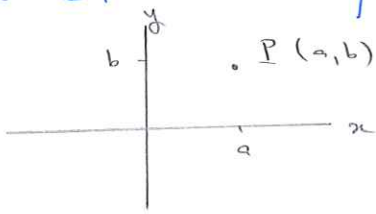
Ch 12. Vectors and The Geometry of Space:

12.1 Three-dimensional Coordinate Systems:

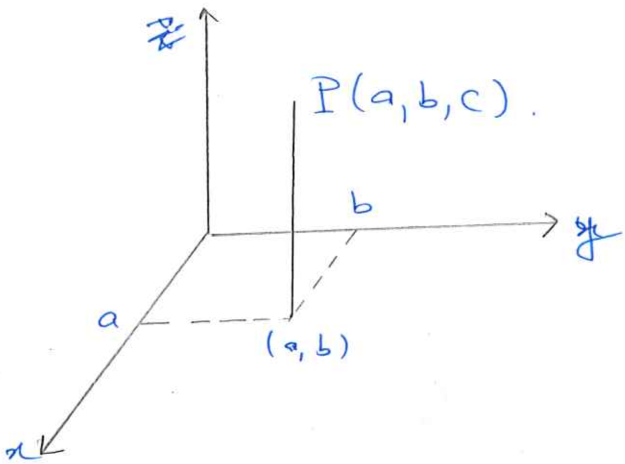
- The point $P(a,b) \in \mathbb{R}^2$ "Two dimensional space".



when $\mathbb{R}^2 = \{(a,b) : a, b \in \mathbb{R}\}$.
(4 quadrant)

- The point $P(a,b,c) \in \mathbb{R}^3$ "Three dimensional space".

a : x - coordinate
b : y - coordinate
c : z - coordinate
(8 octant)



Geometric description in $\mathbb{R}^3$:

1. $x = -3$, The plane perpendicular to x-axis at $x = -3$ and parallel to yz-plane.

2. $y = 4$, The plane $\perp$ to y-axis at $y = 4$ and // to xz-plane

3. $z = 5$, The plane $\perp$ to z-axis at $z = 5$ and // to xy-plane.

4. $x = 2$ and $y = 3$:

The $\boxed{\text{line}}$ in which the planes $x = 2$ & $y = 3$ intersect. This Line passes through $(2, 3, 0)$ and // Z-axis & $\perp$ to xy-plane.

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5. $Z \geq 0$: The half-space consisting of the points on the above the xy-plane.

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6. $x \geq 0$, $y \geq 0$, $Z \geq 0$: The first octant.

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7. $-1 \leq y \leq 1$: The $\boxed{\text{slab}}$ between $y = -1$ & $y = 1$

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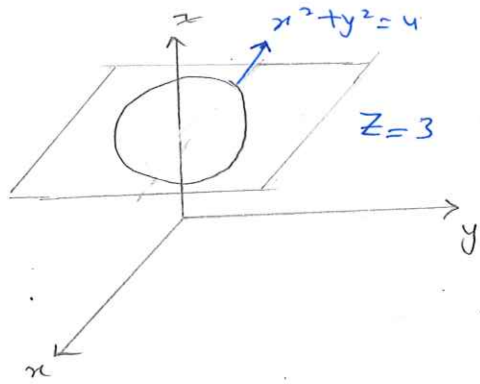
8. $x^2 + y^2 = 4$ & $Z = 3$:



$\boxed{\text{Circle}}$ in the plane $Z = 3$
center $(0, 0, 3)$

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9. $x^2 + y^2 \leq 1$, $Z = 3$:

Disk in the plane $Z = 3$, with center $(0, 0, 3)$

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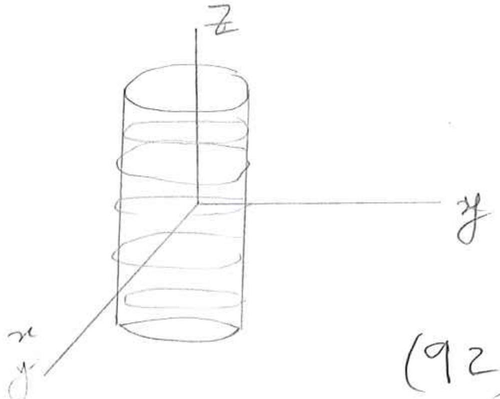
10. $x^2 + y^2 \leq 1$

cylindrical Solid with $r = 1$ and Z axis $\perp$ center.



11. $z = 0$, $x \leq 0$, $y \geq 0$ second quadrant in xy-plane

Distance and spheres in Space:

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Example: Find the distance between $P_1(1,1,1)$ & $P_2(2,3,5)$

$$|P_1P_2| = \sqrt{(1-2)^2 + (1-3)^2 + (1-5)^2} = \sqrt{1+4+16} = \sqrt{21}$$

The standard equation for the sphere of Radius a and center (x_0, y_0, z_0) is

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2.$$

Example: Find the center and radius of:

$$x^2+y^2+z^2+3x-4z+1=0.$$

$$\left(x+\frac{3}{2}\right)^2 + y^2 + (z-2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}$$

$\therefore$ Center $\left(-\frac{3}{2}, 0, 2\right)$ & radius $\sqrt{21}/2$.

Example: ① $x^2+y^2+z^2 > 4$: Exterior of the sphere $x^2+y^2+z^2=4$

② $x^2+y^2+z^2 < 4$: The Interior of the sphere $x^2+y^2+z^2=4$

③ $x^2+y^2+z^2 \leq 4$: The sphere with its Interior.

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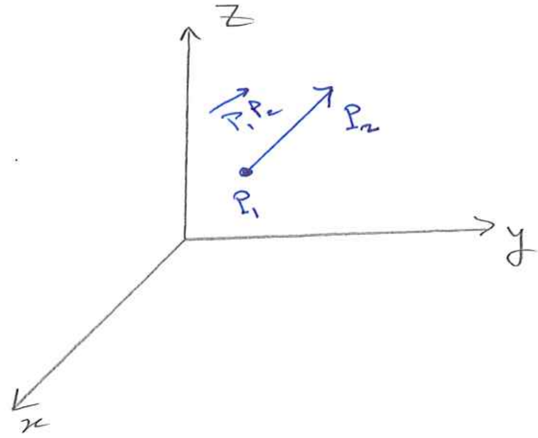
④ $x^2+y^2+z^2=4$, $z \leq 0$: The lower hemisphere cut from the sphere $x^2+y^2+z^2=4$ by xy plane (plane $z=0$)

12.2 Vectors

Def: A vector is a directed line segment from Initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ with length $|P_1P_2| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$

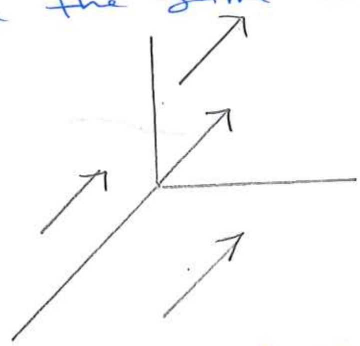


Note: we Usually give names to vectors:

$\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{z}$, ...

OR (Bold) **U**, **V**, **W**, **Z**.

Note: Vectors are equal if they have the same length and direction.



Note: The vector is in Standard position if the Initial point is the Origin. (i.e) $P_1 = (0,0,0)$

$P_2(x_2, y_2, z_2)$

Component form:

Def: If $\vec{v}$ is a vector in 2D equal to the vector in the standard position with terminal point (v_1, v_2), then the Component form of $\vec{v}$ is

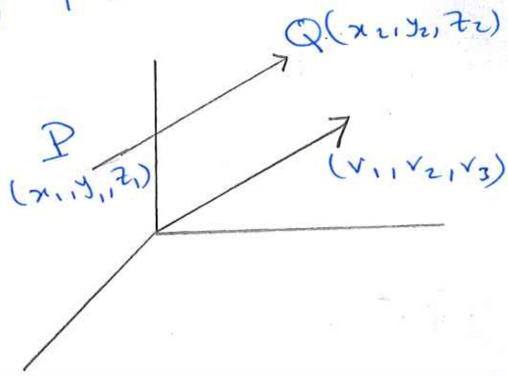
$$\vec{v} = \langle v_1, v_2 \rangle.$$

Similarly: If $\vec{v}$ in a three dimensional vector equal to a vector in the standard position with terminal point (v_1, v_2, v_3), then the Component form of $\vec{v}$ is

$$\vec{V} = \langle v_1, v_2, v_3 \rangle.$$

Note: If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is represented by $\overrightarrow{PQ}$ where $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$, then:

$$\left.\begin{aligned} x_1 + v_1 &= x_2 \\ y_1 + v_2 &= y_2 \\ z_1 + v_3 &= z_2 \end{aligned}\right\} \Rightarrow \begin{aligned} v_1 &= x_2 - x_1 \\ v_2 &= y_2 - y_1 \\ v_3 &= z_2 - z_1 \end{aligned}$$



(i.e) $\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Note: If $\vec{V} = \langle v_1, v_2, v_3 \rangle$ & $\vec{u} = \langle u_1, u_2, u_3 \rangle$ then $\vec{V} = \vec{u}$ iff $v_1 = u_1$, $v_2 = u_2$ & $v_3 = u_3$.

Note: If $\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

then: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Note: The Only vector with length Zero is $\vec{0}$ with no specific direction.

Example: Find (a) The Component form
(b) The length

of the vector with Initial point $P(-3,4,1)$ & $Q(-5,2,2)$

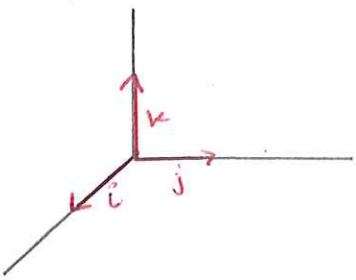
(a) $\overrightarrow{PQ} = \langle v_1, v_2, v_3 \rangle = \langle -5 - -3, 2-4, 2-1 \rangle = \langle -2, -2, 1 \rangle$

(b) length of $\overrightarrow{PQ} = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$

Def: A vector v of length 1 is called a Unit vector.

The standard Unit vectors are:

$i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$.



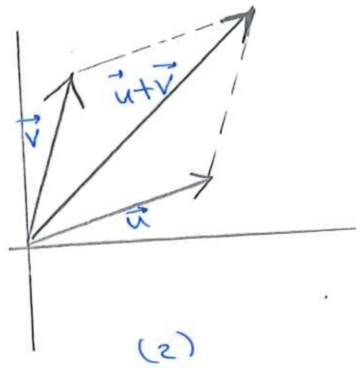
Vectors Algebra Operations:

Def: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$

& k be a scalar, then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

& $$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$$

- 

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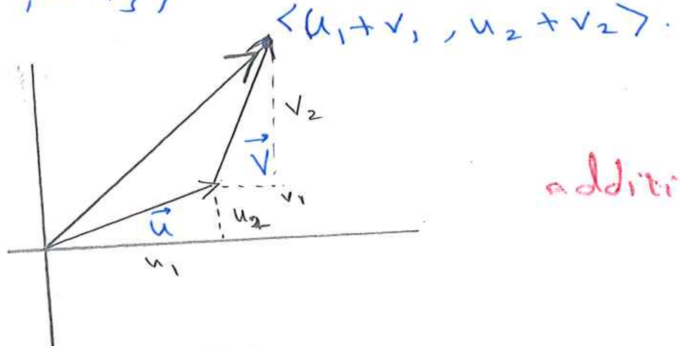
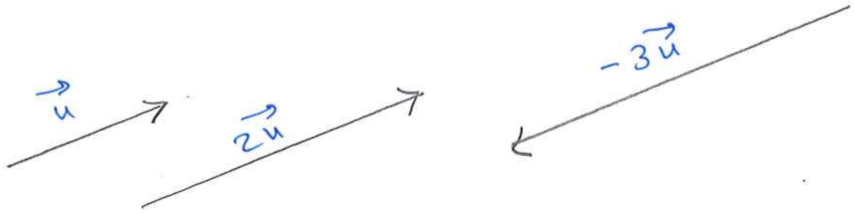
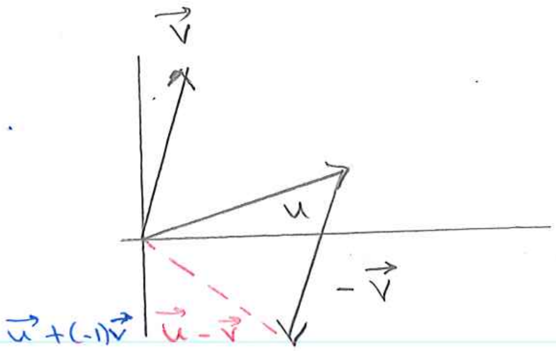


(1)

addition

- 

Scalar multiplication

- 

difference

(*). $|\alpha\vec{u}| = |\langle \alpha u_1, \alpha u_2, \alpha u_3 \rangle| = \sqrt{(\alpha u_1)^2 + (\alpha u_2)^2 + (\alpha u_3)^2}$

$$= \sqrt{\alpha^2(u_1^2 + u_2^2 + u_3^2)} = |\alpha| \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$\Longrightarrow \quad \underbrace{|\alpha\vec{u}|}_{\text{length}} = \underbrace{|\alpha|}_{\text{Abs}} \underbrace{|\vec{u}|}_{\text{length}}$$

Example: Let $\vec{u} = \langle -1, 3, 1 \rangle$ & $\vec{v} = \langle 4, 7, 0 \rangle$.

Find the components of

(a) $2\vec{u} + 3\vec{v}$ (b) $\vec{u} - \vec{v}$ (c) $|\frac{1}{2}\vec{u}|$.

(a) $2\vec{u} + 3\vec{v} = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\vec{u} - \vec{v} = \langle -1-4, 3-7, 1-0 \rangle = \langle -5, -4, 1 \rangle$.

(c) $|\frac{1}{2}\vec{u}| = |\frac{1}{2}|\,|\vec{u}| = \frac{1}{2}\sqrt{(1) + (9) + (1)} = \frac{1}{2}\sqrt{11}$

Properties of vector Operations:

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, a, b ~~and c~~ be scalars

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

3. $(\vec{u}) + \vec{0} = \vec{u}$

4. $0\vec{u} = \vec{0}$

5. $\vec{u} + (-\vec{u}) = \vec{0}$.

6. $1\vec{u} = \vec{u}$.

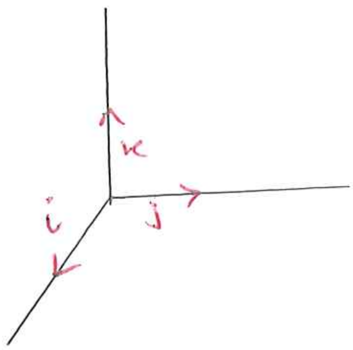
7. $a(b\vec{u}) = ab(\vec{u})$

8. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.

9. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$.

Def: A unit vector $\vec{v}$ is a vector of length 1.

- Standard Unit vectors are:



$\vec{i} = \langle 1, 0, 0 \rangle \Rightarrow |\vec{i}| = 1$

$\vec{j} = \langle 0, 1, 0 \rangle \Rightarrow |\vec{j}| = 1$

$\vec{k} = \langle 0, 0, 1 \rangle \Rightarrow |\vec{k}| = 1$

Note: Any vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ Can be written as a linear combination of the standard Unit vectors.

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle$$

$$= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle$$

$$= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

v_1 is the i-component of $\vec{v}$

v_2 is the j-component of $\vec{v}$

v_3 is the k-component of $\vec{v}$.

- If $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, then:

$$\overrightarrow{PQ} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

Example: $\vec{v} = \langle 2, 5, 3 \rangle = 2\vec{i} + 5\vec{j} + 3\vec{k}$.

Note: If $\vec{V}$ is any non zero vector, then

$\dfrac{\vec{V}}{|\vec{V}|}$ is a unit vector and is called the direction of $\vec{V}$

- $\left|\dfrac{\vec{V}}{|\vec{V}|}\right| = \dfrac{1}{|\vec{V}|} \cdot |\vec{V}| = 1.$

Note: Any non Zero vector $\vec{V}$ Can be written as:

$$\vec{V} = (\text{length})(\text{Direction of } \vec{V})$$

$$\underset{\text{velocity}}{\vec{V}} = \underset{\text{speed}}{|\vec{V}|} \cdot \underset{\text{direction}}{\left(\dfrac{\vec{V}}{|\vec{V}|}\right)}$$

. **Example:** Find a Unit vector $\vec{u}$ in the direction of the vector from $P(2,3,0)$ to $Q(5,2,4)$

$$\overrightarrow{PQ} = (5-2)\vec{i} + (2-3)\vec{j} + (4-0)\vec{k}$$

$$= 3\vec{i} - \vec{j} + 4\vec{k}.$$

$$|\overrightarrow{PQ}| = \sqrt{(3)^2 + (-1)^2 + (4)^2} = \sqrt{9+1+16} = \sqrt{26}.$$

$$\therefore \vec{u} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{3}{\sqrt{26}}\vec{i} - \dfrac{1}{\sqrt{26}}\vec{j} + \dfrac{4}{\sqrt{26}}\vec{k}.$$

Example: If $v = 3\vec{i} - 4\vec{j}$ is a velocity vector.

express $\vec{v}$ as a product of its speed times a Unit vector in the direction of motion.

- speed is the length of $\vec{v}$.

$\therefore \ |\vec{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5.$

Unit vector $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j}.$

so $\vec{v} = 5\left(\dfrac{3}{5}\vec{i} - \dfrac{4}{5}\right)\vec{j}.$

Example: Let $\vec{u} = 2\vec{i} + \vec{j}$, $\vec{v} = \vec{i} + \vec{j}$, $\vec{w} = \vec{i} - \vec{j}$.

Find a & b (scalars) (s.t) $\vec{u} = a\vec{v} + b\vec{w}$.

$2\vec{i} + \vec{j} = a(\vec{i} + \vec{j}) + b(\vec{i} - \vec{j})$

$2\vec{i} + \vec{j} = (a+b)\vec{i} + (a-b)\vec{j}$

$\Rightarrow \left.\begin{aligned} 2 &= a+b \\ 1 &= a-b \end{aligned}\right\} \Rightarrow \quad a = \dfrac{3}{2} \ , \ b = \dfrac{1}{2}.$

Def: The midpoint of the line PQ , where $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ is : $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right).$

$\overrightarrow{OM} = \dfrac{x_1+x_2}{2}\vec{i} + \dfrac{y_1+y_2}{2}\vec{j} + \dfrac{z_1+z_2}{2}\vec{k}.$

12.3 The dot product: ← (Number)

Def: The dot product "$\vec{u}\cdot\vec{v}$" of the vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is

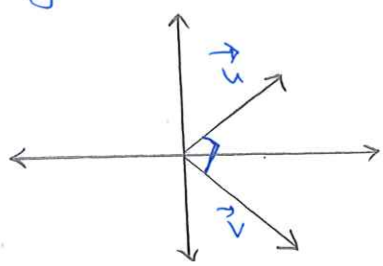
$$\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

Example: If $\vec{u} = \langle 2, -5, 0 \rangle$ & $\vec{v} = \frac{1}{2}\vec{i} + \frac{1}{3}\vec{j} + \vec{k}$

then $\vec{u}\cdot\vec{v} = 2\left(\frac{1}{2}\right) + (-5)\left(\frac{1}{3}\right) + 0(1) = 1 - \frac{5}{3} = \frac{-2}{3}$.

Example: If $\vec{u} = \vec{i} + \vec{j}$, $\vec{v} = \vec{i} - \vec{j}$, then

$\vec{u}\cdot\vec{v} = (1)(1) - (1)(1) = 0$



Def: Vectors $\vec{u}$ & $\vec{v}$ are orthogonal (perpendicular)

iff $\vec{u}\cdot\vec{v} = 0$. عدد

Example: $\vec{0} \perp$ for any nonzero vector $\vec{v}$ since $\vec{0}\cdot\vec{v} = 0$

Example: $\vec{u} = \vec{i}$, $\vec{v} = \vec{k}$

$\vec{u}\cdot\vec{v} = 1(0) + 0(0) + 0(1) = 0 \Rightarrow \vec{i} \perp \vec{k}$.

Thm: The Angle between two vectors

The angle θ between two non Zero vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by:

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \right).$$

Example: Find the angle between $\vec{u}$ & $\vec{v}$ In the following:

1) $\vec{u} = 2\vec{i} - 3\vec{j} + \sqrt{12}\,\vec{k}$

$\vec{v} = -2\vec{i} + 3\vec{j} - \sqrt{12}\,\vec{k}$.

$$\theta = \cos^{-1}\left(\frac{2(-2) + (3)(-3) + (\sqrt{12})(-\sqrt{12})}{\sqrt{(2)^2 + (-3)^2 + (\sqrt{12})^2} \cdot \sqrt{(-2)^3 + (3)^2 + (-\sqrt{12})^2}} \right) = \cos^{-1}\left(\frac{-25}{25}\right) = \cos^{-1}(-1)$$

$\Longrightarrow$ $\theta = \pi$

2) $\vec{u} = \vec{i} + \vec{j}$, $\vec{v} = \vec{i} - \vec{j}$.

$$\theta = \cos^{-1}\left(\frac{0}{\sqrt{2}\cdot\sqrt{2}}\right) = \cos^{-1}(0) = \frac{\pi}{2}. \qquad (\vec{u} \perp \vec{v})$$

Properties of the Dot Product:

If $\vec{u}$ & $\vec{v}, \vec{w}$ are vectors & c is a scalar, then

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

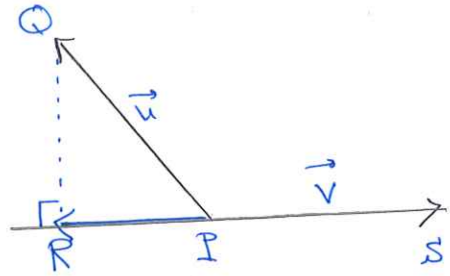
2. $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$.

3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$ (length)^2.

5. $\vec{0} \cdot \vec{u} = 0$.

proof $\boxed{4}$: $\vec{u} \cdot \vec{u} = \langle u_1, u_2 \rangle \cdot \langle u_1, u_2 \rangle = u_1^2 + u_2^2 = |\vec{u}|^2$.



Vector projection:

The vector projection of $\vec{u} = \overrightarrow{PQ}$ onto a nonzero vector $\vec{v} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to $\overrightarrow{PS}$

The projection of $\vec{u}$ onto $\vec{v}$ is defined by:

$$\text{proj}_{\vec{v}} \vec{u} = \underbrace{\left(|\vec{u}| \cos \theta \right)}\ \underbrace{\frac{\vec{v}}{|\vec{v}|}}$$, θ between $\vec{u}$ & $\vec{v}$

$$= \left(\text{scalar Component of } \vec{u}\right) \cdot \left(\text{Direction of } \vec{v}\right)$$

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Note: $\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$.

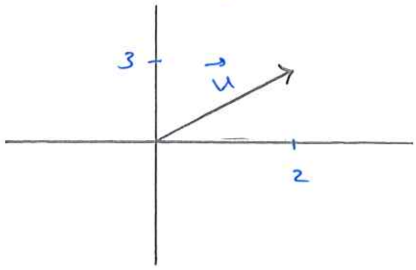
Example: Find the vector projection of $\vec{u} = 6\vec{i} + 3\vec{j} + 2\vec{k}$ onto $\vec{v} = \vec{i} - 2\vec{j} - 2\vec{k}$.

$\text{proj}_{v} \vec{u} = (\text{scalar component of } \vec{u}) \cdot (\text{Direction of } \vec{v})$

$$= \left(\frac{u \cdot \vec{v}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|} = \frac{6(1) + 3(-2) + 2(-2)}{\sqrt{1+4+4}} \cdot \frac{\vec{i} - 2\vec{j} - 2\vec{k}}{\sqrt{1+4+4}}$$

$$= \frac{-4}{3} \cdot \frac{\vec{i} - 2\vec{j} - 2\vec{k}}{3} = -\frac{4}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{8}{9}\vec{k}.$$

Example: For $\vec{u} = a\vec{i} + b\vec{j}$ find the slope.



(for example: $\vec{u} = 2\vec{i} + 3\vec{j}$):

The slope of $\vec{u}$ is $\frac{3}{2}$

(i-e) The slope of $\vec{u} = a\vec{i} + b\vec{j}$ is $\frac{b}{a}$, $a \neq 0$

Example (32) Show that the vector $\vec{V} = a\vec{i} + b\vec{j}$ is parallel to the line $bx - ay = c$.

sol: Slope of $\vec{V} = \frac{b}{a}$, $a \neq 0$

& slope of $ay = bx - c \iff$ Slope of $y = \frac{b}{a}x - \frac{c}{a}$

which is $\frac{b}{a}$. , $a \neq 0$

$\therefore \vec{V} \parallel bx - ay = c$.

Example: (37) Find an equation of a line that is Parallel to the vector $\vec{V} = \vec{i} - \vec{j}$ & passes through $(-2,1)$.

slope $= \frac{-1}{1} = -1 = \frac{b}{a}$

$\therefore y - 1 = -1(x - {-2})$

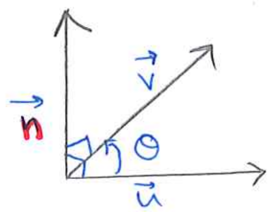
$\therefore y = -x - 1$.

12.4 The Cross Product.

Def: If $\vec{u}$ & $\vec{v}$ are any non Zero vectors, and if we select a unit vector $\vec{n}$ that is perpendicular to the plane $\vec{u}\vec{v}$ by the right hand rule., then:

The cross product denoted by $\vec{u} \times \vec{v}$ is defined by:

$$\vec{u} \times \vec{v} = \underbrace{(|\vec{u}||\vec{v}| \sin\theta)}_{\text{number}} \underbrace{\vec{n}}_{\text{vector}}.$$



Note: Non Zero vectors $\vec{u}$ & $\vec{v}$ are parallel iff $\vec{u} \times \vec{v} = 0$.

Since $\vec{u} \times \vec{v} = 0$ iff $\theta = 0$ or $\theta = \pi$.

Properties of the Cross Product:

If $\vec{u}, \vec{v}$ & $\vec{w}$ are any vectors, r, s are Scalars, then:

1) $(r\vec{u}) \times (s\vec{v}) = rs(\vec{u} \times \vec{v})$
2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
3) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
4) $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.
5) $\vec{0} \times \vec{u} = \vec{0}$
6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

Example: $\vec{u} = 2\vec{i} + \vec{j} - \vec{k}$, $\vec{v} = \vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{w} = \vec{j} + \vec{k}$.

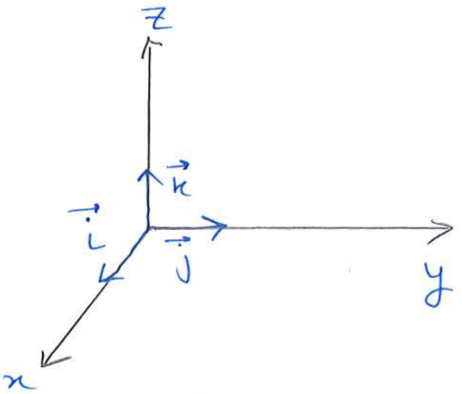
$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

$$= ((2)(0) + (1)(1) + (-1)(1))\left[\vec{i} - 2\vec{j} + 3\vec{k}\right] - \left[2 - 2 - 3\right]\left[\vec{j} + \vec{k}\right]$$

$$= 0\,\vec{v} + 3\,\vec{w} = 3\vec{j} + 3\vec{k}.$$

Note: $\boxed{1}$ $\vec{i} \times \vec{j} = (|i||j| \sin\theta)\vec{n}$

$$= (1 \cdot 1 \cdot \sin 90) \cdot \vec{k} = \vec{k}$$



$\boxed{2}$ $\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$

$\boxed{3}$ $\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$.

Calculating the Cross product as a Determinant:

If $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$

$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - v_2 u_3)\vec{i} - (u_1 v_3 - v_1 u_3)\vec{j} + (u_1 v_2 - v_1 u_2)\vec{k}$$

proof: $\vec{u} \times \vec{v} = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$

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Example: Find $\vec{u} \times \vec{v}$ if $\vec{u} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{v} = \vec{i} - 4\vec{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 0 & -4 \end{vmatrix} = 12\vec{i} + 9\vec{j} + 3\vec{k}.$$

Example: Find $|\vec{u} \times \vec{v}| = \sqrt{(12)^2 + (9)^2 + (3)^2} = \sqrt{234}.$

Example: $\vec{v} \times \vec{u} = -12\vec{i} - 9\vec{j} - 3\vec{k}.$

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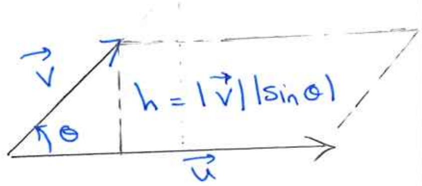
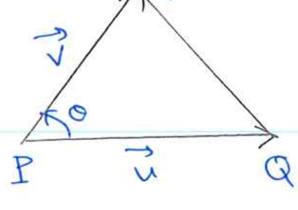
Area of Parallelogram:

$$\text{Area} = (\text{base}) * (\text{height}).$$

$$= |\vec{u}| . |\vec{v}| |\sin\theta|$$

$$= |\vec{u} \times \vec{v}|.$$





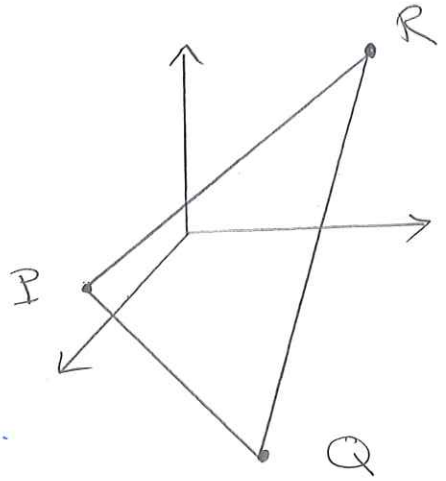
Area of triangle PQR $= \frac{1}{2}(\text{base})(\text{height})$

$$= \frac{1}{2} |\vec{u}| |\vec{v}| |\sin\theta|$$

$$= \frac{1}{2} |\vec{u} \times \vec{v}|.$$

<u>Example:</u> Find the area of the triangle with vertices:

$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$



Let $\vec{u} = \overrightarrow{PQ} = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{v} = \overrightarrow{PR} = -2\vec{i} + 2\vec{j} + 2\vec{k}$.

Then $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = 6\vec{i} + 6\vec{k}$.

$\Rightarrow$ Area of triangle $= \dfrac{|\vec{u} \times \vec{v}|}{2} = \dfrac{\sqrt{(6)^2+(6)^2}}{2} = \dfrac{\sqrt{72}}{2} = 3\sqrt{2}$

<u>Example:</u> Find a Unit vector perpendicular to the plane of.

$P(1,-1,0)$, $Q(2,1,-1)$, $R(-1,1,2)$.

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ $\perp$ Triangle (to the plane)

Then $\overrightarrow{PQ} \times \overrightarrow{PR}$ is the vector perpendicular.

$\Rightarrow$ Unit vector $= \dfrac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \dfrac{6\vec{i} + 6\vec{k}}{6\sqrt{2}}$

$= \dfrac{\vec{i}}{\sqrt{2}} + \dfrac{\vec{k}}{\sqrt{2}}$.

<u>Note:</u> Also $-\dfrac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} \perp$ plane.

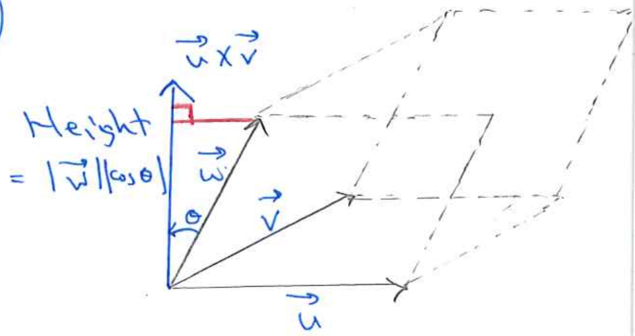
Triple Scalar or Box Product:

(parallelogram-Sided Box)

(parallelepiped)

Volume of the Box = $\left(\underset{\text{base}}{\text{Area of}}\right)(\text{height})$

$= |\vec{u}\times\vec{v}|\,|\vec{w}|\,|\cos\theta|.$



$$\Longleftrightarrow \left|(\vec{u}\times\vec{v})\cdot\vec{w}\right| = |\vec{u}\times\vec{v}|\,|\vec{w}|\,|\cos\theta|.$$

Remark: $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Example: Find the volume of the box determined by

$\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = -2\vec{i} + 3\vec{k}$, $\vec{w} = 7\vec{j} - 4\vec{k}$.

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $\left|(\vec{u}\times\vec{v})\cdot\vec{w}\right| = 23$ units cubed.

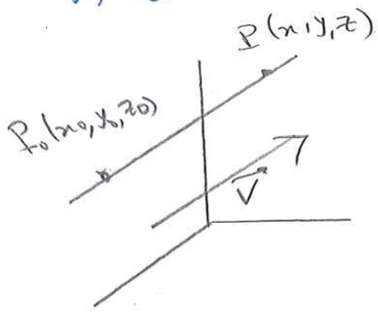
12.5 Lines and planes in Space:

Parametrization of a Line in space.

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

$$\left.\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}\right\} \quad -\infty < t < \infty.$$

proof: L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\vec{v}$.



Thus: $\overrightarrow{P_0P} = t\vec{v}$

$$\Rightarrow (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$\Rightarrow \begin{aligned} x - x_0 &= tv_1 \\ y - y_0 &= tv_2 \\ z - z_0 &= tv_3 \end{aligned} \Rightarrow \begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}$$

Example: Find the parametric equation of the following:

(1) A Line through $(-2, 4, 0)$ parallel to $\vec{v} = 2\vec{i} + 4\vec{j} - 2\vec{k}$.

$$\quad\ \ v_1 \quad\ v_2 \quad\ v_3$$

Assume $P_0(-2, 4, 0)$, then:

$$x = -2 + 2t$$

$$y = 4 + 4t \qquad\qquad -\infty < t < \infty$$

$$z = 0 + -2t$$

---

(2) A Line passes through $P(-3, 2, -3)$ and $Q(1, -1, 4)$

The vector $\overrightarrow{PQ}$ // to the line.

$$\overrightarrow{PQ} = (1 = 3)\vec{i} + (-1 - 2)\vec{j} + (4 = 3)\vec{k}.$$

$$= 4\vec{i} - 3\vec{j} + 7\vec{k}.$$

Then Assuming $P_0(x_0, y_0, z_0) = (-3, 2, -3)$, then the parametric equations are:

$$x = -3 + 4t$$
$$y = 2 - 3t$$
$$z = -3 + 7t$$

__Or__: If we assume $P_0(x_0, y_0, z_0) = (1, -1, 4)$, then:

$$x = 1 + 4t$$
$$y = -1 - 3t$$
$$z = 4 + 7t.$$

Example: Parametrize the Line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

1) We first parametrize a Line through the points, which are

$$x = -3 + 4t \qquad \vec{PQ} = 4\vec{i} - 3\vec{j} + 7\vec{k}$$

$$y = 2 - 3t$$

$$z = -3 + 7t$$

2) we find the t-values of the endpoints and restrict t.

. when $t = 0$, the parametric equation give the point $P(-3, 2, -3)$

. when $t = 1 \Rightarrow Q(1, -1, 4)$, so we add a restriction:

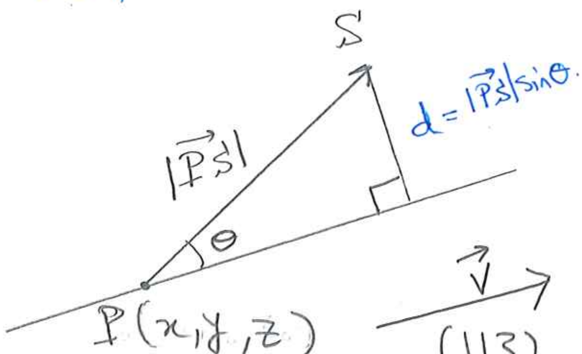
$$x = -3 + 4t, \; y = 2 - 3t, \; z = -3 + 7t, \; 0 \leq t \leq 1$$

Distance from a point to a Line in Space

Def: The distance from a Point S to a Line through P parallel to $\vec{v}$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{|\vec{PS}|\,|\vec{v}| \sin\theta \,|\vec{n}|}{|\vec{v}|}$$

$$= |\vec{PS}| \sin\theta$$



Example: Find the distance from the point $S(1,1,5)$

to the line $L: \; x = 1+t \leftarrow v_1 = 1$

$$y = 3 - t \leftarrow v_2 = -1$$

$$z = 2t \leftarrow v_3 = 2$$

Note that when $t=0$, L passes through $P(1,3,0)$

parallel to $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$, with:

$$\vec{PS} = (1-1)\vec{i} + (1-3)\vec{j} + (5-0)\vec{k} = -2\vec{j} + 5\vec{k}.$$

$$\therefore \vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \vec{i} + 5\vec{j} + 2\vec{k}.$$

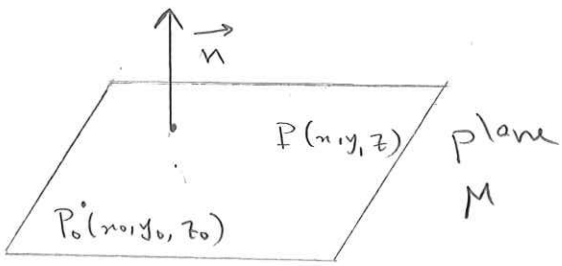
then
$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{(1)^2 + (5)^2 + (2)^2}}{\sqrt{(1)^2 + (-1)^2 + (2)^2}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

An Equation for a plane in Space:

The equation of plane in space can be determined by

1) Point $P_0(x_0, y_0, z_0)$ on the plane

2) vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ (s.t) orthogonal (normal) to the plane.



The plane M is the set of all points $P(x, y, z)$

for which $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$, (i.e)

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} \perp A\vec{i} + B\vec{j} + C\vec{k}$$

Hence $\overrightarrow{P_0P} \cdot \vec{n} = 0$

$$\Rightarrow (x - x_0) \cdot A + (y - y_0)B + (z - z_0)C = 0$$

$$\Rightarrow \boxed{Ax + By + Cz = Ax_0 + By_0 + Cz_0}$$ Equation of plane

$\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$

Example: Find Equation of plane that

(1) passes through $P_0(\overset{x_0}{0}, \overset{y_0}{2}, \overset{z_0}{-1})$ and normal to $\vec{n} = \overset{A}{3}\vec{i} - \overset{B}{2}\vec{j} - \overset{C}{\vec{k}}$

$$3x - 2y + -z = 3(0) + (-2)(2) + (-1)(-1) = \boxed{-3}$$

---

(2) passes through $P_0(1, -1, 3)$ and // to the plane $3x + y + z = 7$

ناخذ نورمال سوينا ال plane

// لأنه

we find $\vec{n} \perp 3x + y + z = 7$

$$\Rightarrow \vec{n} = 3\vec{i} + \vec{j} + \vec{k}$$

$$\Rightarrow 3x + y + z = 3(1) + (1)(-1) + (3)(1)$$

$$= 5$$

check !!

Example: Find an equation for the plane through:

$A(0,0,1)$, $B(2,0,0)$, $C(0,3,0)$

1. we find $\vec{n}$ (s.t) $\vec{n}$ $\perp$ plane.

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \underset{A}{3\vec{i}} + \underset{B}{2\vec{j}} + \underset{C}{6\vec{k}}.$$

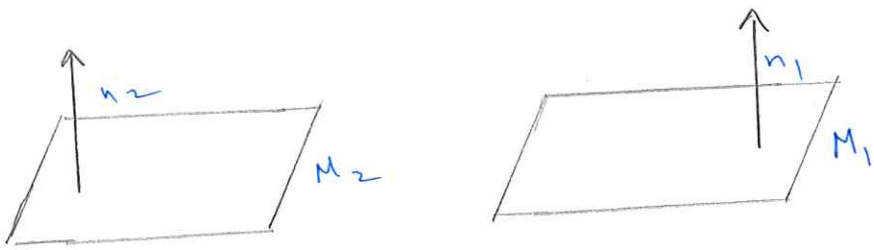
$\overrightarrow{AB} \times \overrightarrow{AC} \perp$ plane.

$x_0 \; y_0 \; z_0$

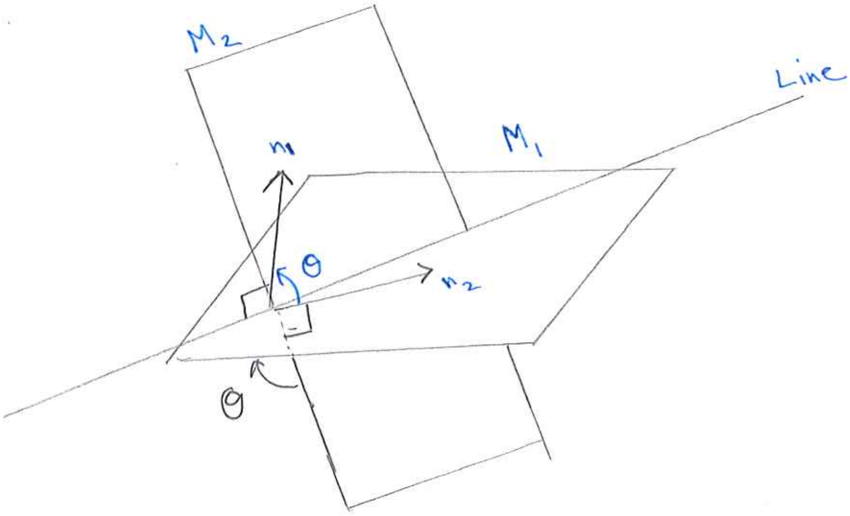
2. Use $\vec{n}$ with one of the points: , say, $A(0,0,1)$

$$\Rightarrow \quad 3x + 2y + 6z = (0)(3) + (0)(2) + (1)(6) = 6$$

Note: Two planes in space are parallel iff their normals are parallel.



Note: If two planes are not parallel, then they Intersect in a Line.



Note: The angle θ between the two planes in space is the same as the angle between the Corresponding Normals.

Lines of Intersection

Example: Find a vector parallel to the Line of Intersection of the planes $3x - 6y - 2z = 15$ & $2x + y - 2z = 5$

The Line of Intersection of two planes $\perp$ $\vec{n}_1$ & $\vec{n}_2$

$\vec{n}_1$ for plane 1, $\vec{n}_2$ for plane 2

& $\vec{n}_1 \times \vec{n}_2 \perp \vec{n}_1$ & $\vec{n}_2$

$\Rightarrow \vec{n}_1 \times \vec{n}_2 \parallel$ Line of Intersection of two planes

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\vec{i} + 2\vec{j} + 15\vec{k}$$

Example: Find parametric equation for the Line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ Intersect.

1) $\vec{v} = \vec{n}_1 \times \vec{n}_2 \parallel$ Line of Intersection

2) Choose any point $P_0(x_0, y_0, z_0)$ (by Letting $z = 0$,

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$\Rightarrow \boxed{x = 3,\ y = -1} \Rightarrow P_0(x_0, y_0, z_0) = (3, -1, 0)$

$\Rightarrow x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t$

Example: Find the point where the Line:

$$x = \frac{8}{3} + 2t \ , \quad y = -2t \ , \quad z = 1+t$$

intersects the plane $3x + 2y + 6z = 6$.

The point $\left(\frac{8}{3} + 2t, -2t, 1+t\right)$ Lies in the plane if its coordinates satisfy the equation of the plane, (i.e),

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$$

solve it for $t \Rightarrow \boxed{t = -1}$

$$\therefore (x, y, z)\Big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1-1\right) = \left(\frac{2}{3}, 2, 0\right).$$

The Distance from a point to a plane:

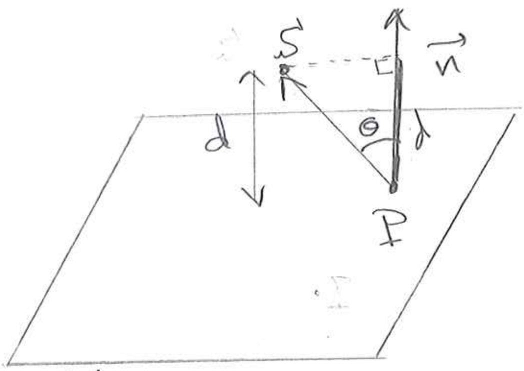
If P is a point on a plane with normal $\vec{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\vec{n}$

absolute value

that is $d = \left| \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$, $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$

proof:

$$\cos\theta = \frac{d}{|\overrightarrow{PS}|}$$



M

$$\Rightarrow d = |\overrightarrow{PS}| \cos\theta$$

$$= \frac{|\vec{n}|\,|\overrightarrow{PS}|\cos\theta}{|\vec{n}|}$$

$$= \left| \frac{\overrightarrow{PS} \cdot \vec{n}}{|\vec{n}|} \right|$$

Example: Find the distance from the point $S(0,0,0)$ to the plane $3x + 2y + 6z = 6$.

$\vec{n} = 3\vec{i} + 2\vec{j} + 6\vec{k}$.

$|\vec{n}| = \sqrt{9+4+36} = 7$

we need P known, the easiest to find from the plane's equation are the intercepts

Assume P the x Intercept. (i-e)

$3x + 0 + 0 = 6 \Rightarrow x = 2 \Rightarrow \boxed{P(2,0,0)}$

Then $\overrightarrow{PS} = (0-2)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k}$

$= -2\vec{i}$.

Hence, $\overrightarrow{PS} \cdot \vec{n} = (-2)(3) + 0(2) + 0(6)$

$= -6$.

$\therefore d = \left| \dfrac{-6}{7} \right| = \dfrac{6}{7}$.

Angles Between planes:

The angle between two intersecting planes is defined to be the acute angle between their normal vectors

Example: Find the angle between the planes:

$5x + y - z = 10$ & $x - 2y + 3z = -1$

$\vec{n}_1 = 5\vec{i} + \vec{j} - \vec{k} \Rightarrow |\vec{n}_1| = \sqrt{25+1+1} = \sqrt{27}$

$\vec{n}_2 = \vec{i} - 2\vec{j} + 3\vec{k} \Rightarrow |\vec{n}_2| = \sqrt{1+4+9} = \sqrt{14}$

$$\therefore \theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right) = \cos^{-1}\left(\frac{5-2-3}{\sqrt{27}\sqrt{14}}\right)$$

$$= \cos^{-1}(0) = \frac{\pi}{2}$$

Example: Discussion 10

Find parametric equation for the line passes through $P(2,3,0)$ and $\perp$ to the vectors:

$\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ & $\vec{v} = 3\vec{i} + 4\vec{j} + 5\vec{k}$.

vector $\vec{w} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{w} \parallel$ line.

$\therefore x = 2 - 2t$, $y = 3 + 4t$, $z = -2k$

$t \leq \mathbb{R}$.

12.6 Cylinders and Quadric Surfaces:

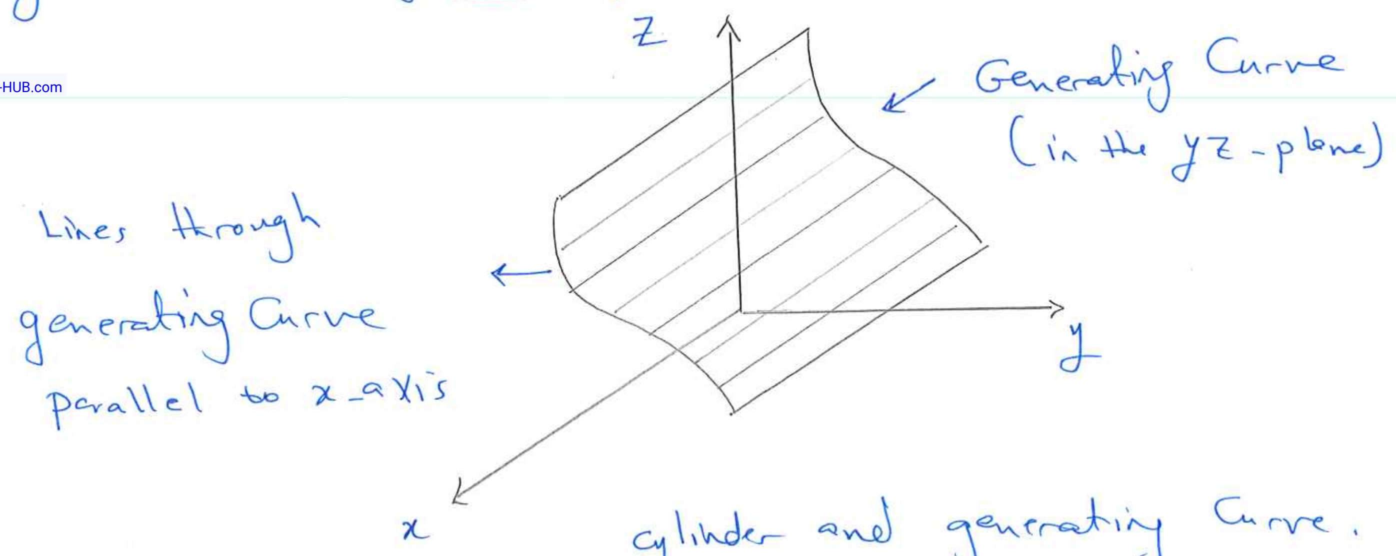
Until now we studied two types of surfaces: spheres and planes. We will extend our study & study Cylinders and Quadric Surfaces.

Cylinders:

A cylinder is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line.

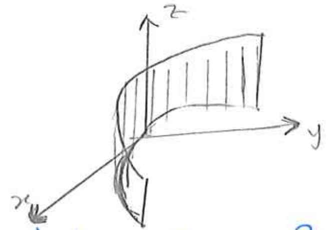
The curve is called a generating curve for the cylinder.

Note: In solid geometry, where cylinder means circular cylinder, the generating curves are circles.



cylinder and generating curve.

Example: Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2$, $z = 0$.



The point $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the xy-plane.

Then for any value of z, the point $Q(x_0, x_0^2, z)$ lies on the cylinder because it lies on the line $x = x_0$, $y = x_0^2$ through P_0 parallel to z-axis.

In this case, we call the cylinder "The cylinder $y = x^2$"

Note: The points on the surface are the points whose coordinates satisfy "$y = x^2$". This makes $y = x^2$ an equation for the cylinder.

Note: Any curve $f(x,y) = c$ in xy-plane defines a cylinder parallel to z-axis.

Any curve $f(x,z) = c$ in xz-plane defines a cylinder parallel to y-axis

Note: The equation $x^2+y^2=1$ defines the Circular cylinder made by the Lines parallel to the z axis that pass through the Circle $x^2+y^2=1$ in xy-plane

Quadric Surfaces:

A quadric Surface is the graph in space of a second - degree equation in x, y and z.

we will focus on the special equation:

$$Ax^2+By^2+Cz^2+Dz=E$$

where A, B, C, D, E are Constants.

The Basic Quadric Surfaces are:

1) ellipsoids, 2) paraboloids, 3) elliptical Cones, 4) hyperboloids

Note: sphere is a special Case of ellipsoid.
when $a=b=c$

Example: The ellipsoid:

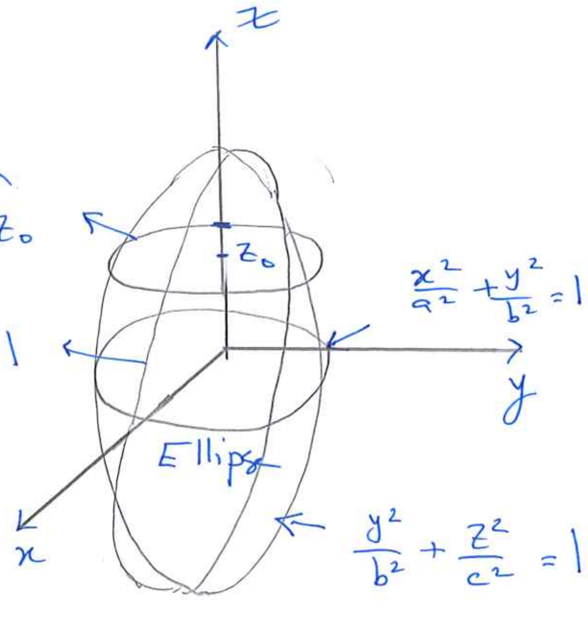
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Cuts the Coordinate axes at $(\pm a, 0, 0), (0, \pm b, 0)$ & $(0, 0, \pm c)$, then it lies within the rectangular box defined by the inequalities:

$$|x| \le a \quad , \quad |y| \le b \quad \text{and} \quad |z| \le c.$$

Moreover, The surface is symmetric with respect to each of the Coordinate planes because each variable in the defining equation is squared.



The Curves in which the three coordinate planes cut the surface are ellipses.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{when } z = 0 \quad (\text{in } xy \text{ plane})$$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad , \text{when } y = 0 \quad (\text{in } xz \text{ plane})$$

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad , \text{when } x = 0 \quad (\text{in } yz\text{-plane})$$

The Curve cut from the surface by the plane $Z = Z_0$

with $|Z_0| < c$ is the ellipse:

$$\frac{x^2}{a^2\left(1-(Z_0/c)^2\right)} + \frac{y^2}{b^2\left(1-(Z_0/c)^2\right)} = 1$$

Note: If any two of the semiaxes a, b, c are equal

the surface is an ellipsoid of revolution

Note: If all of the semiaxes a, b, c are equal

the surface is sphere.

Example: The hyperbolic paraboloid:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{Z}{c} \quad , c > 0.$$

has symmetry with respect to the planes $x = 0$ and $y = 0$

The Cross sections are:

$x = 0 \Rightarrow$ The parabola $Z = \frac{c}{b^2} y^2$. (opens upward)

$y = 0 \Rightarrow$ The parabola $Z = -\frac{c}{a^2} x^2$. (opens down ward)

If we cut the surface by a plane $Z = Z_0 > 0$.

The cross-section is a hyperbola

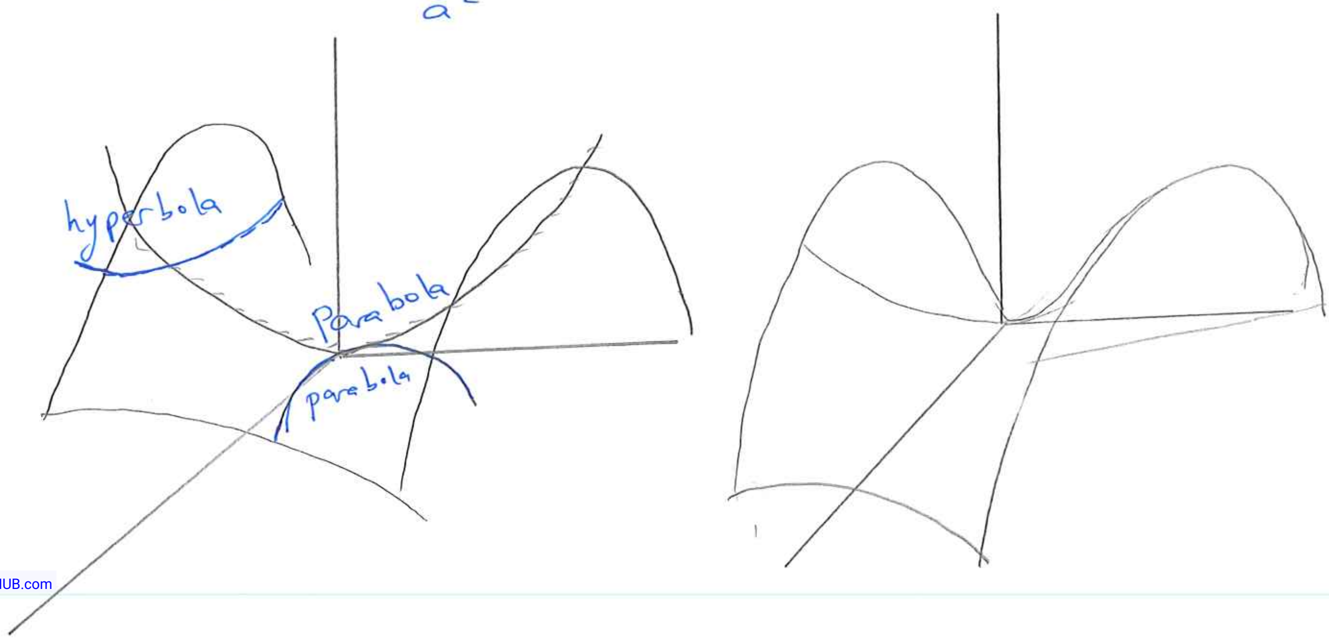
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{Z_0}{c}$$

with its focal axis parallel to the y-axis

& its vertices on the parabola $Z = \frac{c}{b^2} y^2$.

- If $Z_0 < 0$, the focal axis is parallel to the x-axis and the vertices lie on the parabola

$$Z = -\frac{c}{a^2} x^2.$$

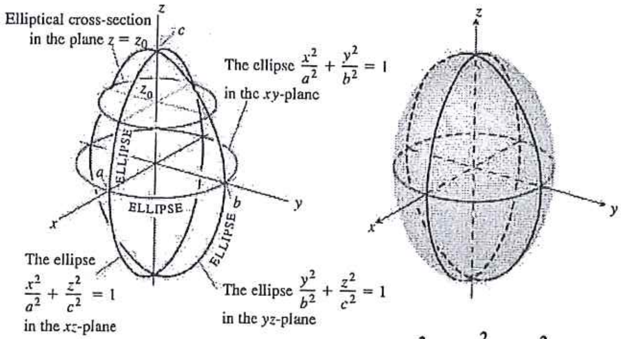


Note: Near the Origin, the surface is shaped like a saddle point since:

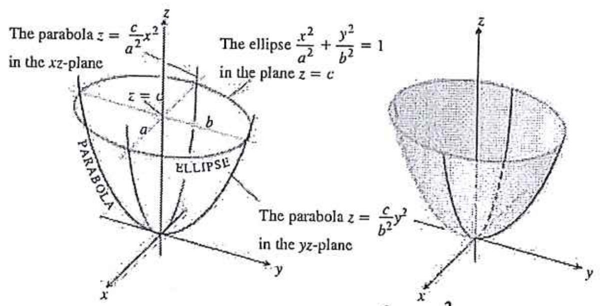
(Traveling along the surface in yz-plane, the Origin is Min)

(" " " " in xz-plane, the origin is Max).

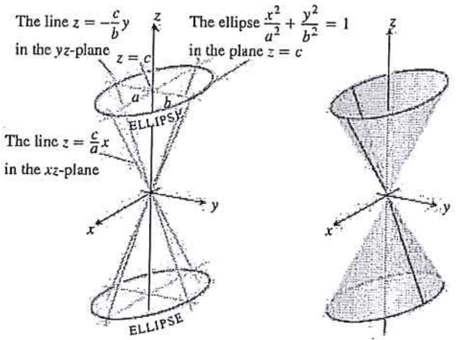
TABLE 12.1 Graphs of Quadric Surfaces



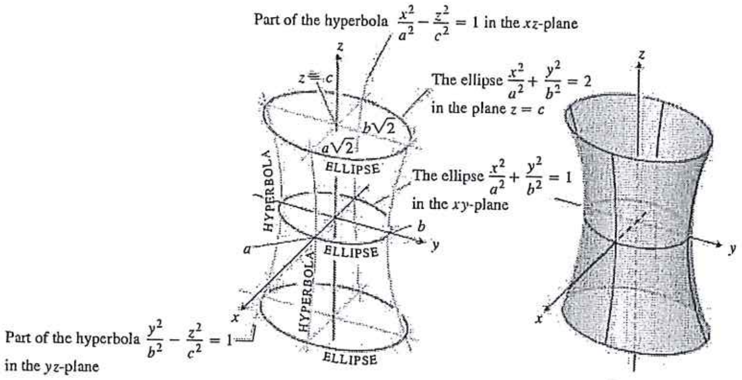
ELLIPSOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



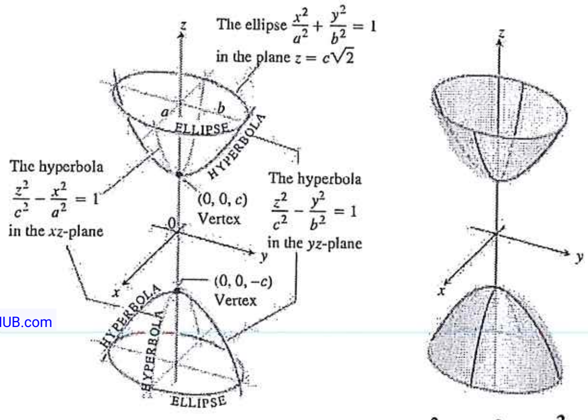
ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$



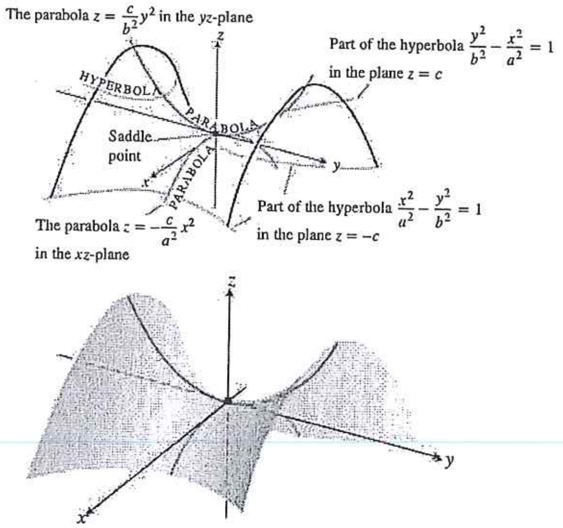
ELLIPTICAL CONE $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}, \quad c > 0$

(127)