

$$\sum_{n=1}^{\infty} a_n = A \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = B$$

Then:

① $$\sum_{n=1}^{\infty} a_n + b_n = A + B$$

② $$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

③ $$\sum_{n=1}^{\infty} K a_n = KA$$; If $\sum a_n$ div then $\sum\limits_{K \neq 0} K a_n$ div

④ Assume:

$\sum c_n$ div $\Rightarrow$ Then ; If $\sum a_n$ conv and $\sum b_n$ div then

$\sum (a_n + c_n)$ div ; $\sum a_n + b_n$ div and

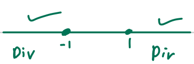
$\sum K c_n$ div $(K \neq 0)$; $\sum a_n + b_n$ div

Def: هندسي Geometric series has the form:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar^2 + ar^3 + ar^4 \ldots$$

Conv : If $|r| < 1$ $\quad (-1 < r < 1)$

Div : If $|r| \geq 1$



r: ratio الاساس → بلاقيه عن طريق قسمة الحد على يلي قبله

Test ③: Geometric series (strong) conv to:

اذا كانت conv

$$\text{Sum} = \frac{a}{1 - r}$$

الحد الأول (a), ratio (r)

Exp: $$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots$$

geometric → $r = \frac{2}{3} \in (-1, 1)$ → conv

$$S_n = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

Exp: find sum of:

① $1 + \frac{1}{2} + \frac{1}{4} + \cdots \left(\frac{1}{2}\right)^{n-1}$

$r = \frac{\frac{1}{2}}{1} = \frac{1}{2}$ this series is conv since $r \in (-1,1)$

$\text{sum} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2 \Rightarrow \sum \frac{1}{2}^{n-1}$ converge to $\frac{1}{2}$

② $1 - \frac{1}{3} + \frac{1}{9} + \cdots \left(\frac{-1}{3}\right)^{n-1}$

$r = \frac{\frac{-1}{3}}{1} = \frac{-1}{3}$, $\frac{\frac{1}{9}}{\frac{-1}{3}} = \frac{-1}{3}$ $\longrightarrow$ بناخذ الجزء واحد عشان نتأكد انها جيو

Geometric series $\Rightarrow$ conv since $r = \frac{-1}{3} \in (-1,1)$

$r = -\frac{1}{3}$

$\text{sum} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$

② $\sum_{n=1}^{\infty} \left(\frac{-3}{2}\right)^n$, $\frac{-3}{2} + \left(\frac{-3}{2}\right)^2 + \left(\frac{-3}{2}\right)^3 \cdots$

geomtric with

$r = \frac{-3}{2} \notin (-1,1)$

Div, geom series.

③ $\sum_{n=1}^{\infty} \left(\frac{7}{2^n} + \frac{4}{3^n}\right)$

$7\sum_{n=1}^{\infty} \frac{1}{2^n} + 4\sum \frac{1}{3^n}$

$7\left[\frac{\frac{1}{2}}{1-\frac{1}{2}}\right] + 4\left[\frac{\frac{1}{3}}{1-\frac{1}{3}}\right]$

$7\left[\frac{\frac{1}{2}}{\frac{1}{2}}\right] + 4\left[\frac{\frac{1}{3}}{\frac{2}{3}}\right]$

$7 + 2 = 9$

④ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$

$3\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} = 3\left[\underset{a}{\frac{1}{2}} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} \cdots\right]$

$\frac{\frac{-1}{4}}{\frac{1}{2}} = \frac{-1}{2}$, $\frac{\frac{1}{8}}{\frac{-1}{4}} = -\frac{1}{2} \cdots$

Geometric series since $r = \frac{-1}{2} \in (-1,1)$

$Sum = 3\left[\frac{a}{1-r}\right]$

$= 3\left[\frac{\frac{1}{2}}{1+\frac{1}{2}}\right] = 1$

Exp: Write the following decimals as ratio of two integers.

① $0.\overline{7} = 0.7 + 0.07 + 0.007 + 0.0007 \cdots$

$= \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} \cdots$ geometric with $r = \frac{1}{10} \in (-1,1)$

(conv)

$= \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$

② $0.\overline{19} = 0.19 + 0.019 + 0.0019 + 0.00019 \cdots$

$= \frac{19}{100} + \frac{19}{10000} + \frac{19}{1000000} \cdots$ geometric with $r = \frac{1}{100} \in (-1,1)$

$= \frac{a}{1-r} = \frac{\frac{19}{100}}{1-\frac{1}{100}} = \frac{\frac{19}{100}}{\frac{99}{100}} = \frac{19}{99}$

③ $0.0\overline{7} = \frac{0.\overline{7}}{10} = \frac{\frac{7}{9}}{10} = \frac{7}{90}$

④ $0.00\overline{7} = \frac{7}{900}$

Exp: Consider $\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots$ geometric with $r = -(x+1)$

① find values of x so that this series converges

$$|r| < 1$$
$$|-(x+1)| < 1$$
$$|x+1| < 1$$
$$-1 < x+1 < 1$$
$$-2 < x < 0$$

② find it's sum :

$$\frac{a}{1-r}$$

$$= \frac{1}{1-(-(x+1))}$$

$$= \frac{1}{1+x+1}$$

$$= \frac{1}{x+2}$$

$\therefore x = -\frac{1}{2}$ conv to $\frac{1}{-\frac{1}{2}+2} = \frac{2}{3}$

Ch 10.3: Integral Test: (conv, Div)

Consider $\sum_{n=k}^{\infty} a_n$, where

- a_n positive terms
- $a_n = f(n)$ is cont, $+$, $\downarrow$ on $[k, \infty)$

Then $\sum_{n=k}^{\infty} a_n$ and $\int_{k}^{\infty} f(x)\,dx$ both conv or both Div

Exp: check conv, div of

① $\sum_{n=1}^{\infty} \frac{1}{n^2}$, positive term for $\forall\ n = 1, 2, 3$

, $f(x) = \frac{1}{x^2}$ is $+$, $\downarrow$ on $[1, \infty]$, cont

Test ② $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2} = 0$

$f(x) = \frac{1}{x^2}$ cont, $+$, $\downarrow$ on $[1, \infty]$

and $\int_{1}^{\infty} \frac{dx}{x^2}$ conv to $\frac{1}{2-1} = 1$

so $\frac{1}{n^2}$ conv by I.T , we can't say conv to 1 we just say conv here.

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$f(x) = \frac{1}{\sqrt{x}}$ cont, $+$, $\downarrow$ on $[1, \infty]$

and $\int_{1}^{\infty} \frac{dx}{\sqrt{x}} \Rightarrow$ div $\Rightarrow \frac{1}{\sqrt{n}}$ div

div by I.T :-

Rule: p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv if } p > 1 \\ \text{Div if } p \leq 1 \end{cases}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic series $\Rightarrow$ Div

* Reminder

Exp*: $\int_{1}^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & , \text{if } p > 1 \\ \infty & , \text{if } p \leq 1 \end{cases}$

③ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, $\displaystyle\lim_{n\to\infty} a_n = 0$

$f(x) = \dfrac{1}{x^2+1}$ cont , + , $\downarrow$ on $[1, \infty]$

المقام ما بصفر

and $\displaystyle\int_1^{\infty} \frac{dx}{x^2+1}$

$\displaystyle\lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} \Rightarrow \lim_{b\to\infty} \tan^{-1} \Big|_1^b$

$\displaystyle\lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1 \right]$

$\dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4} \Longrightarrow \sum \dfrac{1}{n^2+1}$ conv by I.T

④ $\displaystyle\sum_{n=3}^{\infty} \frac{5-n}{3n-\frac{1}{2}}$

$\displaystyle\lim_{n\to\infty} a_n = \frac{-1}{3} \neq 0$

so $\displaystyle\sum \frac{5-n}{3n-\frac{1}{2}}$ div by n^{th} term Test

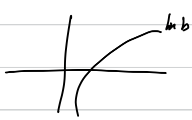
⑤ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$, $\displaystyle\lim_{n\to\infty} a_n = 0$

$f(x) = \dfrac{1}{2x-1}$ cont , + , $\downarrow$ on $[1, \infty]$

and $\displaystyle\int_1^{\infty} \frac{dx}{2x-1} = \frac{1}{2} \lim_{b\to\infty} \int_1^b \frac{2\,dx}{2x-1}$

$\displaystyle= \frac{1}{2} \lim_{b\to\infty} \ln|2x-1| \Big|_1^b$

$\displaystyle= \frac{1}{2} \lim_{b\to\infty} \left[\ln 2b-1 - 0 \right]$

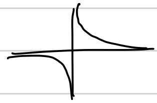


$= \infty$

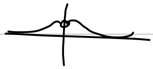
$\sum \dfrac{1}{2n-1}$ div by I.T

Exp: $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\frac{1}{n} = \lim \frac{\sin\frac{1}{n}}{\frac{1}{n}}$; $u = \frac{1}{n}$, $n \to \infty$, $u \to 0^+$



$\lim_{u\to 0^+} \frac{\sin u}{u}$



$= 1$ so Div by n^{th} term test

ch 10.4:

DCT:

$d_n \leq a_n \leq c_n$

$\downarrow$ if $\sum d_n$ div $\Rightarrow$ $\sum a_n$ div

$\downarrow$ if $\sum c_n$ conv $\Rightarrow$ $\sum a_n$ conv

$\sum a_n \to$ con / div

div: $\sum d_n$ given

con: $\sum c_n$ given

$a_n \geq 0$, $d_n \geq 0$, $c_n \geq 0$ $\forall$ large n

Exp: Check for conv/div

① $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n}{n^2}$ ← a_n

$$\frac{3-1}{n^2} \leq \frac{3+\sin n}{n^2} \leq \frac{3+1}{n^2}$$

$\hookrightarrow$ conv by integral test $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2}$, $\displaystyle\int_1^{\infty} \frac{4}{n^2} = 1$ conv

Hince $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n}{n^2}$ con by DCT

② $\displaystyle\sum_{n=1}^{\infty} \frac{7n}{5n+1}$

div by n^{th} term test

$$\lim_{n\to\infty} \frac{7n}{5n+1} = \frac{7}{5} \neq 0$$

③ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

$$< \left(\frac{n}{3n+1}\right)^n \leq \left(\frac{n}{3n}\right)^n$$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \to$ geometric

Hince $\sum \frac{n}{3n+1}$ conv by DCT conv to $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

P-Series Test

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv if $p > 1$

Div if $p \leq 1$

④ $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$, use DCT

$\frac{1}{n} \leqslant \frac{3}{n+\sqrt{n}} \ngtr$; for large n, $n > \sqrt{n}$

$n > \sqrt{n}$

$2n > n+\sqrt{n}$

$3n > 2n > n+\sqrt{n}$

$3n > n+\sqrt{n}$

$n > \frac{n+\sqrt{n}}{3}$

$\frac{1}{n} < \frac{3}{n+\sqrt{n}}$

Div by p-series (Harmonic series)

Hince $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ Div by DCT

✱ Limit comparsion Test:

$\sum_{n=1}^{\infty} a_n = ??$, find b_n s.t $\sum b_n$ Known

; $a_n > 0$, $b_n > 0$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ (finite) then both ($\sum a_n$ and $\sum b_n$) are conv or both div
- $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ conv then $\sum a_n$ conv
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div then $\sum a_n$ div

Exp: Check for conv/div

① $\sum_{n=1}^{\infty} \tan\frac{1}{n}$ ← a_n

بنجيب b_n

$b_n = \sum_{n=1}^{\infty} \frac{1}{n} \to$ div

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \frac{\frac{\sin\frac{1}{n}}{\cos\frac{1}{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \quad \lim_{n\to\infty} \cos\frac{1}{n}$

$= 1 \cdot 1 = 1 > 0$

Hince $\sum_{n=1}^{\infty} \tan\frac{1}{n}$ div by LCT

② $$\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} \leftarrow a_n$$

$; b_n = \frac{1}{n^2}$ conv by p-series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2^n}{n^2 2^n}}{??}$$

$$= \lim_{n\to\infty} \frac{\frac{n+2^n}{n^2 2^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n+2^n}{2^n} = \frac{\infty}{\infty}$$

لوبيتال

$$= \frac{1+2^n \ln 2}{2^n \ln 2}$$

$$= \frac{2^n (\ln 2)^2}{2^n (\ln 2)^2} = 1 > 0$$

Hince $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ conv by LCT

طريقة DCT

$$\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n 2^n} + \frac{1}{n^2} \leq \sum \frac{1}{2^n} + \sum \frac{1}{n^2}$$

conv conv

conv

Hince $\sum \frac{n+2^n}{n^2+2^n}$ conv by DCT

③ $$\sum_{n=1}^{\infty} \frac{n+2}{n^3+n^2+5}$$

, find $b_n = \frac{1}{n^2} \rightarrow$ فرق القوى

$\sum \frac{1}{n^2}$ conv

$$\lim_{n\to\infty} \frac{\frac{n+2}{n^3+n^2+5}}{\frac{1}{n^2}} = \frac{n+2}{n^3+n^2+5} \cdot \frac{n^2}{1} = 1 > 0$$

Hince $\sum \frac{n+2}{n^3+n^2+5}$ conv by LCT

④ $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$, $b = \frac{1}{\sqrt{n}} \to$ فرق القوى

div - p series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{\frac{1}{\sqrt{n}}}$$

$$= \frac{\sqrt{n+1}}{\sqrt{n^2+1}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} \overset{\to \text{cont}}{=} \sqrt{\lim_{n\to\infty} \frac{n^2+n}{n^2+2}} = \sqrt{1} = 1 > 0$$

Hince $\sum \sqrt{\frac{n+1}{n^2+1}}$ div by LCT

⑤ $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$, use LCT

use DCT

LCT:

find b_n : $\frac{n^1}{n^{2.5}} = \frac{1}{n^{\frac{3}{2}}} \Rightarrow \sum \frac{1}{n^{\frac{3}{2}}}$ conv - pserise

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{\frac{3}{2}}}} = 1 > 0$$

$\sum \frac{n+1}{n^2\sqrt{n}}$ conv by LCT

DCT :

$$\frac{n+1}{n^2\sqrt{n}} \leq \frac{n+n}{n^2\sqrt{n}}$$

$$= \sum \frac{2n}{n^2\sqrt{n}}$$

$$= \sum \frac{2}{n^{\frac{3}{2}}} \to \text{conv}$$

Hince $\sum \frac{n+1}{n^2\sqrt{n}}$ conv by DCT

*10.5:

$$\sum_{n=1}^{\infty} a_n \begin{cases} \to \text{Conv} \\ \to \text{Div} \end{cases}$$

* Ratio TesT:

Assume $a_n > 0$. find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \to$ rho

اي اشي

- If $\rho < 1$ then $\sum a_n$ conv
- If $\rho > 1$ then $\sum a_n$ Div
- If $\rho = 1$ the TesT fails.

اذا في ! المضروب

بنستخدم R T

*** Exp:** $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{\frac{(n+1)!}{e^{n+1}}}{\frac{n!}{e^n}} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{e^{n+1}}_{\,e}} \cdot \frac{\cancel{e^n}}{\cancel{n!}}$$

$(n+1)!$

$= (n+1)\, n!$

$$= \lim_{n\to\infty} \frac{n+1}{e} = \infty > 1$$

$\sum \frac{n!}{e^n}$ div by RT

② $\sum_{n=1}^{\infty} \frac{n^2}{e^n} \leftarrow a_n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{e^{n+1}}_{\,e}} \cdot \frac{\cancel{e^n}}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{n^2+2n+1}{n^2}$$

$$= \frac{1}{e} \cdot 1 = \frac{1}{e} < 1$$

$\sum \frac{n^2}{e^n}$ conv by RT

③ $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \quad \text{TesT fails.}$$

4

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{\ln n} \quad a_n > 0$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+2}}{\ln(n+1)}}{\dfrac{3^{n+1}}{\ln n}} = \frac{\cancel{3^{n+2}}}{\ln(n+1)} \cdot \frac{\ln n}{\underset{3}{\cancel{3^{n+1}}}} = \lim_{n\to\infty} \frac{3}{1} \frac{\ln n}{\ln(n+1)}$$

$$= 3 \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \rightarrow \frac{\infty}{\infty} = \lim_{n\to\infty} 3 \frac{\frac{1}{n}}{\frac{1}{n+1}} = 3 > 1$$

$\sum \frac{3^{n+2}}{\ln n}$ div by RT

5

$$\sum_{n=1}^{\infty} \frac{n!}{10^n} \quad a_n > 0 \quad \leftarrow ! \text{ factoral}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{10^{n+1}}}{\dfrac{n!}{10^n}} = \lim_{n\to\infty} \frac{\overset{n+1}{\cancel{(n+1)!}}}{\underset{10}{\cancel{10^{n+1}}}} \cdot \frac{\cancel{10^n}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n+1}{10} = \infty > 1$$

$\sum \frac{n!}{10^n}$ div by RT

* Root Test: $\sqrt{RT}$

assume $a_n \geq 0$ for large n

$\sum a_n$
تحتوي على قوة
n

Find $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

- If $\rho < 1$ then $\sum a_n$ conv
- If $\rho > 1$ then $\sum a_n$ Div
- If $\rho = 1$ the Test fails.

Exp.

① $\sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ ← قوة n

$$\lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^{\cancel{n}}} = \lim_{n\to\infty} \frac{1}{n} - \frac{1}{n^2} = 0 - 0 = 0 < 1$$

$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ conv by $\sqrt{RT}$

② $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

$$\lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{\ln n}{n}\right)^{\cancel{n}}} = \lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{لوبيتال}}{=} \frac{\frac{1}{n}}{1} = 0 < 1$$

$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ conv by $\sqrt{RT}$

③ $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ ← a_n +

$$\lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^3}} = \frac{3}{\sqrt[n]{n^3}} = \frac{3}{(\underbrace{\sqrt[n]{n}}_{\to 1})^3} = \frac{3}{1} = 3 > 1$$

$\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ Div by $\sqrt{RT}$

④ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ → conv - p series

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \frac{1}{\sqrt[n]{n^2}} = \frac{1}{(\sqrt[n]{n})^2} = \frac{1}{1} = 1$$

The Test fails.

⑤ $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$, RT في !

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}}{\frac{(n+3)!}{3!\,n!\,3^n}} = \lim_{n\to\infty} \frac{\cancel{(n+4)!}^{\,n+4}}{\cancel{3!}\,\cancel{(n+1)!}_{\,n+1}\,\cancel{3^{n+1}}_{\,3}} \cdot \frac{\cancel{3!}\,\cancel{n!}\,\cancel{3^n}}{\cancel{(n+3)!}}$$

$$= \lim_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1$$

$\sum \frac{(n+3)!}{3!\,n!\,3^n}$ conv by RT

⑥ $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$, $\sqrt{RT}$, $a_n +$

قاعدة

$$\lim_{n\to\infty} \sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \sqrt[\cancel{n}]{\left[\left(1-\frac{1}{n}\right)^n\right]^{\cancel{n}}} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e} < 1$$

$\sum \left(1-\frac{1}{n}\right)^{n^2}$ conv by $\sqrt{RT}$

* Given this recursive Terms:

$a_1 = 2$, $a_{n+1} = \frac{2}{n} a_n$

Does $\sum_{n=1}^{\infty} a_n$ conv/div?

Apply RT:

معطى

$$\frac{a_{n+1}}{a_n} = \frac{2}{n}$$

$$\lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

$\sum a_n$ conv by RT

① حل

$a_1 = 2$

$a_{n+1} = \frac{2}{n} a_n$

$a_2 = 2 a_1 = 2.2 = 2^2$

$a_3 = \frac{2}{2} a_2 = \frac{2^3}{2(1)}$

$a_4 = \frac{2}{3} a_3 = \frac{2^4}{2.3}$

$a_n = \frac{2^n}{(n-1)!}$ => check $\sum a_n$

$\sum \frac{2^n}{(n-1)!}$, apply RT

$$= \lim_{n\to\infty} \frac{\cancel{2^{n+1}}\,2}{\cancel{n!}_{\,n}} \cdot \frac{\cancel{(n-1)!}}{\cancel{2^n}} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

10.6: Alternating Series:

The Alternating series has the form:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots \quad *$$

Q: when * conv/Div?

Th (Alternating series TesT) AST

$$\sum_{n=1}^{\infty} a_n = \overbrace{\sum_{n=1}^{\infty} (-1)^{n+1} u_n}^{*} = u_1 - u_2 + u_3 - u_4 \cdots$$

* AST:

$\sum (-1)^{n+1} u_n$ conv If

1. $u_n \geq 0 \ \forall n$ and $u_n = |a_n|$
2. $u_n \downarrow$ for large n $\quad u_{n+1} \leq u_n$
3. $\lim_{n\to\infty} u_n = 0$ "if not $\Rightarrow \sum (-1)^{n+1} u_n$ div by n^{th} term test"

Exp:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, Alternating Harmonic series $\rightarrow$ conv , but $\sum \frac{1}{n} \rightarrow$ div

Apply AST:

$a_n = (-1)^{n+1} \cdot \frac{1}{n}$

$u_n = |a_n| = \frac{1}{n}$

1. $u_n \geq 0$ for all n
2. $u_n \downarrow$ for all n
3. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hince $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv by AST

② $\sum (-1)^n (0.2)^n$, Alternating

a_n

$u_n = (0.2)^n = \left(\frac{1}{5}\right)^n$

① $u_n \geq 0$ for all n

② $u_n \downarrow$ for all n

③ $\lim_{n \to \infty} \left(\frac{1}{5}\right)^n = 0$

Hince $\sum (-1)^{n+} (0.2)^n$ is conv by AST

③ $\sum_{n=1}^{\infty} (-1)^n n$, Alternating → Applay AST

$u_n = n$

① $u_n \geq 0$

② u_n is not decreasing

③ $\lim_{n \to \infty} u_n = \lim_{n \to \infty} n = \infty \neq 0$

$\sum (-1)^n n$ div by n^{th} term test.

④ $\sum_{n=3}^{\infty} (-1)^n \frac{2n}{3n-4}$, Alternating → Applay AST

① . .

② . .

③ $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{2n}{3n-4} = \frac{2}{3} \neq 0$

Hince $\sum (-1)^n \frac{2n}{3n-4}$ div by n^{th} term test.

* Def: (Abs convergnece)

$\sum a_n$ conv Abs if $\sum |a_n|$ conv

$\sum u_n$

Exp: check if $\sum a_n$ conv abs?

① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

$\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$
↳ conv p-series

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

$\Rightarrow \sum (-1)^{n+1} \frac{1}{n^2}$ conv Abs

Th:
If $\sum |a_n|$ conv $\Rightarrow$ $\sum a_n$ conv
means
If $\sum a_n$ conv Abs $\Rightarrow$ $\sum a_n$ conv
($\sum |a_n|$ conv)
العكس غير صحيح

② $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ $\Rightarrow$ conv by AST

$\sum |a_n| = \sum \frac{1}{n} \Rightarrow$ div

اذا كانت conv abs يعني اكيد الجارية conv والعكس غير صحيح

$\sum (-1)^{n+1} \frac{1}{n} \Rightarrow$ conv but not abs since $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series

Def: (Converge Conditionally)

The infinite series $\sum a_n$ conv cond.
If it conv. by AST but not Abs

Exp. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv. cond

Exp. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv by AST since ①✓ ②✓ ③✓

but not Abs since $\sum |a_n| = \sum \frac{1}{\sqrt{n}}$ → div p-series
This series conv. cond NOT Abs

* Remark:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p} = \begin{cases} \text{conv Abs if } p>1 \\ \text{conv condi if } 0<p\leq 1 \\ \text{div if } 0\geq p \end{cases}$$ → Alternating p-series TesT

Exp. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$ Conv. Abs $\Rightarrow$ Conv

since $\sum |a_n| = \sum \frac{1}{n^3} \Rightarrow$ conv p-series

Exp. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{\frac{2}{3}}} \Rightarrow$ conv condi $\Rightarrow$ conv but not abs $\quad 0<p<1$
by Ast

$\sum |a_n| = \sum \frac{1}{n^{\frac{2}{3}}}$ div p-series

Exp. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{-2}} = \sum (-1)^n n^2$ div by n^{th} term test

---

Th (Alternating Estimation Th)

- Assume $\sum_{n=1}^{\infty} a_n = \sum^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 \dots = L$ conv

$S_n \rightarrow$ جمع الحدود.
$L \rightarrow$ الحساب

- If we approximate L by $S_n = u_1 - u_2 + u_3 \dots + (-1)^{n+1} u_n$

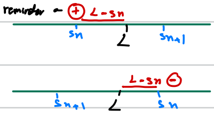
*** then:**

① the reminder $L - S_n$ has same sign as a_{n+1}

② The error $|L - S_n| < \overset{\text{الموجب}}{u_{n+1}} = |a_{n+1}|$

③ $\min\{s_n, s_{n+1}\} < L < \max\{s_n, s_{n+1}\} \quad \forall n$

* $\sum (-1)^{n+1} u_n = L$



error $= |\text{Reminder}| = |L - s_n| < \underbrace{u_{n+1}}_{} = \underbrace{|a_{n+1}|}_{}$

الموجب منهم

Exp: $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 + \cdots$

$= \frac{\frac{2}{3}}{1 - -\frac{2}{3}} = \frac{4}{10} = 0.4 = L$

If we approximate $L = 0.4$ by $S_3 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 = \frac{14}{27} \approx 0.519$

$n = 3$

$L = 0.4$

$S_3 = 0.519$

① Reminder: $L - s_n = 0.4 - 0.519 = -0.119$

$a_{n+1} = a_{3+1} = a_4 = -\left(\frac{2}{3}\right)^4$

② error: $|L - s_n| = 0.119$

$E = 0.119 < u_4 = |a_4| = \left(\frac{2}{3}\right)^4 = 0.198$

③ $S_4 = \frac{1}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 \approx 0.321$

$\min\{S_3, S_4\} < L < \max\{S_3, S_4\}$

$\min\{0.519, 0.321\} < 0.4 < \max\{0.519, 0.321\}$

$0.321 < 0.4 < 0.519$

الاصغر من الثنتين

الأكبر

Exp. Approximate the sum of $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}$ with error of magnitude less than 5×10^{-6}

- We use S_n to approx the sum
- so we need to find $n \Rightarrow \frac{1}{(2n)!} < 5\times10^{-6}$

$$(2n)! > \frac{1}{5\times10^{-6}} = \frac{10^6}{5} = \frac{10\times10^5}{5} = 200000$$

$$(2n)! > 200000 \Rightarrow n \geq 5$$ بنجرب

- $S_5 = 1 - \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} \approx 0.54$

- Error $= |L - S_n| = |L - S_5| < u_6 = \frac{1}{(10)!} = 0.275\times10^{-6}$

Exp. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}$

$\lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = 3$ div by n^{th} term test

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$

$u_n = \frac{n}{n^3+1}$

① $u_n \geq 0$

② $u_n \downarrow$

③ $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^3+1} = 0 \Rightarrow$ conv by AST

To check if its conv abs

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow$ conv p-series

$\sum |a_n|$ conv by DCT

$\sum |a_n|$ conv by DCT

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ conv abs

$n^3 + 1 > n^3$

$\frac{1}{n^3+1} < \frac{1}{n^3}$

$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$

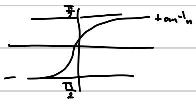
Exp: $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$

- If the alternating series conv Abs , then it conv
- That is if $\sum |a_n|$ conv then $\sum a_n$ conv. Abs

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$$

if we use IT

$\frac{\tan^{-1} n}{n^2+1}$: cont , + , ↓ on $[1,\infty)$



$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx = \lim_{n\to\infty} \int_1^{b} \frac{\tan^{-1} x}{x^2+1}\, dx$$

$u = \tan^{-1} x$

$du = \frac{dx}{x^2+1}$

$x = 1 \Rightarrow u = \tan^{-1} 1 = \frac{\pi}{4}$

$x = \infty \Rightarrow u = \tan^{-1} b$

$$\lim_{b\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} b} u\, du$$

$$\lim_{b\to\infty} \frac{u^2}{2} \Big|_{\frac{\pi}{4}}^{\tan^{-1} b}$$

$$\frac{1}{2} \lim_{b\to\infty} \left[(\tan^{-1} b)^2 - \left(\frac{\pi}{4}\right)^2 \right)$$

$\frac{1}{2} \left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2 \right] = \frac{3\pi^2}{3^2}$ so $\sum (-1)^n \frac{\tan^{-1} n}{n^2+1}$ conv Abs.

10.7 : Power series.

: are infinit sum of Polys

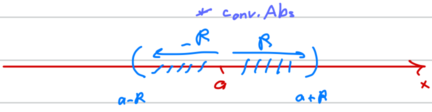
$$* = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$$

a_i : coefficients

a : center

R: Radius of convergnce

I_c: Interval of convergnc



I_c: $|x-a| < R$

* Note: To find R and I_c => we applay RT (Ratio Test)

$-R < x-a < R$

$a-R < x < a+R$

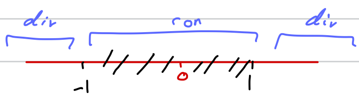
Exp ① $\sum_{n=0}^{\infty} x^n$ ==> نقارنها بالشكل العام $a_n(x-a)$ بالتالي $0 = a$

$a = 0$ center

$\sum_{1}^{\infty} x^n = 1 + x + x^2 + x^3 \dots$ => geometric series

$r = x$

If $|x| < 1$ => $\sum_{n=0}^{\infty} x^n$ Converges to



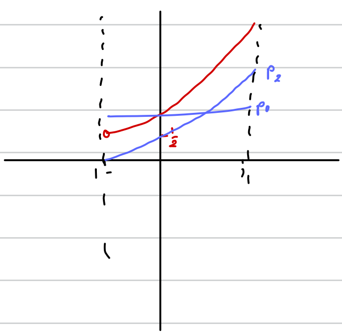
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 = \frac{1}{1-x}$$

$x \in [-1, 1]$

take $x = \frac{1}{2}$ => $\sum (\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2$

take $x = 2$ => $\sum 2^n$ div by n^{th} term test since $\lim_{n\to\infty} 2^n = \infty$

$f(x) = \frac{1}{1-x}$ => we can approximate $f(x)$ by



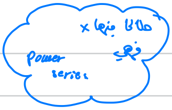
$P_0(x) = 1$

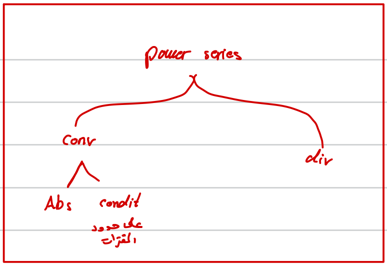
$P_1(x) = 1 + x$

$P_2(x) = 1 + x + x^2$

$P_3(x) = 1 + x + x^2 + x^3$

Exp: find R and IC





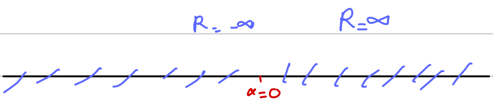
① $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \rightarrow a=0$

Apply RT:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$(-1)^n, (-1)^{n+1}$ ما الهم داعي لوجود بالقيمة المطلقة

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$$



$R = \infty$

$IC = \mathbb{R}\ (-\infty, \infty)\ \} \Rightarrow \frac{(-1)^n x^n}{n!}$ conv abs for every x

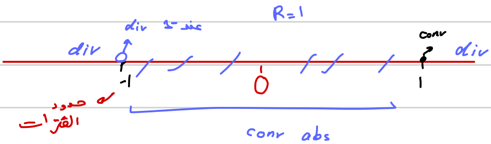
div لان مش قيم عندها

② $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \rightarrow a=0$

Apply RT

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$$

$$= |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$$



$|x| < 1$

$-1 < x < 1$

* To check conv condit

$\Rightarrow$ check end points

$x=-1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum \frac{(-1)^{2n-1}}{n} = \sum (-1)^{2n} (-1)^{-1} \frac{1}{n} = 1 \cdot \frac{1}{-1} \cdot \frac{1}{n} = -\sum \frac{1}{n} \Rightarrow$ div

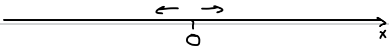
$x=1 \Rightarrow (-1)^{n-1} \frac{(1)^n}{n} = \sum \frac{(-1)^{n-1}}{n}$ Alternating harmonic series (Conv)

$x=1 \Rightarrow \sum \frac{(-1)^{n-1}}{n}$ is not conv Abs since $\sum |a_n| = \sum \frac{1}{n} \rightarrow$ div

$x=1 \Rightarrow \sum \frac{(-1)^{n-1} x^n}{n}$ conv but not Abs

Conv Abs in $(-1,1)$

Exp: $\sum_{n=1}^{\infty} (\ln n)\, x^n \rightarrow a=0$



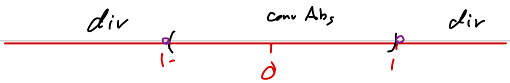
Apply RT $\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{\ln(n+1)\ x^{n+1}}{\ln n \quad x^n}\right| = |x| \lim_{n\to\infty} \frac{\ln n+1}{\ln n} = \frac{\infty}{\infty}$

$= |x| \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$

$= |x| \boxed{?}\ 1$

$-1 < x < 1$

↳ conv. Abs on $(-1,1)$ $\Rightarrow$ conv on $(-1,1)$



$R = 1$

$IC = (-1,1)$

$x = 1 \Rightarrow \sum \ln n\, x^n = \sum \ln n\ 1^n = \sum \overset{a_n}{\ln n}$ div by n^{th} term test

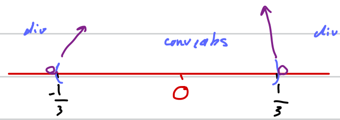
$x = -1 \Rightarrow \ln n\, x^n = \ln n\, (-1)^n \rightarrow$ Alternating $\Rightarrow \lim \ln n = \infty \neq 0$ div

$\nexists\ x$ s.t $\sum \ln n (x)^n$ conv condit

Exp:

Find IC, R, conv Abs, condit

$\sum_{n=0}^{\infty} 3^n x^n \Rightarrow \sum a_n (x-a)^n \Rightarrow a=0$



Apply RT:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{3^{n+1} x^{n+1}}{3^n x^n} = \lim_{n\to\infty} 3|x| = 3|x|$

$3|x| \boxed{<} 1$

$|x| < \frac{1}{3}$

$-\frac{1}{3} < x < \frac{1}{3}$

الأطراف: $x = -\frac{1}{3} \Rightarrow \sum 3^n \left(-\frac{1}{3}\right)^n = \sum (-1)^n \Rightarrow DNE \Rightarrow$ div by nth term test

$x = \frac{1}{3} \Rightarrow \sum 3^n \left(\frac{1}{3}\right)^n = \sum 1^n = \infty \Rightarrow$ div

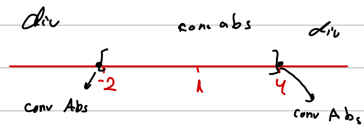
Hence, $\sum 3^n x^n$ conv Abs $\forall x \in \overset{IC}{\left(-\frac{1}{3}, \frac{1}{3}\right)}$, $R = \frac{1}{3}$

another way * $\sum_{n=0}^{\infty} 3^n x^n = \sum (3x)^n = 1 + 3x + (3x)^2 \dots$ geometric $R = 3x$

Exp: $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^3} \Rightarrow a_n (x-a)^n \Rightarrow a = 1$

Apply RT: $\sum_{n=1}^{\infty} \left|\frac{a_{n+1}}{a_n}\right| = \sum_{n=1}^{\infty} \left|\frac{(x-1)^{n+1}}{3^{n+1}(n+1)^3} \cdot \frac{3^n n^3}{(x-1)^n}\right| = \left|\frac{(x-1)n^3}{3(n+1)^3}\right| = \frac{|x-1|}{3} \lim \frac{n^3}{(n+1)^3} = \frac{|x-1|}{3} \lim \left(\underbrace{\frac{n}{n-1}}_{\to 1}\right)^3 = \frac{|x-1|}{3}$

$\frac{|x-1|}{3} < 1 \Rightarrow -3 < x - 1 < 3 \Rightarrow -2 < x < 4$



at $x = -2$: $\sum \frac{(-3)^n}{3^n n^3} = \frac{(-1)^n \cancel{3^n}}{\cancel{3^n} n^3} = \frac{(-1)^n}{n^3} \Rightarrow$ conv by AST

$\lim |a_n| = \lim \frac{1}{n^3} \Rightarrow$ conv by p-series

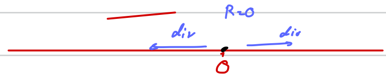
at $x = 4$: $\sum \frac{3^n}{3^n n^3} = \frac{1}{n^3} \Rightarrow$ conv by p-series

IC: $[-2, 4]$, $R = 3$

$\nexists$ no point's (X) where the power series conv condit power series لأن x لها

Exp: $\sum_{n=0}^{\infty} n!\,x^n \Rightarrow \sum a_n(x-a)^n \Rightarrow a=0$

Apply RT: $\lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty}\left|(n+1)\,x\right| = |x|\lim_{n\to\infty} n+1 = \infty > 1$

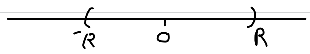


This infint series diverges for every x expet $x=0$

since when $x=0 \Rightarrow \sum n!\,x^n = \sum_{n=0}^{\infty} 0 = 0+0+0\cdots = 0$ conv

Th: Assume $\sum_{n=0}^{\infty} a_n x^n = A(x)$ and $\sum_{n=0}^{\infty} b_n x^n = B(x)$ converge abs on $|x|<R$

$0=0$



Then $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$ converges Abs to $A(x)B(x)$ on $|x|<R$

Th: If $\sum_{n=0}^{\infty} a_n x^n$ conv Abs on $|x|<R$

$\hookrightarrow a=0$

Then: $\sum a_n (f(x))^n$ conv Abs on $|f(x)|<R$ for any cont function f

Exp: • $\sum_{n=0}^{\infty} x^n = 1+x+x^2\cdots = \frac{1}{1-x}$ if $|x|<1$

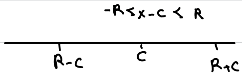
This means $\sum_{n=0}^{\infty} x^n$ conv Abs to $\frac{1}{1-x}$ on $|x|<1$
$-1<x<1$

• $\sum_{n=0}^{\infty} (4x^2)^n = 1+4x^2+(4x^2)^3\cdots = \frac{1}{1-4x^2}$ if $|4x^2|<1$
$= -\frac{1}{2}<x<\frac{1}{2}$

Th: (Term by Term Differentiation)

$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_n + a_1(x-c) + a_2(x-c)^2\cdots$

conv Abs on $|x-c|<R$



if f has all derevatives on $|x-c|<R$

اشتق الفوق

Then $f'(x) = \sum n a_n (x-c)^{n-1} = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2\cdots$

$f''(x) = \sum n(n-1)a_n(x-c)^{n-2} = 0+0+2a_{n_2} + 6a_3(x-c)\cdots$

Th: (Term by Term integration Th)

Assume $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ conv Abs on $|x-c| < R$

$= a_0 + a_1(x-c) + a_2(x-c)^2 \dots$

Then $\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + C$ on $|x-c| < R$

Exp: Idintify this function. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $|x| \leq 1$

$\boxed{a}$ $\sin^{-1}x$ $\boxed{b}$ $\cos^{-1}x$ $\boxed{c}$ $\tan^{-1}x$ $\boxed{d}$ $\sec^{-1}x$ ....

compare with $\sum a_n (x-a)^n$ $\Rightarrow a=0$

تعويض $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}$

conv Abs

-1 0 1

$f'(x) = 1 - x^2 + x^4 - x^6 \dots$ on $|x|<1$ → geomitric series

$= \frac{1}{1-(-x)^2} \Rightarrow f'(x) = \frac{1}{1+x^2}$ $r = |x^2| = x^2 < 1 \Rightarrow x < 1 \Rightarrow$ conv

$\int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx$

$f(x) = \tan^{-1}x + C$

$f(x)$ نعوض $f(0) = 0+0+0\dots$ $\Rightarrow f(0) = 0$

$0 = \tan^{-1} 0 + C$

$C = 0$

$f(x) = \tan^{-1}(x)$

10.8 : Taylor and Maclaurin series.

Def. Let f be a smoth function "all derivatives exist" on an interval that contains the interior point a. Then

* The Taylor series generated by f at $x=a$ is

المشتقة رقم n

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \dots$$

* The Maclaurim series generated by f is $(a=0)$

$$\sum \frac{f^{n}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

Exp: Find maclaurin series for: $Ms = Ts$ at $a=0$

① $f(x) = \cos x$

$$\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)x^4}{4!} \dots$$

$f(x) = \cos x \Rightarrow f(0) = 1$, $f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$, $f'''(x) = \sin x \Rightarrow f''(0) = 0$

$f^{(4)} = \cos x \Rightarrow f^{4}(0) = 1$, $f^{(5)}(x) = -\sin x \Rightarrow f^{5}(0) = 0$

$\vdots \qquad \vdots$

خصوص فوق : $\displaystyle\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} \dots$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

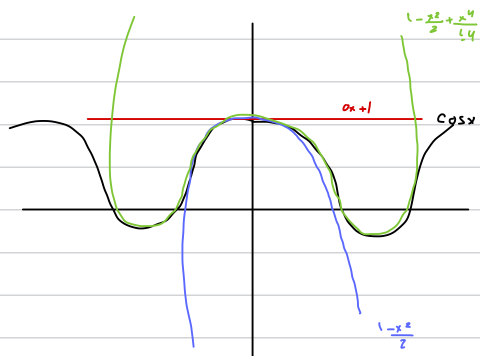
$f(x) = \cos x$

Approximation.

$P_0(x) = 1$ poly of degree 0

$P_2(x) = 1 - \frac{x^2}{2!}$ 〃 〃 〃 2

تقريب احسن $P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ 〃 〃 〃 4



بنحصل تقريب لـ cosx عن طريق poly كل ما كثرنا حدود

كلما كان التقريب احسن

$$\underbrace{\cos x = 1 - \frac{x^2}{2!}}_{\text{conv}} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \text{poly of degree } n \text{ to approximate } \cos x$$

$\lim_{n\to\infty} P_n(x) = \cos x$ about 0

Exp: ② $f(x) = \sin x$

$f = \sin x \Rightarrow f(0) = 0$ $\qquad$ $f' = \cos x \Rightarrow f'(0) = 1$

$f'' = -\sin x \Rightarrow f''(0) = 0$ $\qquad$ $f''' = -\cos x \Rightarrow f'''(0) = -1$

$f^{4} = \sin x \Rightarrow f^{4}(0) = 0$ $\qquad$ $f^{5} = \cos x \Rightarrow f^{5}(0) = 1$

$\vdots$ $\qquad\qquad$ $\vdots$

$$\sum \frac{f^{n}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

نعوض. $P_n = \sin x = 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \frac{x^7}{7!}$

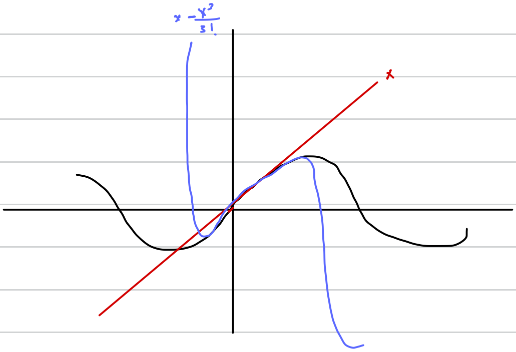
$$\underbrace{\sin x = \text{Ms}}_{\text{conv}} \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Approximation

$P_1(x) = x$ ✓

$P_3(x) = x - \frac{x^3}{3!}$ ✓✓ $\qquad$ at $x = 0$

$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ ✓✓✓



Exp: ③ $f(x) = e^x$

$f = e^x \Rightarrow f(0) = 1$

$f' = e^x \Rightarrow f'(0) = 1$

$f'' = e^x \Rightarrow f''(0) = 1$

$\vdots$

$$e^x = \sum \frac{f^{n}(0)}{n!} x^n = f(0)x + f'(0) + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

يساوي لان السلسلة conv

$$\underbrace{e^x = \text{Ms}}_{\text{conv}}$$

Ms:

① $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ فردي وزوجي

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!}$ فردي

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ زوجي

Exp: find Tylor series of $f(x) = 2^x$ at $\underset{a=1}{x=1}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \cdots$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \cdots$$

$f = 2^x \Rightarrow f(1) = 2$

$f' = 2^x \ln 2 \Rightarrow f'(1) = 2\ln 2$

$f'' = 2^x (\ln 2)^2 \Rightarrow f''(1) = 2(\ln 2)^2$

$f''' = 2^x (\ln 2)^2$

$f^n = 2^x (\ln 2)^n$

نعوض

$$f = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 \cdots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(n-1)^n$$

Exp: find Ms for $\cosh x$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[e^x + e^{-x}\right]$$

وحل كل x بنحط -x

$$= \frac{1}{2}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{3!} \cdots + \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} \cdots\right)\right]$$

$$= \frac{1}{2}\left[1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{3!} \cdots + 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} \cdots\right]$$

عامل مشترك 2

$$= \frac{\cancel{1}}{\cancel{2}}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} \cdots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

طريقة ثانية الاشتقاق
زي الباقي.

Exp: find the first non zero Terms in the Ms of

① $\frac{1}{3}(2x + x\cos x)$

$$= \frac{2}{3}x + \frac{x}{3}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]$$

$$= \frac{2}{3}x + \frac{x}{3} - \frac{x^3}{3(2!)} + \frac{x^5}{3(4!)} - \frac{x^7}{3(6!)} \cdots$$

$$= x - \frac{x^3}{3(2!)} + \frac{x^5}{3(4!)} - \frac{x^7}{3(6!)} + \cdots$$

The first non zero terms :

$$x ,\ \frac{-x^3}{3(2!)} ,\ \frac{x^5}{3(4!)} ,\ \frac{-x^7}{3(6!)}$$

② $e^x \sin x$

بالحد الثالث

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right]$$

نضرب بالأول

بالثاني الخ

$$= x + x^2 + \frac{x^3}{2!} + \cancel{\frac{x^4}{3!}} + \frac{x^5}{4!} + -\frac{x^3}{3!} - \cancel{\frac{x^4}{3!}} - \frac{x^5}{3!(2!)} \cdots + \frac{x^5}{5!} + \frac{x^6}{5!} \cdots$$

نأخذ اقل درجات

الحدود الاربعة

$$= x + x^2 + \left(\frac{x^3}{2!} - \frac{x^3}{3!}\right) + \left(\frac{x^5}{4!} - \frac{x^5}{3!(2!)} + \frac{x^5}{5!}\right)$$

1 2 3 4

③ $\cos 2x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$\cos 2x$:

$$= \sum \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!}$$

10.9:

f has all derivatives on $[a,b]$, $c \in (a,b)$

Taylor series of f about $x=c$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 \dots$$

Maclurine series = Taylor series at $c=0$

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Exp: Find Maclurine series of $f(x) = \frac{2}{(1-x)^3}$

$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} \dots$

or : $f(x) = \frac{2}{(1-x)^3} = \left(\frac{1}{1-x}\right)''$

$= (1 + x + x^2 + x^3 + x^4 + x^5 \dots)''$

$= (0 + 1 + 2x + 3x^2 + 4x^3)'$

$= (0 + 0 + 2 + 6x + 12x^2 \dots)$ نشتق

$f = 2(1-x)^{-3} \Rightarrow f(0) = 2$

$f' = -6(1-x)^{-4}(-1) \Rightarrow f'(0) = 6$

$f'' = -24(1-x)^{-5}(-1) \Rightarrow f''(0) = 24$

$f''' = -120(1-x)^{-6}(-1) \Rightarrow f'''(0) = 120$

$2 + 6x + \frac{24x^2}{2!} + \frac{120x^3}{3!} + \dots$

$2 + 6x + 12x^2 + 20x^3 + \dots = \sum_{n=0}^{\infty} (n+2)(n+1)x^n$

Exp:

كيف نوجد الايرور *

بس بدنا نستخدم أول حدين

assume $\sqrt{1+x} \approx 1 + \frac{x}{2}$; Estimate the error if $|x| < 0.01$

"use alternating series Estimation Theorem"

$-0.01 < x < 0.01$

Maclurine series $\sqrt{1+x}$ $\quad c = 0$ ← لان الفترة يلي معنا حول الصفر

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f' = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow \frac{1}{2}$

$f'' = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow \frac{1}{4}$

$f''' = \frac{3}{8}(1+x)^{-2.5} \Rightarrow \frac{3}{8}$

$$\sqrt{1+x} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

Error $< \left| -\frac{1}{8}x^2 \right| = \frac{x^2}{8}$; $|x| < 0.01$

$$\frac{x^2}{8} < \frac{(0.01)^2}{8} = 1.25 \times 10^{-5}$$

by ASET

Th (Taylor's Theorem)

Assume $f, f', f'', \dots f^{(n)}$ are continous on $[a,b]$ and $f^{(n)}$ is differentiable on (a,b), Then There exists a number $c \in (a,b)$ such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots \frac{f^{(n)}(a)}{n!}(b-a)^n$$

Taylor Th:

$f, f', f'', f^{(n+1)}$ cont on $[a,b]$ Then, $\exists$ a number $c \in (a,b)$ s.t

$$f(b) = \underbrace{f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots \frac{f^{(n)}(a)}{n!}(b-a)^n}_{\text{تقريب}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}_{\text{الباقي reminder}}$$

Note:

MVT (mean value theorem): f cont on $[a,b]$; f diff on (a,b) $\Rightarrow$ $\exists\, c \in (a,b)$ s.t $f'(c) = \frac{f(b)-f(a)}{b-a}$

MVT is special case from Taylor Theorem

$f(b) = f(a) + f'(c)(b-a)$

$f(b) - f(a) = f'(c)(b-a)$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

* replace b by x

$$f(x) = \overbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}^{P_n} + \overbrace{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{(n+1)}}^{R_n}$$

$P_n(x)$: poly of degree n

$f(x) = P_n(x) + R_n(x)$ $\qquad c \in (a,b)$

$P_n(x) \approx f(x)$ with error $= |R_n(x)|$ $\qquad c \in (a,x)$ متغير

* Remark: (Convergence of Taylor series)

If $\lim_{n\to\infty} R_n(x) = 0$ $\forall x \in$ Interval , Then Taylor series generated by f at $x=a$ converge to f

0= lim reminder اذا بتكون تقترب

that is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Exp: Show that Taylor series generated by $f(x)=e^x$ at $x=0$ converges to $f(x)$ $\forall x$

Taylor series of $f(x)=e^x$ at $x=0$ is maclurine series.

$$= \underbrace{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots+\frac{x^n}{n!}}_{P_n(x)} + R_n(x)$$

قاعدة الإيرور هي الحمرة

$$e^x = 1+x+\frac{x^2}{2!}\cdots\frac{x^n}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

$f(x)=e^x$ $\qquad$ $f''(x)=e^x$

$f'(x)=e^x$ $\qquad$ $f^{(n+1)}=e^x$

$f^{(n+1)}(c)=e^c$

$$\star \quad R_n(x) = \frac{f^{(n+1)}(c)\,x^{n+1}}{(n+1)!} = \frac{e^c\,x^{n+1}}{(n+1)!}$$

ناخذ الليمت للإيرور

$$\lim R_n(x) = \lim_{n\to\infty} \frac{e^c x^{n+1}}{(n+1)!} = e^c \lim_{n\to\infty} \underbrace{\frac{x^{n+1}}{(n+1)!}}_{\text{ch } 10.1} = e^c(0)=0 \quad \forall x$$

Hence, maclurine series converges to $e^x \Rightarrow e^x = 1+x+\frac{x^2}{2!}\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

توضيح تقريب

$$e^1 = 1+1+\frac{1}{2!}\cdots \sum_{n=0}^{\infty}\frac{1^n}{n!}$$

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}$$

$$= 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}\cdots = 2.718 = e$$

Th (Remainder Estimation Th)

TF $\quad f(x) = P_n(x) + R_n(x)$

$$R_n(x) = \frac{f^{n+1}(c)\,(x-a)^{n+1}}{(n+1)!}$$

Assume $|f^{n+1}(t)| \leq M$ for all $t \in (a, x)$

$t = c$

رقم موجب

upper bound for f^{n+1}

Then, Remainder $\left[\, |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \,\right]$ *

If * holds for all n, Then the Taylor series generated by $f(x)$ converges to $f(x)$

Exp: show that Maclurine series for $\sin x$ converges to $\sin x \;\forall x$

(TF) $\quad f(x) = P_n(x) + R_n(x)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$$

الدرجة يلي يوصلها حيث الحدود

بالتالي $\quad f(x) = P_{(2n+1)}(x) + R_{(2n+1)}(x)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} + \frac{(-1)^{n+1} x^{(2n+2)}}{(2n+2)!}$$

$$|R_{2n+1}| = \left| \frac{f^{2n+2}(c)\, x^{2n+2}}{(2n+2)!} \right|$$

; $f(x) = \sin x \quad f''(x) = -\sin x$

$f'(x) = \cos x \quad \vdots \quad |f^{2n+2}| \leq \boxed{1} \leftarrow M$

$$|R_{2n+1}| = \left| \frac{f^{2n+2}(c)\, x^{2n+2}}{(2n+2)!} \right| \leq \frac{1}{(2n+2)!} |x^{2n+2}|$$

القيمة المطلقة دايماً اكبر من 0

$$0 \leq |R_{2n+1}| \leq \frac{|x^{2n+2}|}{2n+2!}$$

S.th

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} |R_{2n+1}(x)| \leq \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \right| \to 0 \text{ ch 10.1}$$

by s.th $\Rightarrow \lim_{n\to\infty} |R_{2n+1}(x)| = 0$

Hence, maclurine Series generated by $\sin x$ converges to $\sin x \;\forall x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Exp: Show that Taylor series for $\cos x$ at $x=0$ converges to $\cos x$ for every value of x

TF $f(x) = P_{2n}(x) + R_{2n}(x)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots (-1)^n \frac{x^{2n}}{2n!} + R_{2n}(x)$$

$$|R_{2n}(x)| = \frac{f^{2n+1}(c)\, x^{2n+1}}{(2n+1)!} \quad ; \quad f(x) = \cos, \quad |f^{(2n+1)}(c)| \leq 1$$

$$0 \leq |R_{2n}(x)| \leq \left| \frac{x^{2n+1}}{(2n+1)!} \right|$$

s.th

$$\lim_{n\to\infty} \overset{0}{0} \leq \lim_{n\to\infty} |R_{2n}(x)| \leq \lim_{n\to\infty} \left| \frac{x^{2n+1}}{(2n+1)!} \right| \to 0$$

ch 10.1

$$\lim_{n\to\infty} R_{2n}(x) = 0$$

Hence $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

Exp: for what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no more than 3×10^{-4} ?

TF $= P_n(x) + R_n(x)$

$$\sin x \approx \underbrace{x - \frac{x^3}{3!}}_{\text{تقريب بدال حدين}} + \frac{x^5}{5!}$$

Alternating series Estimation Theorem

$$\text{error} < \left| \frac{x^5}{5!} \right|$$

$$\text{error} < 3 \times 10^{-4}$$

$$\left| \frac{x^5}{5!} \right| < 3 \times 10^{-4}$$

$$|x^5| < 3 \times 10^{-4} \times 5!$$

$$|x| < 0.514$$

$-0.514 < x < 0.514$; $\Rightarrow$ $x = 0.1 \Rightarrow$ Estimate $\sin(0.1)$

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!}$$ — تعويض

Exp: Estimate The error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of sin x at $x = 0.1$

الدرجة

maclaurin $x=0$

TF

$f(x) = P_3(x) + R_3(x)$

$\sin x = x - \frac{x^3}{6} + \frac{f^{(4)}(c)\, x^4}{4!}$

$\sin x$

$\vdots$

$f^{(4)}(c) \leq 1$

$\text{error} = |R_3(x)| = \left| \frac{f^{4}(c)\, x^4}{4!} \right| \leq \frac{|x^4|}{4!}$

$\text{error} \leq \frac{x^4}{4!} = \frac{(0.1)^4}{4!} < 4.2 \times 10^{-6}$

نعوض قيمة x

Ch 10.10 The Binomial Series and Applications of Taylor Series.

$(1+x)^2 = (1+x)(1+x) = 1+2x+x^2$

$(1+x)^3 = (1+x)^2(1+x)$

$(1+x)^{20} = \ldots$

* The Binomial series of $f(x) = (1+x)^m$ is "using Taylor series expa"

$$(1+x)^m = 1+\sum_{k=1}^{\infty} \binom{m}{k} x^k \quad , |x| < 1 \ , \ m \text{ is constant}$$

↳ where the series conv abs

where $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2}$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for} \quad k \geq 3$$

$$(1+x)^m = 1+\sum_{k=1}^{\infty} \binom{m}{k} x^k$$

Exp: find the first four terms of the binomial series for

① $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$; $m = \frac{1}{2}$

$$= 1+\sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} x^k$$

$$= 1+\binom{\frac{1}{2}}{1}x^1 + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 \ldots$$

$\frac{1}{2}-3+1$

$m-k+1$

$$= 1+\frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3$$

$$= 1+\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

② $\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$, $m = \frac{1}{3}$

$= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{3}}{k} x^{\frac{1}{3}}$

$= 1 + \binom{\frac{1}{3}}{1} x + \binom{\frac{1}{3}}{2} x^2 + \binom{\frac{1}{3}}{3} x^3 + \cdots$

$= 1 + \frac{x}{3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} x^3 \cdots$

$= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81}$

Exp: find Poly that approximates $F(x) = \int_0^x \sin t^2 \, dt$ on $[0,1]$, with error less than 10^{-3}

Maclorine of $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \cdots$

$F(x) = \int_0^x \sin^2 t \, dt = \int_0^x \left[t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots \right] dt$

$= \frac{t^3}{3} - \frac{t^7}{(7)(3!)} + \frac{t^{11}}{(11)(5!)} - \frac{t^{15}}{(15)(7!)} + \cdots \Big|_0^x$

$= \frac{x^3}{3} - \frac{x^7}{7 \times 3!} - \frac{x^{11}}{11 \times 5!} - \frac{x^{15}}{15 \times 7!} + \cdots) - 0$

احنا احترنا بنجرب باول حد بعدها ان ما زبط بناخذ حدود اكثر

$x \in [0,1]$

$F(x) \approx \frac{x^3}{3}$ with error $< \left| \frac{x^7}{7 \times 3!} \right| < \frac{1}{7 \times 3!} \approx 0.0238 > 0.001$ × ما زبط

بناخذ اول حدين

$F(x) \approx \frac{x^3}{3} - \frac{x^7}{7 \times 3!}$ with error $< \left| \frac{x^{11}}{11 \times 5!} \right| < \frac{1}{(11)(5!)} \approx 0.0007 < 0.001$

Hence, $P_7(x) = \frac{x^3}{3} - \frac{x^7}{7 \times 3!} \approx F(x) = \int_0^x \sin t^2 \, dt$ on $[0,1]$

لو بدنا نكمل بكون الايرور افضل

$P_{11} = \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!}$ with error $< \left| \frac{x^{15}}{15 \times 7!} \right| < \frac{1}{15 \times 7!} \approx 0.000013$

Approximation Nonelementary integrals.

Exp: use series to estimate the following integrals with an error of magnitude less than 10^{-3}

1

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} = \int_0^{0.1} (1+x^4)^{-\frac{1}{2}}\, dx \qquad , m = -\frac{1}{2}$$

بنحط نفس الحد ع مهو.

$$= \int_0^{0.1} 1 + \sum_{k\geq 1}^{\infty} \binom{-\frac{1}{2}}{k} (x^4)^k$$

$$= \int_0^{0.1} 1 + \binom{-\frac{1}{2}}{1} x^4 + \binom{-\frac{1}{2}}{2} x^8 + \binom{-\frac{1}{2}}{3} x^{12} + \cdots]\, dx$$

$$= \int_0^{0.1} [1 - \frac{x^4}{2} + \frac{3}{8}x^8 + \cdots]\, dx$$

$$= x - \frac{x^5}{10} + \frac{3x^9}{8 \cdot 9} + \cdots \Big|_0^{0.1}$$

نجرب اول حد :

$$\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} \approx x\Big|_{0.1} = 0.1 \quad \text{with error } |E| \lesssim \left|\frac{x^5}{10}\right| \approx \frac{0.1^5}{10} \approx 0.000001 < 10^{-3}$$

اذا ذبط اول حد بزبط الحدود يلي بعده والإيرور بكون اقل واحس

2

$$\int_0^{0.2} \frac{e^{-x}-1}{x} \qquad\qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

بناخذ M.S

$$= \int_0^{0.2} \frac{1}{x}[\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - \cancel{1}]\, dx$$

$$= \int_0^{0.2} -1 + \frac{x}{2} - \frac{x^2}{6} + \cdots dx$$

$$= -x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{4(4!)} \Big|_0^{0.2}$$

نجرب اول حد

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx \approx -x\Big|_{x=0.2} = 0.2 \text{ with } |E| < \frac{x^2}{4} = \frac{0.2^2}{4} = 0.04 = 0.01 > 10^{-3}$$ ✗

نكمل تاني حد

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx \approx -x + \frac{x^2}{4}\Big|_{x=0.2} = -0.2 + 0.01 = -0.19 \text{ with } |E| < \frac{x^3}{18}\Big|_{x=0.2} = \frac{(0.2)^3}{18} = \frac{0.008}{18} = 0.0004 < 10^{-3}$$

approximation

Taylor series for $\ln x$ at $a = 1$

$$= (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} - \frac{(x-1)^4}{4!} \cdots$$

✱ indeterminate forms.

Exp: use series to find this limite

① $$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim \frac{(x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} \cdots}{x-1}$$

$$= \lim_{x\to 1}\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^2}{6} + \cdots\right)$$

نعوض 1 $= \lim 1 - 0 - 0 \cdots = 1$

② $$\lim \frac{e^x - (1+x)}{x^2} = \frac{0}{0}$$

ممكن نشتق بس هو بدو نستخدم series.

$$= \lim_{x\to 0}\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots - 1 - x}{x^2}$$

$$= \lim_{x\to 0}\frac{\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots}{x^2}$$

$$= \lim_{x\to 0} \frac{1}{2} + \frac{x}{6} + \frac{x^2}{4!} + \cdots \underset{\text{نعوض}}{=} \frac{1}{2}$$

③ $$\lim_{y=0}\frac{y - \tan^{-1}y}{y^3}$$ → maclurin series for $\tan^{-1}(x)$

أو بطريقة أسهل

$$\tan^{-1}x = \int f' x\,dx = \int \frac{1}{1-x^2}\,dx \to a = 1,\ r = -x^2 \Rightarrow \frac{a}{1-r} = \frac{1}{1+x^2}$$

$$= \int \left[1 - x^2 + x^4 - x^6 + \cdots\right] dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7!} \Rightarrow$$ انتبه بدون sin !

$$\lim \frac{y - \tan^{-1}y}{y^3} = \frac{0}{0}$$

$$\lim \frac{y - y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} \cdots)}{y^3}$$

$$\lim \frac{\frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} \cdots}{y^3}$$

$$\lim_{y=0} \frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} \cdots = \frac{1}{3}$$

Exp: Euler formula

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = i\, i^2 = -i$

$i^4 = i^2 i^2 = (-1)(-1) = 1$

$i^5 = i\, i^4 = i$

Ms $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$

$$e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^5}{5!} + \frac{(i\phi)^6}{6!} + \dots$$

$$= 1 + i\phi - \frac{\phi^2}{2!} - \frac{\phi^3 i}{3!} + \frac{\phi^4}{4!} + \frac{\phi^5 i}{5!}$$

$$= \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \dots\right) + i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \dots\right)$$

$$= \cos\phi + i \sin\phi \quad \text{Euler's formula}$$

حقيقي   complex

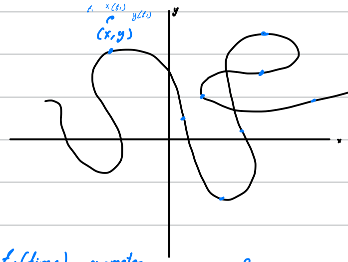
* $e^{i\pi} = -1$??

$e^{i\pi} = \cos\pi + i \sin\pi = -1$

$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

Ch 11.1 Parametrization of plane Curves



it may be a function or not function

→ Curve (parametric curve)

t : (time) aramotor. P

parameter eq. $\leftarrow \begin{cases} x = x(t) = f(t) \\ y = y(t) = g(t) \end{cases}$; $t \in I$ (parameter interval)

I

$I=[a,b]$	$I=(a,b)$	$I=[a,b)$	$I=(a,b]$
$a \leq t \leq b$	$a < t < b$	$a \leq t < b$	$a < t \leq b$
$t=a \Rightarrow (x(a), y(a))$ starting point / intial point	$\nexists$ T.P	$\exists$ I.P	$\nexists$ I.P
$t=b \Rightarrow (x(b), y(b))$ Terminal point / end point	$\nexists$ I.P	$\nexists$ T.P	$\exists$ T.P

Exp: Assume particle moves along the curve whose parametric equations are

① $x = t^2$, $y = t + 1$

find Ⓐ cartesian equation (relation between x and y)
ما في t

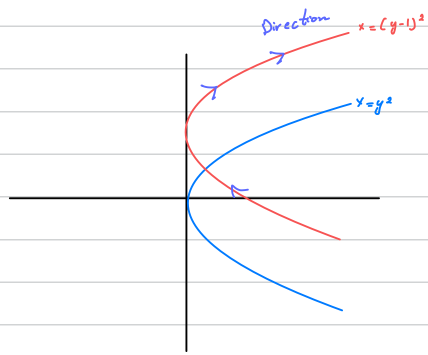
Ⓑ Sketch the curve

Ⓒ find direction

Ⓐ $x = t^2$; $t = y - 1$

$x = (y-1)^2$

Ⓑ



Ⓒ

Direction

$-\infty < t < \infty \Rightarrow \nexists$ IP

$\nexists$ TP

اخذ نقاط $t_0 = 0 \Rightarrow (x, y) = (0^2, 0+1) = (0, 1)$

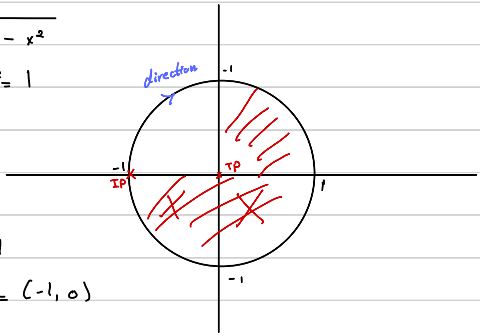
$t_1 = 1 \Rightarrow (x, y) = (1^2, 1+1) = (1, 2)$

② $x = t$, $y = \sqrt{1 - t^2}$, $-1 \leq t \leq 0$

Ⓐ curtisen eq : $x^2 = t^2$, $y = \sqrt{1 - x^2}$

Ⓑ $y = \sqrt{1 - x^2}$

$y^2 + x^2 = 1$



IP , $t = -1$

$(x, y) = (-1, 0)$

TP, $t = 0$

$(x, y) = (0, 1)$

③ Ⓐ $x = -\sec t$, $y = \tan t$; $-\frac{\pi}{2} < t < \frac{\pi}{2}$

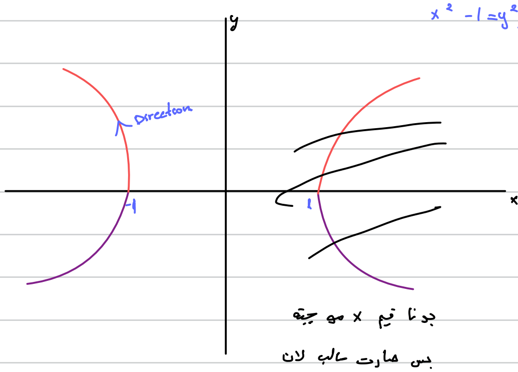
$x^2 - y^2 = 1$

$\sec^2 t - \tan^2 t = 1$

$x^2 - 1 = y^2 \longrightarrow y = -\sqrt{x^2 - 1}$

$\searrow y = \sqrt{x^2 - 1}$

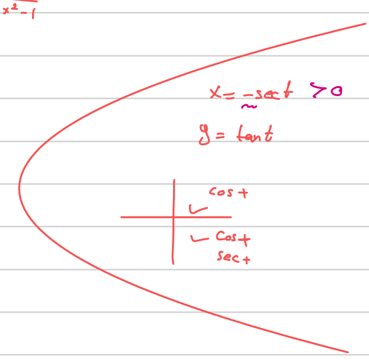
Ⓑ



بدنا قيم x موجبة

بس صارت سالب لان

$-\sec t$



$\nexists$ TP

$\exists$ TP

Ⓒ Direction

$t_0 = 0$

$(x, y) = (-\sec 0, \tan 0) = (-1, 0)$

$t_1 = \frac{\pi}{4}$

$(x, y) = (-\sec\frac{\pi}{4}, \tan\frac{\pi}{4}) = (-\sqrt{2}, 1)$

④ $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0$

Ⓐ $x + y = 2t$

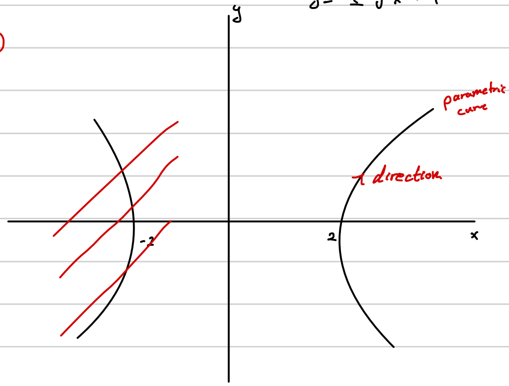
$x - y = \frac{2}{t}$

$(x-y)(x+y) = 4$

$x^2 - y^2 = 4$, $y^2 = x^2 - 4$

$y = \pm\sqrt{x^2 - 4}$

Ⓑ



$x = t + \frac{1}{t}$

$y = t - \frac{1}{t}$

$t > 0$

when $t > 0 \Rightarrow x > 0$

Ⓒ $t_0 = \frac{1}{2}$, $(x,y) \Rightarrow (2.5, -1.5)$

$t_1 = 2$, $(x,y) \Rightarrow (2.5, 1.5)$

Remark : e can write more than one parametrization for the same curve.

$$\left(\begin{array}{l} x = f(t) \\ y = g(t) \\ t \in I \end{array} \right) \begin{array}{l} \to \text{parameter eq} \\ \\ \to \text{parameter interval} \end{array}$$

Exp يلي فوق : parametrization 1 $= \{ x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0 \}$

كلهم بعطو نفس الرسمة

: parametrization 2 $= \{ x = \sqrt{4 + t}$, $y = t$, $-\infty < t < \infty \}$

: parametrization 3 $= \{ x = 2\sec t$, $y = 2\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2} \}$

لا يوجد نقطة بداية ونهاية

* parametrization line: Through the points (a,b), (c,d) (x_0, y_0)

معادلة الخط المستقيم $y - y_0 = m(x - x_0)$

$y - b = m(\underbrace{x-a}_{t})$

$m = \dfrac{d-b}{c-a}$

let $t = x - a$

$x = t + a$

$y = b + mt$

Exp: find Parametrization for

① line through $(\overset{a}{-1}, \overset{b}{3})$, $(3, -2)$

$x = t + a = t - 1$

$y = b + mt = 3 + \boxed{\frac{-5}{4}}t$, $m = \dfrac{-2-3}{3--1} = \dfrac{-5}{4}$

$-\infty < t < \infty$

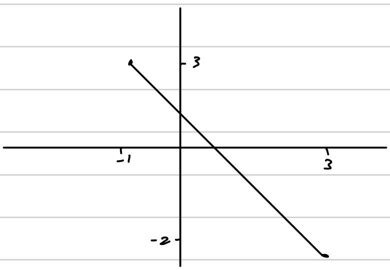
② segment with endpoint $(-1,3), (3,-2)$

TP في IP

$x = t - 1$

$y = 3 - \frac{5}{4}t$

$0 < t < 4 \implies$ x حتى نعوض t عشان نطلع



IP $t_0 = 0 \implies (x,y) = (-1,3)$

TP $t_1 = 4 \implies (x,y) = (3,-2)$

حطي اي اشي

t محل $\implies 4t$

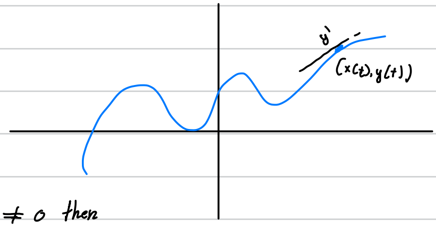
$P_1 \begin{cases} x = -1 + t \\ y = 3 - \frac{5}{4}t \\ 0 \le t \le 4 \end{cases} \implies P_2 \begin{cases} x = -1 - 4t \\ y = 3 - 5t \\ 0 \le t \le 1 \end{cases}$

ch 11.2 Calculus with Parametric curve

Parametrization

$x = f(t)$ } parametric Eq's
$y = g(t)$

$t \in I$ } parameter Interval



Assume f, g, g' are diff at t with $\frac{dx}{dt} \neq 0$ then

① $y' = \frac{dy}{dx} = \frac{\frac{dy}{\cancel{dt}}}{\frac{dx}{\cancel{dt}}}$

② $y'' = \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$

Exp: $x = 2t^2 + 3$
$y = t^4$

find:

Ⓐ slop at $t = -1$ $\qquad x_0 = 2(-1)^2 + 3 = 5$

Ⓑ tangent line at $t = -1$ $\qquad y_0 = (-1)^4 = 1$

Ⓒ $\frac{d^2y}{dx^2}$ at $t = -1$

Ⓐ slop $= \left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=-1} = \left.\frac{4t^3}{4t}\right|_{t=-1} = \frac{-4}{-4} = 1$

$y' = t^2$

Ⓑ $y - y_0 = m(x - x_0)$
$y = y_0 + 1(x - x_0)$
$y = 1 + x - 5$
$y = x - 4$

Ⓒ $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{2t}{4t} = \frac{-2}{-4} = \frac{1}{2}$

Exp: find the slop of the curve at $t=2$ whose parametric eq.

$x^3+2t^2=9$ implicit diff $\Rightarrow$ $3x^2x'+4t=0 \Rightarrow \frac{dx}{dt}=\frac{-4t}{3x^2}$

$2y^3-3t^2=4$ $\qquad 6y^2y'-6t=0 \Rightarrow y'=\frac{6t}{6y^2}=\frac{t}{y^2}$

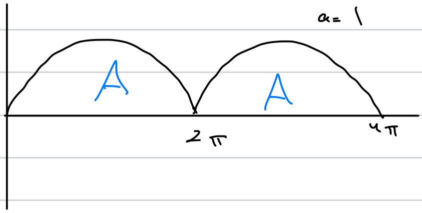
$$\text{slope}=\left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=2}=\left.\frac{\frac{t}{y^2}}{\frac{-4t}{3x^2}}\right|_{t=2}$$

$\Rightarrow$ نعوض $t=2$ $\quad x^3+2(2)^2=9 \Rightarrow x=1$

$2y^3-3(2)^2=4 \Rightarrow y=2$

$$=\frac{\frac{2}{2^2}}{\frac{-4(2)}{3\times1^2}}=\frac{-3}{16}$$

cycloid,

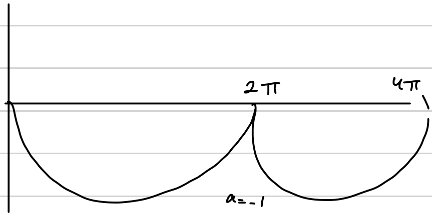


Exp: find the area under under one arc of the cycloid

$x=a(t-\sin t)$, $y=a(1-\cos t)$ when $a=1$

at $a=1 \Rightarrow x=t-\sin t$, $y=1-\cos t$

$A=\int_0^{2\pi} y\,dx$ or $A=\int_0^{2\pi} x\,dy$ ← نستخدم الأولى مثلاً

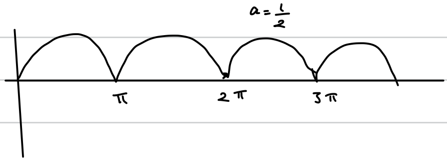


$x=t-\sin t$

$dx=(1-\cos t)dt$

$$A=\int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt$$

$$=\int_0^{2\pi}(1-2\cos t+\cos^2 t)\,dt$$



$$=\int_0^{2\pi}\left(1-2\cos t+\frac{1+\cos 2t}{2}\right)dt$$

$$=\left. t-2\sin t+\frac{1}{2}t+\frac{\sin 2t}{4}\right|_0^{2\pi}$$

$$=\left(2\pi-2\sin 2\pi+\frac{1}{2}(2\pi)+\frac{\sin 4\pi}{4}\right)-0$$

$$=2\pi+\pi=3\pi$$

The area $=2A=6\pi$

Exp: $x = a\cos t \quad , y = b\sin t \qquad , 0 \le t \le 2\pi$

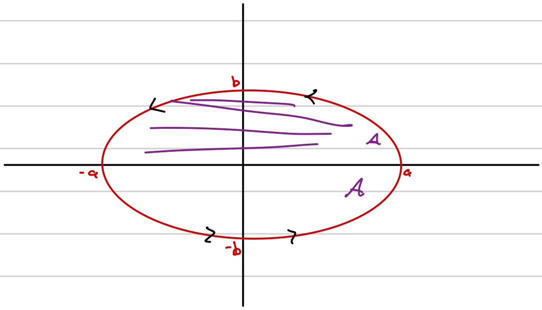
find area inside

ما عنا حدود
بالتالي بدنا نتخلص من t

we need to find Carlesion eq (relative between x, y)

$\frac{x}{a} = \cos t \qquad , dx = -a\sin t\, dt$

$\frac{y}{b} = \sin t$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow$ Ellipse

$x = 0 \Rightarrow y^2 = b^2 \Rightarrow y = \pm b$

$y = 0 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$

Area $= 2A$

$= 2\int_0^{\pi} y\, dx$

$= 2\int_0^{\pi} b\sin t\,(-a\sin t)\, dt$

$= 2ab\int_0^{\pi} \sin^2 t\, dt$

$= 2ab\int_0^{\pi} \frac{1-\cos 2t}{2}\, dt$

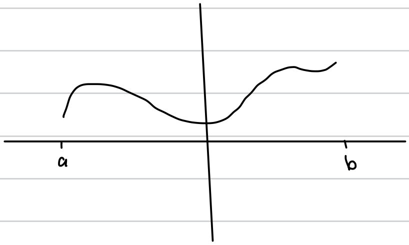
$= ab\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi}$

$= ab\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$

$= ab\pi$

Arc length with parametrization.

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \\ t \in [a,b] \end{array}\right\} \text{parametrization}$$



Assume f', g' are cont, and not zero (both)

Then length of this curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 \left[1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}\right]}\, dt$$

$$= \int_a^b \frac{dx}{\cancel{dt}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \cancel{dt}$$

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Exp: $x = \cos t$, $y = t + \sin t$, $\overset{a}{0} \le t \le \overset{b}{\pi}$

find the length of this curve.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad , \quad \frac{dx}{dt} = -\sin t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \sin^2 t$$

$$\frac{dy}{dt} = 1 + \cos t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 1 + 2\cos t + \cos^2 t$$

$$= \int_0^\pi \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}\, dt$$

$$= \int_0^\pi \sqrt{2 + 2\cos t}\, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{1 + \cos t}\, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{1 + \cos t \cdot \frac{1 - \cos t}{1 - \cos t}}$$

$$= \sqrt{2} \int_0^\pi \sqrt{\frac{1 - \cos^2 t}{1 - \cos t}}\, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{\frac{\sin^2 t}{1 - \cos t}}\, dt$$

موجب بدون قيمة مطلقة

$$= \sqrt{2} \int_0^\pi \frac{\sin t}{\sqrt{1 - \cos t}} \quad , u = 1 - \cos t \quad , du = \sin t\, dt$$

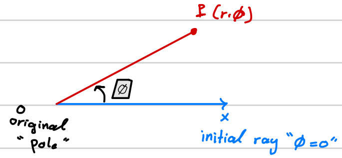
$$= \sqrt{2} \int_0^2 \frac{du}{\sqrt{u}} \qquad t = 0 \Rightarrow u = 0 \qquad t = \pi \Rightarrow u = 2$$

$$= \sqrt{2}\, u^{\frac{1}{2}} \times 2 \Big|_0^2$$

$$= 4$$

11.3 Polar Coordinates

$P(r, \emptyset)$ is polar coordinate



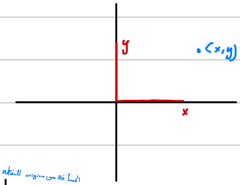
- r is the directed distance from O to P
  "r can be negative"

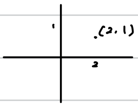
- $\emptyset$ is directed angle from the initial ray to OP.

Coordinates

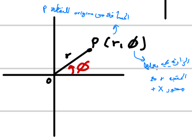
↙ Cartesian coordinate (x, y) — are uniquely represented ←

↘ polar coordinate $(r, \emptyset)$ — is not unique →

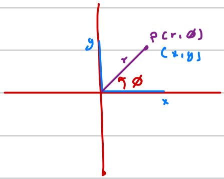




النقطة وحيدة مش غيرها



* التحويل بينهم



التحويل $\Longrightarrow$

$\cos \emptyset = \frac{x}{r} \implies x = r\cos\emptyset$

$\sin \emptyset = \frac{y}{r} \implies y = r\sin\emptyset$

$x^2 + y^2 = r^2$

$\tan \emptyset = \frac{y}{x}$

} العلاقات المهمة

---

* polar coordinate are not unique:

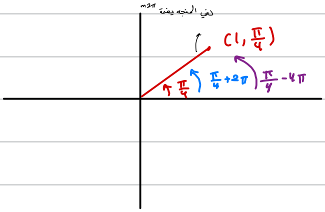
$(r, \emptyset) = (1, \frac{\pi}{4})$

$= (1, \frac{\pi}{4} + 2\pi)$

$= (1, \frac{\pi}{4} - 4\pi)$

$\vdots$

كلهم نفس مكان النقطة



Polar coordinate

$(r, \emptyset) = (r, \emptyset + 2\pi m)$

(same direction)

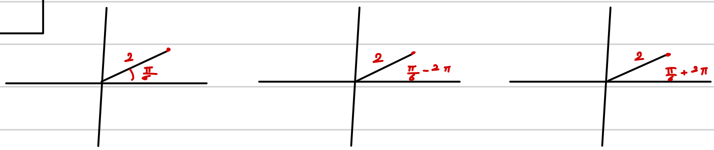
Exp:

> Polar coordinate
>
> $(r, \phi) = (r, \phi + 2\pi m)$
>
> (same direction)

اذا بدي الف زوجي
بحافظ على الاتجاه

$(2, \frac{\pi}{6}) = (2, \frac{\pi}{6} - 2\pi) = (2, -\frac{11\pi}{6})$

$= (2, \frac{\pi}{6} + 2\pi)$



Exp:

> Polar coordinate
>
> $(r, \phi) = (-r, \phi + \pi + 2\pi m)$
>
> (direction reverse)

اذا بدي الف فردي بعكس الاتجاه

بدنا نمشي فردي

$P\ (3, 0) \Rightarrow (-3, \pi)$ عكسنا الاتجاه الموجب

$P\ (-3, 0) \Rightarrow (-3, 2\pi)$ حافظنا على الاتجاه السالب

$P\ (2, \frac{2\pi}{3}) \Rightarrow (-2, \frac{5\pi}{3})$ عكسنا الاتجاه الموجب

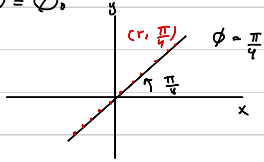
---

* what is the meaning of: $\phi = \phi_0$

Exp: $\phi = \frac{\pi}{4}$ (line)

r changes

اي نقطة على line تمثل $(r, \frac{\pi}{4})$



$\phi = \phi_0 \Rightarrow$ line maks angle of ϕ_0 with $x+$

* what is the meaning of: $r = r_0$

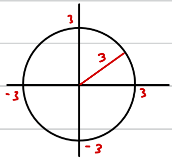
$r = 2$

$x^2 + y^2 = 4 \Rightarrow$ circle with radius 2 and center origin

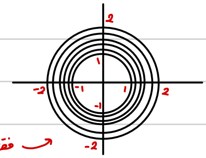
$r = r_0 \Rightarrow$ Circle with radius $|r_0|$ and center origin

Examples :

$r = 3$

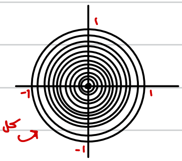


$1 \leq r \leq 2$



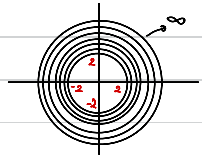
فقط منطقة الدوائر

$0 < r \leq 1$

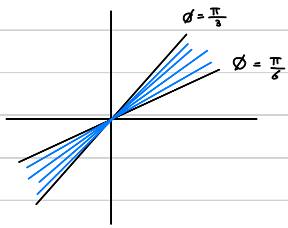


كل المنطقة

$r \geq 2$

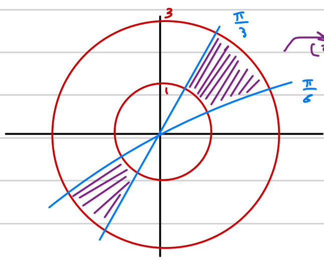


$\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$



$1 \leq r \leq 3$

and $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$



اي نقطة بالمنطقة
المشتركة (البنفسجية)
تحقق الشروط

Exps:

$x = r\cos\phi \quad ; \quad \frac{y}{x} = \tan\phi$

$y = r\sin\phi$

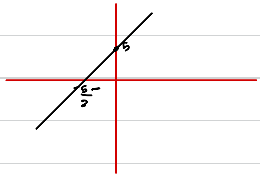
$x^2 + y^2 = r^2$

① $r = \dfrac{5}{\sin\phi - 2\cos\phi}$ Find the CE $\leftarrow$ Cartision eq.

$r\sin\phi - 2r\cos\phi = 5$

$y - 2x = 5$

$y = 2x + 5$



② $xy = 2$ find PE

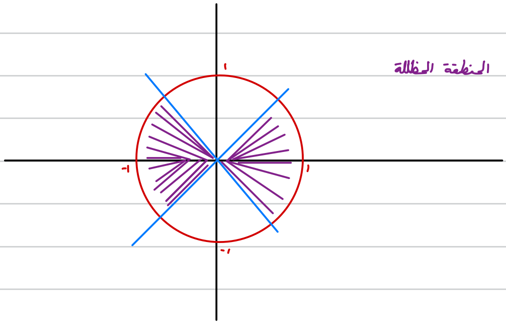
$r\cos\phi \cdot r\sin\phi = 2$

$r^2 \cos\phi \sin\phi = 2$ ✓

$r^2\, 2\cos\phi\sin\phi = 4$ ✓

$r^2 \sin 2\phi = 4$ ✓

③ $-\frac{\pi}{4} \leq \phi \leq \frac{\pi}{4}$, $-1 \leq r \leq 1$ graph:

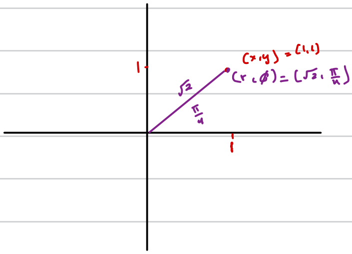


④ $(r, \phi) = (\sqrt{2}, \frac{\pi}{4})$ find (x, y)

$x = r\cos\phi = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$

$y = r\sin\phi = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$

$(r, \phi) = (\sqrt{2}, \frac{\pi}{4}) = (x, y) = (1, 1)$



5. $(x, y) = (-3, 0)$ find (r, ϕ) عدد لانهائي ; $0 \le \phi \le 2\pi$, $r \ge 0$

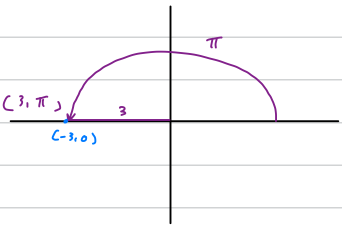
* $r^2 = x^2 + y^2$

$r^2 = (-3)^2 + 0^2$

$r^3 = 9$

$|r| = 3$

$r = \pm 3 \xRightarrow{\text{بناء على الشرط}} r = 3$



* $x = r\cos\phi$ $y = r\sin\phi$

$-3 = 3\cos\phi$ $0 = 3\sin\phi$

$-1 = \cos\phi$ $0, \sin\phi$

$\phi = \pi \in [0, 2\pi]$

$(r, \phi) = (3, \pi)$

6. find CE and sketch $r^2 \sin 2\phi = 2$

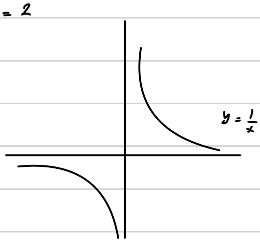
$r^2 \sin 2\phi = 2$

$r^2\, 2\sin\phi\cos\phi = 2$

$r\sin\phi \; r\cos^2 = 1$

$xy = 1$

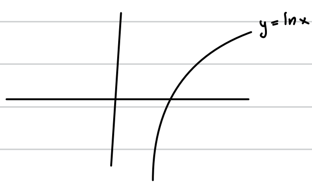
$y = \frac{1}{x}$



7. $r\sin\phi = \ln r + \ln\cos\phi$

$r\sin\phi = \ln r\cos\phi$

$y = \ln x$



8. $(x-5)^2 + y^2 = 25$ find PE

$x^2 - 10x + 25 + y^2 = 25$

$x^2 + y^2 = 10x$

$r^2 = 10 r\cos\phi$

$r = 10\cos\phi$ في حال $0 \ne r$

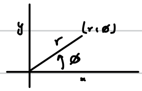
11.4 Graphing in polar coordinate

$y=f(x) \Longrightarrow$ slope : $\left.\frac{dy}{dx}\right|_{x_0,y_0}$

$r=f(\emptyset)$

$x=r\cos\emptyset \quad , \quad y=r\sin\emptyset$

$=f(\emptyset)\cos\emptyset \quad , \quad =f(\emptyset)\sin\emptyset$



$$\left.\frac{dy}{dx}\right|_{(r,\emptyset)} = \frac{\frac{dy}{d\emptyset}}{\frac{dx}{d\emptyset}} = \frac{r\cos\emptyset + \sin\emptyset\, r'}{-r\sin\emptyset + \cos\emptyset\, r'} = \frac{r'\sin\emptyset + r\cos\emptyset}{r'\cos\emptyset - r\sin\emptyset} \quad \checkmark$$

Exp: find slope of $r=\cos 2\emptyset$ at $\emptyset=0$

$\hookrightarrow r' = -2\sin 2\emptyset$

$r=\cos(2\cdot 0)=\cos 0=1 \Rightarrow (r,\emptyset)=(1,0)$

$$\frac{dy}{dx} = \left.\frac{r'\sin\emptyset + r\cos\emptyset}{r'\cos\emptyset - r\sin\emptyset}\right|_{(1,0)} = \left.\frac{-2\sin 2\emptyset\,\sin\emptyset + r\cos\emptyset}{-2\sin 2\emptyset\,\cos\emptyset - r\sin\emptyset}\right|_{(1,0)}$$

$$= \frac{0 + (1)\cos 0}{0 - (1)(0)} = \frac{1}{0} \text{ undefined} \Rightarrow$$ المماس عند النقطة (1,0) سيكون عمودي

Exp: find slope of $r=\cos 2\emptyset$ at $\emptyset=\frac{\pi}{2}$

$\hookrightarrow r' = -2\sin 2\emptyset$

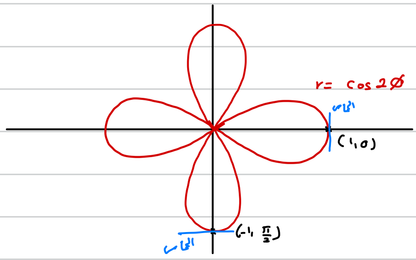
$r=\cos(2\cdot\frac{\pi}{2})=\cos o=\pi \Rightarrow (r,\emptyset)=(-1,\frac{\pi}{2})$

$$\frac{dy}{dx} = \left.\frac{r'\sin\emptyset + r\cos\emptyset}{r'\cos\emptyset - r\sin\emptyset}\right|_{(-1,\frac{\pi}{2})} = \left.\frac{-2\sin 2\emptyset\,\sin\emptyset + r\cos\emptyset}{-2\sin 2\emptyset\,\cos\emptyset - r\sin\emptyset}\right|_{(-1,\frac{\pi}{2})}$$

$$= \frac{-\sin\pi\sin\frac{\pi}{2} + (-1)\cos\frac{\pi}{2}}{-\sin\pi\cos\frac{\pi}{2} - (-1)\sin\frac{\pi}{2}} = \frac{0}{1}$$

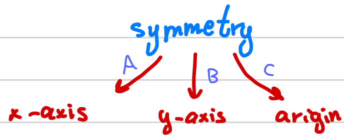
المماس عند النقطة $(-1,\frac{\pi}{2})$ سيكون أفقي

الرسمة



★ To draw $r = f(\emptyset)$ ⟹ it's important to know the symmetry

★ If $r=f(\emptyset)$ is symmetric about A and B so its symmetric about C
If $r=f(\emptyset)$ is symmetric about A and C so its symmetric about B
If $r=f(\emptyset)$ is symmetric about C and B so its symmetric about A

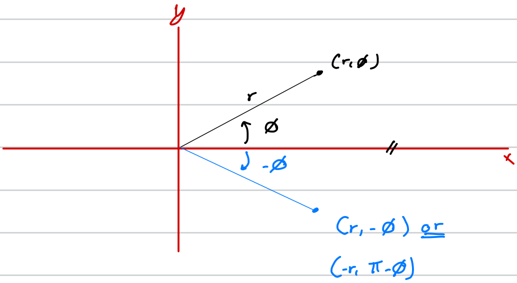


★ If $r = f(\emptyset)$ is not symmetric about A and B so its not symmetric about C
If $r = f(\emptyset)$ is not symmetric about A and C so its not symmetric about B
If $r = f(\emptyset)$ is not symmetric about C and B so its not symmetric about A

A : x - axis

if $(r, \emptyset)$ on the graph

then $(r, -\emptyset)$ or $(-r, \pi-\emptyset)$ on the graph

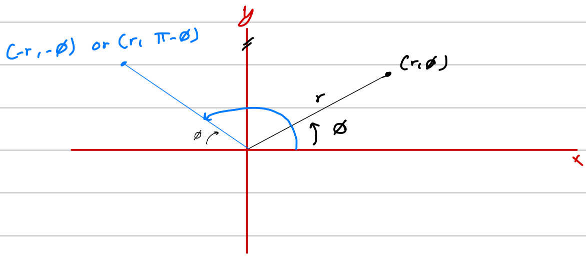


B : y - axis

if $(r, \emptyset)$ on the graph

Then $(r, \pi-\emptyset)$ or $(-r, -\emptyset)$

on the graph

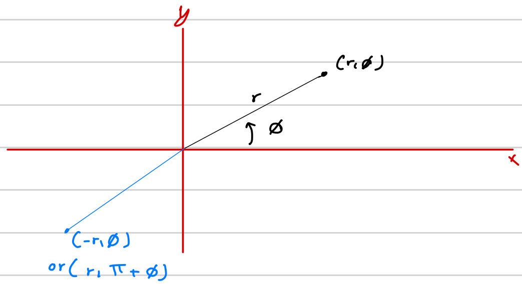


C : origin

if $(r, \emptyset)$ on the graph

Then $(-r, \emptyset)$ or $(-r, \pi+\emptyset)$

on the graph



Exp: sketch

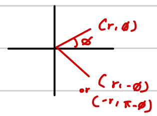
$r = 1 + \sin\theta$ ----> identify the summetry

$\sin(A+B) = \sin A\cos B + \sin B\cos A$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

① x-axis

بدنا نشوف في
تماثل حول x
او لا



تعويض

substitute $(-\theta) \Rightarrow 1 + \sin(-\theta)$

$= 1 - \sin\theta \neq r$ ما زبطت النقطة الأولى

نجرب الثانية

تعويض

substitute $(\pi - \theta) \Rightarrow 1 + \sin(\pi - \theta)$ $A = \pi$, $B = -\theta$

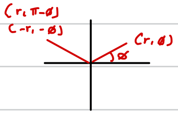
$= 1 + \sin\pi\cos(-\theta) + \sin(-\theta)\cos\pi$

$= 1 + 0 + 1 . \sin\theta$

$= 1 + \sin\theta \neq -r$ ما زبطت

① y-axis

بدنا نشوف في
تماثل حول y
او لا



تعويض

substitute $(-\theta) \Rightarrow 1 + \sin(-\theta)$

$= 1 - \sin\theta \neq -r$ ما زبطت النقطة الأولى

نجرب الثانية

تعويض

substitute $(\pi - \theta) \Rightarrow 1 + \sin(\pi - \theta)$ $A = \pi$, $B = -\theta$

$= 1 + \sin\pi\cos(-\theta) + \sin(-\theta)\cos\pi$

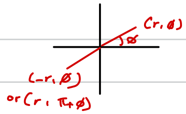
$= 1 + 0 + 1 . \sin\theta$

$= 1 + \sin\theta = r$ زبطت

Hence, $r = 1 + \sin\theta$ is symmetric about y-axis

① y-axis

بدنا نشوف في
تماثل حول y
او لا



تعويض

substitute $(\theta) \Rightarrow 1 + \sin\theta = r \neq -r$

ما زبطت النقطة الأولى

نجرب الثانية

تعويض

substitute $(\pi + \theta) \Rightarrow 1 + \sin(\pi + \theta)$ $A = \pi$, $B = -\theta$

$= 1 + \sin\pi\cos(\theta) + \sin(\theta)\cos\pi$

$= 1 + 0 + 1 . \sin\theta$

$= 1 - \sin\theta \neq r$ ما زبطت

Hence, $r = 1 + \sin\theta$ is symmetric about y-axis

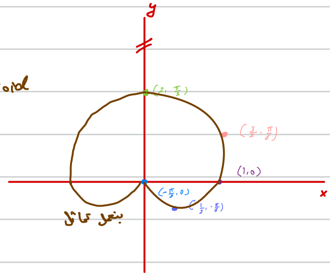
$r = 1 + \sin\phi$

بنسوي جدول

بناخد زوايا في ربعين تحت بعض مثلاً الأول والرابع

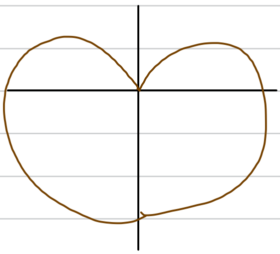
ϕ	$r\ (1+\sin\phi)$
$-\frac{\pi}{2}$	0
$-\frac{\pi}{6}$	$\frac{1}{2}$
0	1
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{2}$	2

Cardioid



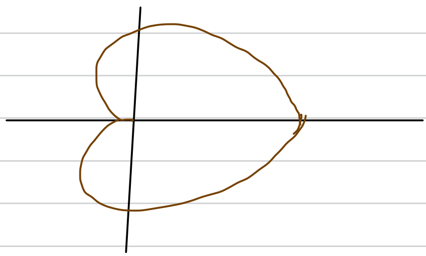
بكفي نرسم في ربعين تحت بعض بعدين نعكسها حول y

$r = 1 - \sin\phi$

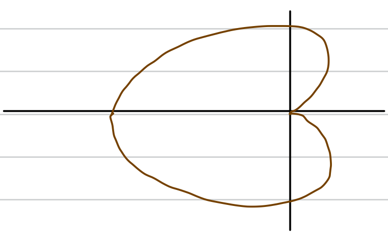


Cardiod

$r = 1 + \cos\phi$



$r = 1 - \cos\phi$



بنرسم بنفس الطريقة


Wishing you
the best of luck
in your exam!