11.7 Conics in PoLar Coordinates

Note Title

Eccentricity

الإختلاف المركزي

DEFINITION

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is

$$e = \boxed{\frac{c}{a}} = \frac{\sqrt{a^2 - b^2}}{a}. = \boxed{\frac{\text{distance between foci}}{\text{distance between vertices}}}$$

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \boxed{\frac{c}{a}} = \frac{\sqrt{a^2 + b^2}}{a}. = \boxed{\frac{\text{distance between foci}}{\text{distance between vertices}}}$$

The **eccentricity** of a parabola is $e = 1$.

Thrm: A conic section is

(a) a *parabola* if $e = 1$,

(b) an *ellipse* of eccentricity e if $e < 1$, and

(c) a *hyperbola* of eccentricity e if $e > 1$.

Example 1. Find the eccentricity of the ellipse $4x^2 + 9y^2 = 36$.

Solution: $\frac{x^2}{9} + \frac{y^2}{4} = 1 \implies a = 3, \quad b = 2$

Hence $c = \sqrt{9-4} = \sqrt{5}$.

$\therefore \quad e = c/a = \boxed{\sqrt{5}/3}$ (Note that $e < 1$).

Example 2. Find the eccentricity of the hyperbola $25y^2 - 16x^2 = 400$.

Sol: $\frac{y^2}{16} - \frac{x^2}{25} = 1 \implies a = 4, \quad b = 5 \implies$

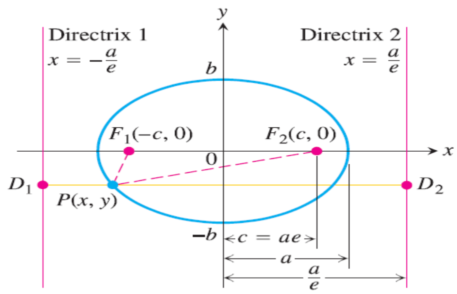
$c = \sqrt{16+25} = \sqrt{41} \implies e = \frac{c}{a} = \boxed{\frac{\sqrt{41}}{4}}$ (> 1)

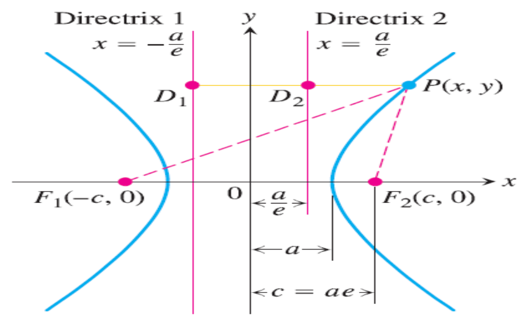
The Focus - Directrix Equation

في البداية افرضنا أنه $P(x,y)$ هي نقطة تقع على قطع مكافئ بؤرته F ودليله D ، لذا فإنه من تعريف القطع المكافئ نحصل على العلاقة $PF = PD$.
لاحظ أنه $e=1$ هنا ، وبالتالي ستكون العلاقة التالية صحيحة .

$$PF = e \cdot PD$$

السؤال ، هل يوجد دليل لكل بؤرة في بقية القطوع المخروطية تحقق نفس العلاقة $PF = e \cdot PD$ ؟
سنجد أنه كل بؤرة لها دليل ويبعد مسافة $\frac{a}{e}$ من المركز .





في الحالتين ، لاحظنا أنه $PF_1 = e\, PD_1$ وأنه $PF_2 = e\, PD_2$ وبالتالي نحصل على النتيجة التالية :

لجميع القطوع المخروطية ، تكون العلاقة التالية صحيحة :

$$PF = e \cdot PD$$

حيث e هي eccentricity ، P هي نقطة على المنحنى ، D هو دليل
و F هي بؤرة هذا الدليل ،

Definition. (Directrices)

1. The directrices of an ellipse or a hyperbola with foci on the x-axis and center $(0,0)$ are

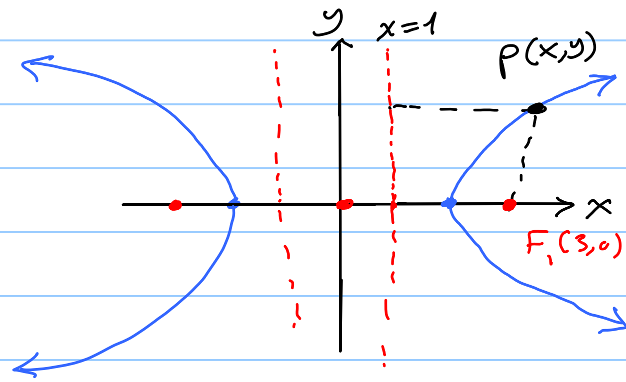
$$x = \pm\frac{a}{e} = \pm\frac{a^2}{c}.$$

2. The directrices of an ellipse or a hyperbola with foci on the y-axis and center $(0,0)$ are

$$y = \pm\frac{a}{e} = \pm\frac{a^2}{c}.$$

لاحظ أنه المسافة بين المركز والدليل تساوي $\frac{a}{e}$ وبالتالي عند وجود إزاحات تراعى معادلات الدليل .

EXAMPLE 1 Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line $x = 1$ as the corresponding directrix.



Sol: حل 1

Center-to-focus distance $c = 3$

Center-to-vertex distance $a = ??$

Center-to-Directrix distance $\frac{a}{e} = 1$

$\Rightarrow \quad a = e$

$\Rightarrow \quad e = \frac{c}{a} \Rightarrow a = \frac{c}{a}$

$\Rightarrow \quad a^2 = c = 3 \Rightarrow a = \sqrt{3}$

$\Rightarrow \quad b^2 = c^2 - a^2 = 9 - 3 = 6$

focal axis: $\boxed{y = 0}$

Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \boxed{\frac{x^2}{3} - \frac{y^2}{6} = 1}$

حل 2 $\quad c = 3, \quad \frac{a}{e} = 1 \Rightarrow a = e$

$e = \frac{c}{a} = \frac{c}{e} \Rightarrow e^2 = c = 3 \Rightarrow e = \sqrt{3}$

افرض أن $P(x,y)$ نقطة على القطع الزائد / لذا فإن

$PF_1 = e \cdot PD_1 \Rightarrow \sqrt{(x-3)^2 + y^2} = \sqrt{3}\,|x-1|$

$\Rightarrow \quad x^2 - 6x + 9 + y^2 = 3(x^2 - 2x + 1) = 3x^2 - 6x + 3$

$2x^2 - y^2 = 6 \Rightarrow \boxed{\frac{x^2}{3} - \frac{y^2}{6} = 1}$

(2) Consider the ellipse centered at the origin whose focus is $(-3, 0)$ with corresponding directrix $x = -5$. Find the eccentricity of the ellipse and its standard-form equation.

sol: $c = 3$ and $\frac{a}{e} = 5$. Moreover $e = \frac{c}{a} = \frac{3}{a} \Rightarrow a = \frac{3}{e}$.

$\Rightarrow 5e = a = \frac{3}{e} \Rightarrow e^2 = \frac{3}{5} \Rightarrow e = \sqrt{3/5}$.

Suppose that $p(x,y)$ is a point on the ellipse. Then $PF = e\,PD$

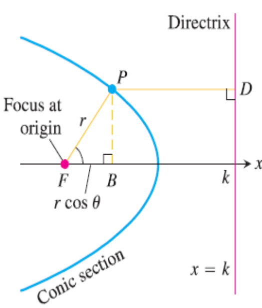
$$(PF)^2 = e^2 (PD)^2 \Rightarrow (x+3)^2 + y^2 = \frac{3}{5}(x+5)^2 \Rightarrow$$

$$5(x^2 + 6x + 9 + y^2) = 3(x^2 + 10x + 25)$$

$\Rightarrow$ $2x^2 + 5y^2 = 30$. Therefore $\boxed{\frac{x}{15} + \frac{y^2}{6} = 1}$

Polar Equations

إذا كان لدينا قطع مخروطي (مكافئ أو زائد أو ناقص) وكانت
إحدى البؤر تقع على نقطة الأصل ، فإن المعادلة القطبية
لهذا القطع المخروطي تأخذ الشكل التالي :

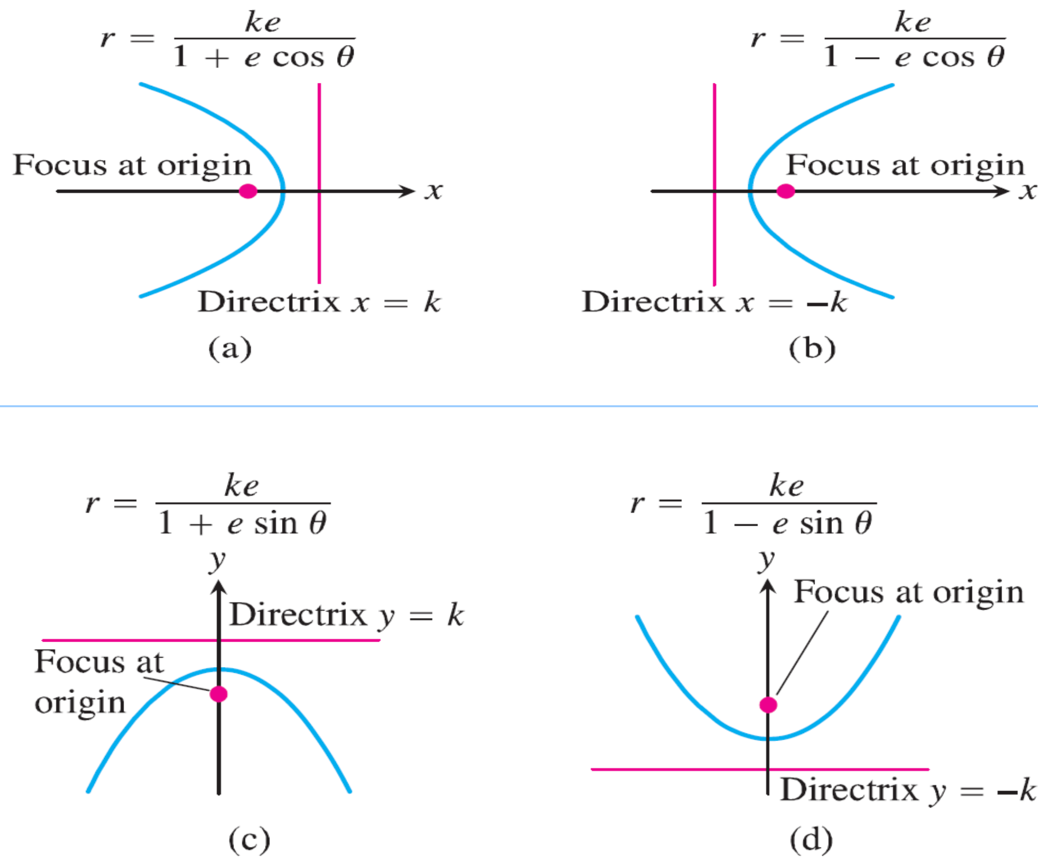


Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e\cos\theta}, \quad ---- (*)$$

where $x = k > 0$ is the vertical directrix.

حيث تكون e هي الإختلاف المركزي eccentricity ، ويكون الدليل على الشق
الموجب لمحور x ومعادلته $x = k$ (انظر الرسمة) . وهنا يجب التنويه
أنه إذا كان دليل البؤرة التي تقع على نقطة الأصل على الشق السالب لمحور x ،
فإن المعادلة الجديدة لها نفس المعادلة (*) بعد تغيير إشارة + في المقام إلى - وتأخذ
$r = \frac{ke}{1 - e\cos\theta}$. أما إذا كان الدليل على الشق الموجب لمحور y نغير $\cos\theta$
في المقام لـ $\sin\theta$ وتأخذ المعادلة الشكل التالي $r = \frac{ke}{1 + e\sin\theta}$.



(a) (b) (c) (d)

Illustration:

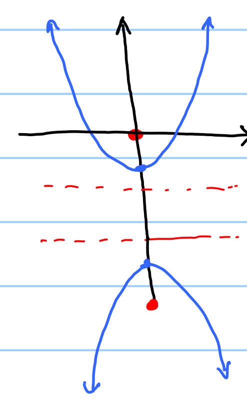
1) $r = \dfrac{k}{2 + \cos\theta} \implies e = \frac{1}{2}$ and conic section is ellipse

2) $r = \dfrac{k}{1 + \cos\theta} \implies e = 1$ and conic section is parabola.

3) $r = \dfrac{2k}{1 + 2\cos\theta} \implies e = 2$ and conic section is hyperbola.

Examples: ① Find the polar equation of the hyperbola with eccentricity $e = 3$, one focus at the origin and corresponding directrix $y = -2$.

sol: $e = 3$, $K = 2$ and $r = \dfrac{Ke}{1 - e\sin\theta}$



$$\implies r = \frac{2 * 3}{1 - 3\sin\theta} = \frac{6}{1 - 3\sin\theta}.$$

Example 2. Sketch the ellipse $r = \dfrac{4}{2 - \cos\theta}$. Include the directrix corresponding to the focus at the origin, label the vertices and center with appropriate polar coordinates.

sol: $r = \dfrac{2}{1 - \frac{1}{2}\cos\theta} \Rightarrow e = \frac{1}{2}$, $ke = 2 \Rightarrow k = 4$

$F_1(0,0)$, $D_1 : x = -4$

Vertices: at $\theta = 0 \Rightarrow r = 4$ and at $\theta = \pi$, $r = \frac{4}{3}$

$\therefore$ $V_1(4, 0)$, $V_2(\frac{4}{3}, \pi)$.

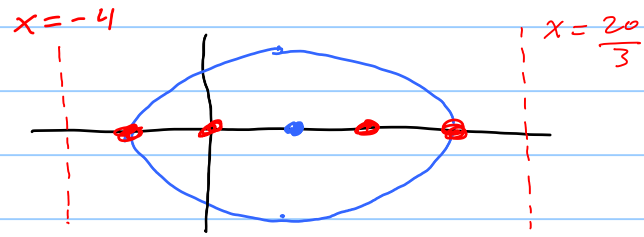
$2a = 4 + \frac{4}{3} = \frac{16}{3} \Rightarrow a = \frac{8}{3}$

[لاحظ أن الرأسين يقعان عند الزوايا $\theta = 0, \pi$ ، وأن a هي نصف المسافة بين الرأسين]

$\therefore$ Center: $C(0, 4 - \frac{8}{3}) = C(0, \frac{4}{3}) \Rightarrow$

center-to-focus distance $c = \frac{4}{3} \Rightarrow$ the other focus $F_2(0, \frac{8}{3})$.

Finally, center-to-directrix distance $= \frac{4}{3} + 4 = \frac{16}{3} \Rightarrow$ the other Directrix is $D_2 : x = \frac{4}{3} + \frac{16}{3} = \frac{20}{3}$.



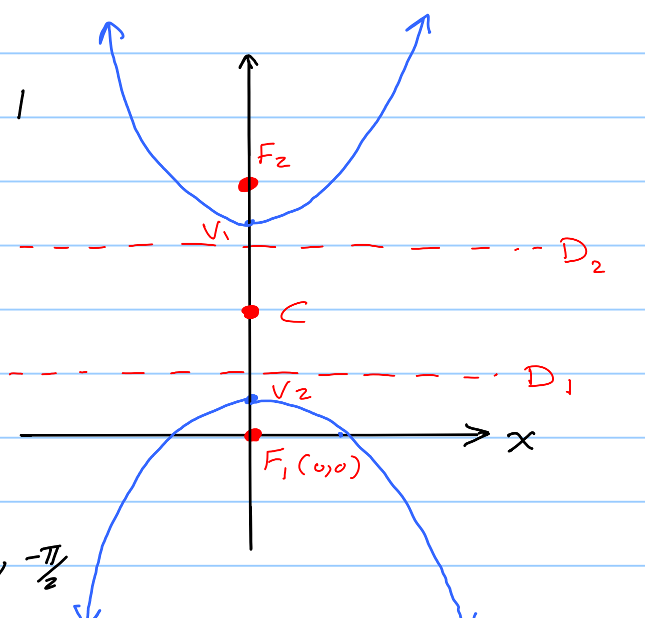
Example 3: Identify the conic section $r = \dfrac{5}{2 + 3\sin\theta}$. Find the directrix corresponding to the focus at the origin. label the vertices, the center and the other focus. Sketch the conic section.

sol: $r = \dfrac{5/2}{1 + \frac{3}{2}\sin\theta} \Rightarrow e = \frac{3}{2} > 1$

$\Rightarrow$ Conic section is $\boxed{\text{hyperbola}}$

$k \cdot e = \frac{5}{2} \Rightarrow k = \frac{5}{2} * \frac{2}{3} = 5/3$

$\therefore$ $F_1(0,0)$ and corresponding directrix D_1 : $\boxed{y = \frac{5}{3}}$



Vertices: The vertices are at $\theta = \pi/2$, $-\pi/2$

at $\theta = \pi/2 \Rightarrow r = 1$, and at $\theta = -\pi/2$, $r = -5$, so

$V_1(1, \pi/2)$ and $V_2(-5, -\pi/2) = (5, -\pi/2 + \pi) = (5, \pi/2)$

$2a = 4$ ((المسافة بين الرأسين)) $\Rightarrow a = 2$

<u>Center:</u> $C(3, \pi/2)$ [$a=2$ مسافة من الرأس للمركز]

$\Rightarrow$ center-to-focus distance $c = 3$

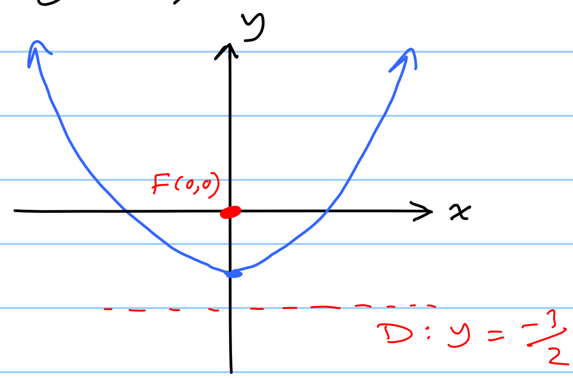
$\Rightarrow$ The other focus $F_2(6, \pi/2)$

Example 4: Find all information about the conic section $r = \dfrac{3}{2 - 2\sin\theta}$

<u>sol:</u> $r = \dfrac{3/2}{1 - \sin\theta} \Rightarrow e = 1$. $\therefore$ Parabola.

$K \cdot e = \frac{3}{2} \Rightarrow K = \frac{3}{2} \Rightarrow$

<u>Focus:</u> $F(0,0)$, <u>Directrix:</u> $y = -3/2$

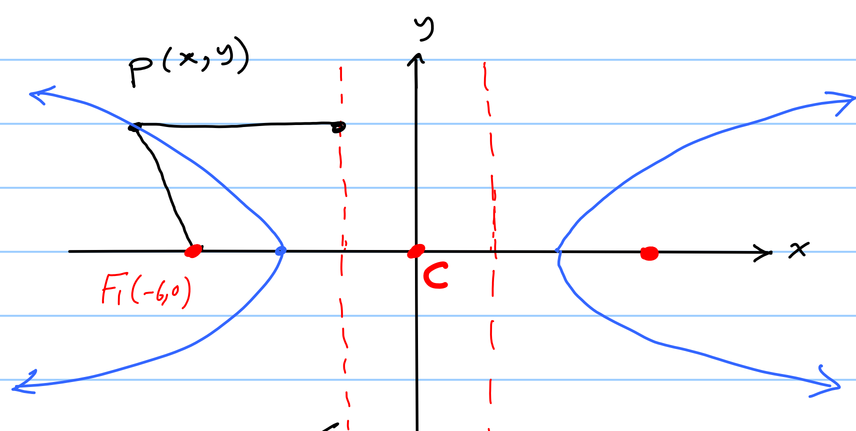


vertex: at $\theta = -\pi/2$, $r = 3/4$

$\therefore V(\frac{3}{4}, -\pi/2)$

ملحق

① Find the standard-form equation for the hyperbola with center at the origin, focus are $(-6, 0)$ and directrix $x = -2$.



لاحظ في البداية أنه يمكن حل السؤال بأكثر من طريقة / بتعوض هنا بحالا
بطريقتين مختلفتين

حل 1 $\quad C = 6, \quad \frac{a}{e} = 2 \Rightarrow a = 2e$

$e = \frac{c}{a} = \frac{c}{2e} \Rightarrow c = 2e^2 \Rightarrow e^2 = \frac{c}{2} = 3 \quad \therefore \quad e = \sqrt{3}$

بفرض أن $p(x,y)$ نقطة على القطع الزائد / فان $\quad PF_1 = e\, PD_1$

$PF_1^2 = e^2\, PD_1^2 \Rightarrow (x+6)^2 + y^2 = 3(x+2)^2$

$x^2 + 12x + 36 + y^2 = 3x^2 + 12x + 12 \Rightarrow 2x^2 - y^2 = 24$

$$\therefore \quad \boxed{\frac{x^2}{12} - \frac{y^2}{24} = 1}$$

حل 2 $\quad C = 6, \quad \frac{a}{e} = 2 \Rightarrow e = \frac{a}{2} \Rightarrow a = 2\sqrt{3}$

$\therefore$

$b^2 = c^2 - a^2 = 36 - 12 = 24.$

Center $(0,0)$ so no shifting

focal axis: y-axis $(x = 0) \Rightarrow$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \boxed{\frac{x^2}{12} - \frac{y^2}{24} = 1}$$

(2) Find all information about the following conic section.

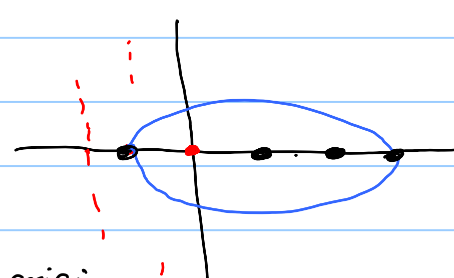
$$r = \frac{25}{10 - 5\cos\theta}$$

sol: $\quad r = \dfrac{25/10}{1 - \frac{1}{2}\cos\theta} \Rightarrow e = \frac{1}{2} \Rightarrow$ Conic section is

ellipse with one focus at origin and focal axis $y=0$

(x-axis). $\quad ke = \frac{25}{10} = \frac{5}{2}$

$\Rightarrow \boxed{k = 5}$



Directrix corresponding to focus at origin has the eq: $\boxed{x = -5}$

vertices at $\theta = 0$ and $\theta = \pi$, so at $\theta = 0$, then $r = 5$ and at $\theta = \pi$, $r = \frac{25}{15} = \frac{5}{3}$

$\therefore$ vertices: (In Polar) $V_1(5, 0)$, $V_2(\frac{5}{3}, \pi)$

$$2a = 5 + 5/3 = \frac{20}{3} \Rightarrow a = \frac{10}{3}.$$

Center: $(5 - \frac{10}{3}, 0) = (\frac{5}{3}, 0)$

تم إيجادهم سابقاً

Directrix: $\frac{a}{e} = \frac{20}{3} \Rightarrow D_1: x = \frac{5}{3} - \frac{20}{3} = -5$

$D_2: x = 5/3 + \frac{20}{3} = \frac{25}{3}$

$$e = \frac{c}{a} \Rightarrow c = e \cdot a = \frac{1}{2} * \frac{10}{3} = \frac{5}{3}$$

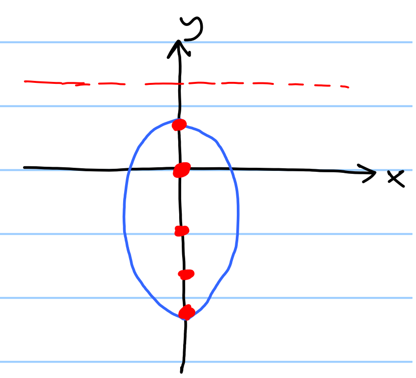
foci: $F_1(0,0)$, $F_2(5/3 + \frac{5}{3}, 0) = (\frac{10}{3}, 0)$

Example 3. Sketch the ellipse $r = \dfrac{5}{3 + 2\sin\theta}$. Include the directrix corresponding to the focus at the origin, label the vertices and center with appropriate polar coordinates.

sol: $r = (5/3)/(1 + \frac{2}{3}\sin\theta) \Rightarrow e = \frac{2}{3} \Rightarrow k \cdot e = 5/3 \Rightarrow k = 5/2$

$\Rightarrow$

$F_1(0,0)$, $D_1: \boxed{y = 5/2}$



vertices: at $\theta = \pi/2 \Rightarrow r = 1$, at $\theta = -\pi/2$

$r = 5 \Rightarrow V_1(1, \pi/2)$, $V_2(5, -\pi/2)$.

$2a = 6 \Rightarrow a = 3 \Rightarrow$ من الرأس الى المركز مسافة $\underline{\underline{3}}$

Center: $(2, -\pi/2) \Rightarrow$

center-to-focus distance $c = 2 \Rightarrow$ other focus: $F_2(4, -\pi/2)$

4) Sketch the Conic section $r = \frac{3}{4 + 4\cos\theta}$

Find all information.

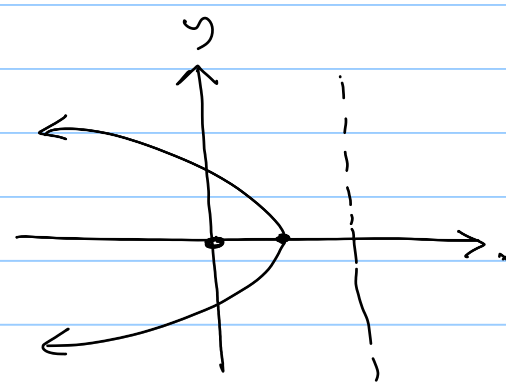
sol: $r = \frac{3/4}{1 + \cos\theta} \Longrightarrow e = 1 \Longrightarrow$ $\boxed{\text{parabola.}}$

$K \cdot e = 3/4 \Longrightarrow K = 3/4$.

Focus: $F(0,0)$ Directrix: $x = 3/4$

Vertex: At $\theta = 0$, $r = \frac{3}{8}$

$\therefore$ $V(\frac{3}{8}, 0)$



End of chapter 11