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MATH1321

Calculus 2

Chapter 8.7

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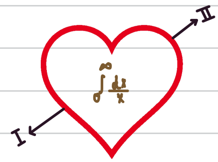
8.7 Improper Integrals

Type I: $\int_a^{\infty} f$, $\int_{-\infty}^{1} f$, $\int_{-\infty}^{\infty} f$

Type II (discontinuity): $\int_0^1 \frac{dx}{x}$ $\int_0^3 \frac{dx}{x-3}$ $\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$



Type I+II $\int_0^{\infty} \frac{dx}{x} = \int_0^1 \frac{dx}{x} + \int_1^{\infty} \frac{dx}{x}$

✱Type I✱ f cont. $[a, b]$. (How to find the improper integrals?)

1) $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$

2) $\int_{-\infty}^{b} f(x)\,dx = \lim_{c\to-\infty} \int_c^b f(x)\,dx.$

3) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$

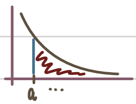
$= \lim_{c\to-\infty} \int_c^a f(x)\,dx + \lim_{b\to\infty} \int_a^b f(x)\,dx.$

Remark 1: 1) If limit exists (الجواب رقم), then we say Improper integral converges to this number.

2) If limit exists $\begin{cases} \to \infty \\ \to -\infty \end{cases}$, then we say Improper integral diverges.

Remark 2: If $f(x) \geq 0$ and $\int_a^{\infty} f(x)\,dx$ converges to the number $L \geq 0$.

The L represents the area under f.

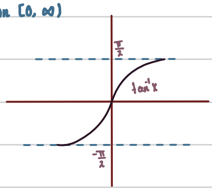


Ex. $\int_0^{\infty} \frac{dx}{x^2+1}$, $f(x) = \frac{1}{x^2+1}$, $f(x) \geq 0$ on $[0, \infty)$

$= \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}\Big|_0^b$

$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2}.$

$\int_0^{\infty} \frac{dx}{x^2+1}$ converges to $\frac{\pi}{2}$.



Ex. $\int_{-\infty}^{0} \frac{dx}{x^2+1} = \frac{\pi}{2}$ (converges to $\frac{\pi}{2}$)

Ex. $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$ (ابشر أعوض أي رقم)

Ex. $\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b$

$= \infty - \ln 1 = \infty$ (div. to ∞).

Ex. $\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx$ $x = \pm 1 \notin (-\infty, -2]$

$= \lim_{b\to-\infty} \int_b^{-2} \frac{2}{x^2-1}\,dx$ only Type I.

we solve the integration first.

$\int_b^{-2} \frac{2}{x^2-1}\,dx$ $\frac{A}{x-1} + \frac{B}{x+1}$ ✱cover method✱

$A = 1$ / $B = -1$.

$= \ln\left|\frac{x-1}{x+1}\right|\Big|_b^{-2}$

$= \ln 3 - \ln \frac{b-1}{b+1}$

$= \lim_{b\to-\infty} \ln 3 - \ln \frac{b-1}{b+1} = \ln 3 - \ln 1 = \ln 3.$

(The integral converges to $\ln 3$).

✶ Type II ✶

Improper Integrals of type II are integrals of functions that become infinite at a point within the interval of integration (Vertical Asymptote).

1) If f is discont. at a, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$.

2) If f is discont. at b, then $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.

3) If f is discont. at c, when $a < c < b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Ex. $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{c \to 4^-} 2\int_0^c \frac{dx}{2\sqrt{4-x}}$

$= \lim_{c \to 4^-} -2\sqrt{4-x}\,\Big|_0^c = \lim_{c \to 4^-} -2(\sqrt{4-c} - 2)$

$= 4$. (converges to 4).

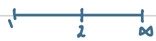
Ex. $\int_0^1 \frac{\theta + 1}{\sqrt{\theta^2 + 2\theta}}\,d\theta$ $\qquad \theta^2 + 2\theta = 0 \longrightarrow \theta = \underline{0}, -2$

$= \int \frac{du}{2\sqrt{u}}$ $\qquad u = \theta^2 + 2\theta$

$= \lim_{a \to 0^+} \sqrt{\theta^2 + 2\theta}\,\Big|_a^1$ $\qquad \frac{1}{2}du = (\theta + 1)\,d\theta$

$= \sqrt{3}$. (converges to $\sqrt{3}$).

Ex. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$ (type I + II)



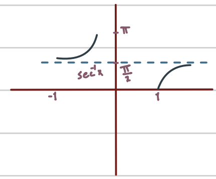
$= \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{c \to 1^+} \int_c^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{b \to \infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{c \to 1^+} \sec^{-1}x\,\Big|_c^2 + \lim_{b \to \infty} \sec^{-1}x\,\Big|_2^b$

$= \lim_{c \to 1^+} \sec^{-1}c + \lim_{b \to \infty} \sec^{-1}b$

$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$. (converges to $\frac{\pi}{2}$).



Ex. $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b \to \infty} \int_0^b \frac{16\tan^{-1}x}{1+x^2}\,dx$ $\qquad u = \tan^{-1}x$

$= \lim_{b \to \infty} 8(\tan^{-1}x)^2\,\Big|_0^b$ $\qquad 16\,du = 16 \cdot \frac{1}{1+x^2}\,dx$

$= 8\left(\left(\frac{\pi}{2}\right)^2 - 0\right) = 2\pi^2$. (converges to $2\pi^2$).

Remark 3: $\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases} = \begin{cases} \text{conv.} & \text{if } p > 1 \\ \text{Div.} & \text{if } p \leq 1 \end{cases}$

$\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases} = \begin{cases} \text{conv.} & \text{if } p > 1 \\ \text{Div.} & \text{if } p \leq 1 \end{cases}$

Ex. $\int_1^\infty \frac{dx}{x^3} = \frac{1}{3-1} = \frac{1}{2}$. (converges to $\frac{1}{2}$).

Ex. $\int_1^\infty \frac{dx}{x^{2/3}} = \infty$. (diverges).

Ex. $\int_1^\infty \frac{dx}{x^2} = 1$. (converges to 1).

Ex. $\int_1^\infty \frac{dx}{\sqrt{x}} = \infty$. (diverges).

Ex. $\int_0^1 \frac{dx}{\sqrt{x}} = \frac{1}{1-\frac{1}{2}} = 2$. (converges to 2).

Ex. How can we test an Improper Integral without solving the limit ?

We use two tests → 1) DCT: Direct Comparison Test.

2) LCT: Limit Comparison Test.

$\int_a^{\infty} f(x)\,dx$ → هدفي أعرف < converge. / diverge.

Let $f(x), g(x)$ are cont. and positive on $[a, \infty)$.

1) DCT: Assume $f(x) \leq g(x)$ on $[a, \infty)$

If $\int_a^{\infty} g(x)\,dx$ con. then $\int_a^{\infty} f(x)\,dx$ is conv.

الكبير ... الصغير

Assume $g(x) \leq f(x)$ on $[a, \infty)$

If $\int_a^{\infty} g(x)\,dx$ div. then $\int_a^{\infty} f(x)\,dx$ div.

الصغير ... الكبير

Ex. Check convergence or divergence for ?

1) $\int_1^{\infty} \frac{\cos x}{x^3}\,dx$ $\qquad -1 \leq \cos x \leq 1$

$\int_1^{\infty} \frac{1}{x^3}\,dx$ $\qquad \frac{\cos x}{x^3} \leq \frac{1}{x^3}$

= convergence to $\frac{1}{2}$.

so $\int_1^{\infty} \frac{\cos x}{x^3}\,dx$ is conv. by DCT.

2) $\int_0^{1} \frac{dx}{\sqrt{x+\sin x}}$ $\qquad x+\sin x \geq x$

$\sqrt{x+\sin x} \geq \sqrt{x}$

$\int_0^{1} \frac{1}{\sqrt{x}}\,dx$ $\qquad \frac{1}{\sqrt{x+\sin x}} \leq \frac{1}{\sqrt{x}}$

= converges to 2.

so $\int_0^{1} \frac{dx}{\sqrt{x+\sin x}}$ is conv. by DCT.

3) $\int_1^{\infty} \frac{dx}{\sqrt{x^2-0.1}}$ $\qquad x^2-0.1 < x^2$

$\sqrt{x^2-0.1} < \sqrt{x^2} = |x| = x$

$\int_1^{\infty} \frac{1}{x}\,dx$ $\qquad \frac{1}{\sqrt{x^2-0.1}} > \frac{1}{x}$

= diverges.

so $\int_1^{\infty} \frac{dx}{\sqrt{x^2-0.1}}$ diverges by DCT.

4) $\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

$\sqrt{t}+\sin t \geq \sqrt{t} \rightarrow \frac{1}{\sqrt{t}+\sin t} \leq \frac{1}{\sqrt{t}}$

$\int_0^{\pi} \frac{1}{\sqrt{t}}\,dt = \lim_{c\to 0^+} 2\int_c^{\pi} \frac{1}{2\sqrt{t}}\,dt = 2\sqrt{\pi}$

= converges to $2\sqrt{\pi}$.

so $\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$ is converge by DCT.

5) $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ $\qquad -1 \leq \sin x \leq 1$

$\int_1^{\infty} \frac{1}{x^2} = 1$ $\qquad 0 \leq \sin^2 x \leq 1$

= converge to 1. $\qquad \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$

so $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converge by DCT.

2) LCT:

find $g(x)$ and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$

then $\int_a^\infty f$ and $\int_a^\infty g$ are both conv. or div.

Ex. Use LCT to check con. or div. of ?

1) $\int_1^\infty \frac{dx}{1+x^3} \longrightarrow f(x) = \frac{1}{1+x^3}$, $g(x) = \frac{1}{x^3}$

$\int_1^\infty \frac{1}{x^3}\,dx$ is con. to $\frac{1}{2}$.

$\longrightarrow \lim\limits_{x\to\infty} \frac{f}{g} = \lim\limits_{x\to\infty} \frac{x^3}{1+x^3} = 1$

so $\int_1^\infty \frac{dx}{1+x^3}$ conv. by LCT.

2) $\int_2^\infty \frac{dx}{\sqrt{x-1}} \longrightarrow f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = \frac{1}{\sqrt{x}}$

$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim\limits_{b\to\infty} 2\sqrt{x}\Big|_2^b = \infty$

so it is diverge.

$\longrightarrow \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x-1}} = \sqrt{\lim\limits_{x\to\infty} \frac{x}{x-1}}$

$= \sqrt{1} = 1.$

so $\int_2^\infty \frac{dx}{\sqrt{x-1}}$ div by LCT.

Ex. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}} = \int_{-\infty}^0 \frac{dx}{\sqrt{x^4+1}} + \int_0^\infty \frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$

لأنه اقتران زوجي

$\frac{1}{\sqrt{x^4+1}} \leq \frac{1}{\sqrt{x^4}}$ عنده مشكلة عند الصفر

$\int_0^\infty \frac{1}{\sqrt{x^4}}\,dx = \int_0^1 \frac{1}{\sqrt{x^4}}\,dx + \int_1^\infty \frac{1}{\sqrt{x^4}}\,dx$

conv. conv. by Remark 3.

$\longrightarrow \int_0^\infty \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{\sqrt{x^4+1}}$

متق + متق = متق

so $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$ converge by DCT.