Chapter 4

Population
$\lambda, \mu_x, \sigma_x^2, \sigma_x, b$

Random Sample

$x_1, x_2, x_3 \ldots x_n$

$\Rightarrow$ Size of sample is n

$\hat{\mu}_x \rightarrow$ sample mean

$\hat{\sigma}_x^2 \rightarrow$ sample var

$\hat{\sigma}_x \rightarrow$ sample s.d

$= \hat{b}, \hat{\lambda}$

$\Rightarrow$ Sample mean $\hat{\mu}_x = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\rightarrow$ Sample var. $S_x^2, \hat{\sigma}_x^2$

$$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_x)^2$$

μ_x = true mean

True mean is known

$$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu}_x)^2$$

True Mean is unknown

$\rightarrow$ Sample standard deviation $= S_x, \hat{\sigma}_x$

$$S_x = \hat{\sigma}_x = \sqrt{S_x^2} = \sqrt{\hat{\sigma}_x^2}$$

$\rightarrow$ Sample coveriance $= \hat{\mu}_{x,y} = C_{xy}$

$$C_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu}_x)(y_i - \hat{\mu}_y)$$

$\rightarrow$ Sample correlation coefficient $= r_{xy}$

$$r_{xy} = \frac{C_{xy}}{S_x S_y}$$

$\rightarrow$ Sample var

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left[x_i^2 - 2x_i\hat{\mu}_x + \hat{\mu}_x^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - 2\hat{\mu}_x\left(\sum_{i=1}^{n} x_i\right) + \hat{\mu}_x \sum_{i=1}^{n} 1\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i - 2\hat{\mu}_x n \hat{\mu}_x + \hat{\mu}_x^2 n\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - 2n\hat{\mu}_x^2 + n\hat{\mu}_x^2\right] = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2\right]$$

$$\Rightarrow \quad S_x^2 = \frac{1}{n(n-1)}\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]$$

$$\Rightarrow \quad C_{xy} = \frac{1}{n[n-1]}\left[n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\right]$$

sample covariance

⇒ Regression Techniques

x_i	x_1	x_2	x_3	x_4	--	x_n
y_i	y_1	y_2	y_3	y_4	--	y_n
$=g(x)$	$g(x_1)$	$g(x_2)$				$g(x_n)$

y_n ← practical, measured

$\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$, $\langle x_n, y_n\rangle$ — random sample

$g(x_n)$ ← theoratical

⇒ Scatter plot

$y = \alpha x + \beta$

$g(x)$

$$\Rightarrow E = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - g(x_i)\right)^2 \leftarrow \text{mean square error}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \alpha x_i - \beta\right)^2$$

$$\frac{dE}{d\beta} = 0 \longrightarrow \frac{1}{n}\sum_{i=1}^{n} 2(y_i - \alpha x_i - \beta)(-1) = 0$$

$$\longrightarrow -\sum_{i=1}^{n} y_i + \alpha \sum_{i=1}^{n} x_i + \beta n = 0$$

$$\boxed{\beta n + \alpha \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i} \text{ — } ①$$

$$\frac{dE}{d\alpha} = 0 \longrightarrow \frac{1}{n}\sum_{i=1}^{n} 2(y_i - \alpha x_i - \beta)(-x_i) = 0$$

$$\longrightarrow -\sum_{i=1}^{n} y_i x_i + \alpha \sum_{i=1}^{n} x_i^2 + \beta \sum_{i=1}^{n} x_i = 0$$

$$\boxed{\beta \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i} \text{ — } ②$$

$$Y = \alpha X^1 + \beta X^0$$

① ②

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\beta = \frac{\begin{vmatrix} \sum y_i & \sum x_i \\ \sum x_i y_i & \sum x_i^2 \end{vmatrix}}{\begin{vmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{vmatrix}} = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$y = \alpha x + \beta$$

$$\beta = \hat{\mu}_y = \alpha \hat{\mu}_x + \beta$$

$$\beta = \hat{\mu}_y - \alpha \hat{\mu}_x$$

$$\alpha = \frac{\begin{vmatrix} n & \sum y_i \\ \sum x_i & \sum x_i y_i \end{vmatrix}}{\begin{vmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{vmatrix}} = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\downarrow$$

$$\frac{n(n-1)\, C_{xy}}{n(n-1)\, S_x^2}$$

$$\alpha = \frac{C_{xy}}{S_x^2}$$

$$\Rightarrow Y = \alpha X + \beta$$

$$Y = \frac{C_{xy}}{S_x^2} X + \hat{\mu}_Y - \frac{C_{xy}}{S_x^2} \hat{\mu}_X$$

$$\left(Y - \hat{\mu}_Y = \frac{C_{xy}}{S_x^2} \left[X - \hat{\mu}_X \right] \right) \times \frac{1}{S_y}$$

$$\frac{Y - \hat{\mu}_Y}{S_Y} = \frac{C_{xy}}{S_x S_y} \left[\frac{X - \hat{\mu}_X}{S_x} \right] \rightarrow \frac{Y - \hat{\mu}_Y}{S_Y} = r_{xy} \left(\frac{X - \hat{\mu}_X}{S_x} \right)$$

→ Engineering statistics

population μ_x → random sample size n

$$\hat{\mu}_x = \underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i}_{\text{estimator}}$$

Regression techniques:-

$X \quad x_1 \quad x_2 \quad x_3 \dots x_n$

$Y \quad y_1 \quad y_2 \quad y_3 \dots y_n$

<u>Example1</u> let the values in the table bellow represent a sample from a population

X	1	2	2.5	2.8	4	
Y	2	3	3	3.2	5.5	(measured)
$g(x_i)$						(theoritical)

$\beta = 0.602028$

$\alpha = 1.1122996$

→ a) find the best fit line $Y = \alpha X + \beta$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\begin{bmatrix} 5 & 12.3 \\ 12.3 & 35.09 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 16.7 \\ 46.46 \end{bmatrix}$$

$$\beta = \frac{\begin{vmatrix} 16.7 & 12.3 \\ 46.46 & 35.09 \end{vmatrix}}{\begin{vmatrix} 5 & 12.3 \\ 12.3 & 35.09 \end{vmatrix} \to 24.16} = 0.602028$$

$$\alpha = \frac{\begin{vmatrix} 5 & 16.7 \\ 12.3 & 46.46 \end{vmatrix}}{24.16} = 1.1129966$$

$$Y = 1.1129966X + 0.602028$$

⇒ (B) Assume a new $x = 1.5$ is provided use answer of A to determine the value of y for $x = 1.5$

$$Y = 1.1129966(1.5) + 0.602028 = 2.271522$$

(C) Determine the ~~...~~ MSE (mean square error) between the values in the table and the best fit line

$$MSE = \frac{1}{n}\sum_{i=1}^{n}\left(Y_{measeurd} - Y_{theo}\right)^2$$

$Y = \alpha X + \beta \rightarrow$ نعوض بال x اصلا

$$\Rightarrow \frac{0.726823862}{5} = 0.145364772$$

⇒ ازا كانت

$Y = \beta X + \alpha$ هيك

$$\begin{bmatrix} \\ \end{bmatrix}\begin{bmatrix} \cancel{\alpha} \\ \beta \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

ع س

(d) Determine the Best fit polynomial curve

$$y = ax^2 + bx + c$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} * \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

تكملة الحل Ex. d

$\Rightarrow g(x_i) = \alpha x_i + \beta$ $\qquad g(a) = a^2 + 2xa - y$

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - g(x_i))^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(y_i - ax^2 - bx - c)^2$$

$$= \frac{1}{5}\left[(2 - a(1)^2 - b(1) - c)^2 + (3 - (2)^2 a - 2b - c)^2 + \cdots\right]$$

$g(x) = ax^b$

$\Rightarrow$ Ex. $g(x_i) = \alpha x_i + \beta$

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - g(x_i))^2 = \frac{1}{n}\sum_{i=1}^{\infty}(y_i - ax_i^b)^2$$

⇒ Note :- Linearization

$Y = ax^b$

$\ln[Y] = \ln[ax^b]$

$\ln[Y] = \ln[a] + b\ln[x]$; $u = \ln y$, $r = \ln x$

$u = \beta + \alpha r$

⇒

x	1	2	2.5	2.8	4
y	2	3	3	3.2	5.5
$r = \ln x$	ln1	ln2	ln2.5	ln2.8	ln4
$u = \ln y$	ln2	ln3	ln3	ln3.2	ln5.5

→ بعدين بعوض هدول القيم في ال matrix

$\ln[a] = \beta$ بالأخير بعوض

$a = e^{\beta}$

$b = \alpha$

⇒ Ex₂) $Y = 1 + ax^b$

$Y - 1 = ax^b$

$\ln|Y-1| = \ln(a) + b\ln(x)$

$Y - 1 = ax^b$

$\ln[Y-1] = \ln[a] + b\ln[x]$

Central limit theorem :-

population μ_x

Sample size n

$x_1, \dots, x_n$

$\hat{\mu}_x = \frac{1}{n} \sum_{i=0}^{n} x_i$

Random variable

$\hat{\mu}_x$ is a random variable with Gaussian Distribution

$E\{\hat{\mu}_x\} = \mu_x$ ← true mean for the population

$Var\{\hat{\mu}_x\} = \frac{\sigma_x^2}{n}$ ← true variance for population

n: size of the sample

Note :- X_1 and X_2 are S. Independent R.V.S

$Y = X_1 + X_2$

$E\{Y\} = E\{X_1\} + E\{X_2\}$

$Var\{Y\} = \sigma_{x_1}^2 + \sigma_{x_2}^2$

→ $Var\{aX + bY\}$:

$= a^2\sigma_x^2 + b^2\sigma_y^2 + \underbrace{2ab\sigma_x\sigma_y}_{2ab\mu_{xy}}$

Example / :- X_1 and X_2 are two samples from a population with $\mu_x = 2$ and $\sigma_x^2 = 16$ let $Y = \frac{X_1 + X_2}{2}$ Determine the mean and variance of Y ?

$E\{Y\} = E\{\frac{1}{2}X_1 + \frac{1}{2}X_2\} = \frac{1}{2}E(X_1) + \frac{1}{2}(X_2) = \frac{1}{2}\mu_x + \frac{1}{2}\mu_x$

$= \mu_x$

$Var\{Y\} = \left(\frac{1}{2}\right)^2 Var\{X_1\} + \left(\frac{1}{2}\right)^2 Var\{X_2\} + 0$ ← X_1 and X_2 are S. Independent (Random sample)

$= \left(\frac{1}{2}\right)^2 \sigma_x^2 + \left(\frac{1}{2}\right)^2 \sigma_x^2 = \frac{2\sigma_x^2}{4} = \boxed{\frac{\sigma_x^2}{2}}$

$= \frac{16}{2} = \boxed{8}$

Example | $Y = \frac{X_1 + X_2 + X_3}{3}$ نفس السؤال ولكن المعادلة

$\Rightarrow E\{Y\} = \frac{1}{3}E\{X_1\} + \frac{1}{3}E\{X_2\} + \frac{1}{3}E\{X_3\}$

$\mu_x \quad \mu_x \quad \mu_x$

لازم نفس الـ population

$= \frac{1}{\cancel{3}}(\cancel{3}\mu_x) = \boxed{2}$

$Var\{Y\} = \frac{1}{9}Var\{X_1\} + \frac{1}{9}Var\{X_2\} + \frac{1}{9}Var\{X_3\} = \frac{3}{9}\sigma_x^2$

σ_x^2

$= \frac{\sigma_x^2}{3}$

Example | (5-10)

X = resistors

$\mu_x = 100, \sigma_x = 10$

random $n = 25$

$X_1, X_2 \dots X_{25}$

$P(\hat{\mu}_x < 95)$

C.L.T : $\hat{\mu}_x$ is gaussian

$E\{\hat{\mu}_x\} = \mu_x = 100$

$Var\{\hat{\mu}_x\} = \frac{\sigma_x^2}{n} = \frac{(10)^2}{25} = 4$

Average [illegible] لما ←

$\rightarrow P(\hat{\mu}_x < 95) = \phi\left(\frac{95-100}{\sqrt{4}}\right) = \phi\left(\frac{-5}{4}\right)$

$P\left(\frac{1}{n}\sum_{i=1}^{25} X_i < 95\right)$

(b) Determine the probability that a random sample of size n has a total resistance less than 2375Ω

$Y = X_1 + X_2 \dots X_{25}$

$P(Y < 2375) = P\left(\sum_{i=1}^{25} x_i < 2375\right)$

$= P\left(\frac{1}{25}\sum_{i=1}^{25} x_i < \frac{2375}{25}\right)$

$P(\hat{\mu} \dots$

⇒ Properties of point estimator.

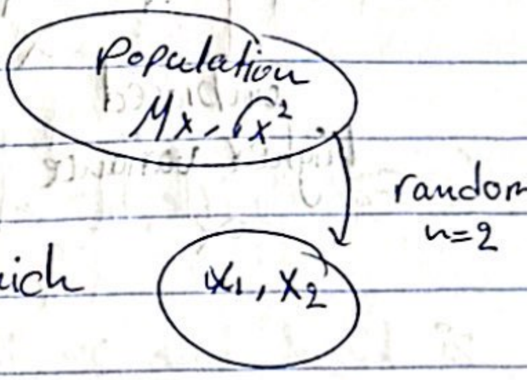
Example: let X_1 and X_2 be a random Sample from a population with a mean μ_X and variance σ_X^2. three estimators are proposed for the mean.

$$\hat{\mu}_1 = \frac{X_1 + 2X_2}{3} \quad , \quad \hat{\mu}_{2} = \frac{X_1 + 2X_2}{3} \quad , \quad \hat{\mu}_3 = \frac{X_1 + X_2}{2}$$

(1) Unbiased: estimator unbiased if

$E\{\hat{\mu}_2\} = \mu_X$

$B = E\{\hat{\mu}_X\} - \mu_X \quad , \quad B = 0$ (unbiased)



⇒ which estimator is better and in which sense?

$$E\{\hat{\mu}_1\} = \frac{1}{3} E\{X_1\} + \frac{2}{3} E\{X_2\} = \mu_X \quad \checkmark \text{ unbiased}$$

$$E\{\hat{\mu}_2\} = \frac{1}{3}\mu_X + \frac{2}{3}\mu_X + 1 = \mu_X + 1 \quad \text{a biased}$$

$$E\{\hat{\mu}_3\} = \frac{1}{2}\mu_X + \frac{1}{2}\mu_X = \mu_X \quad \checkmark \text{ unbiased}$$

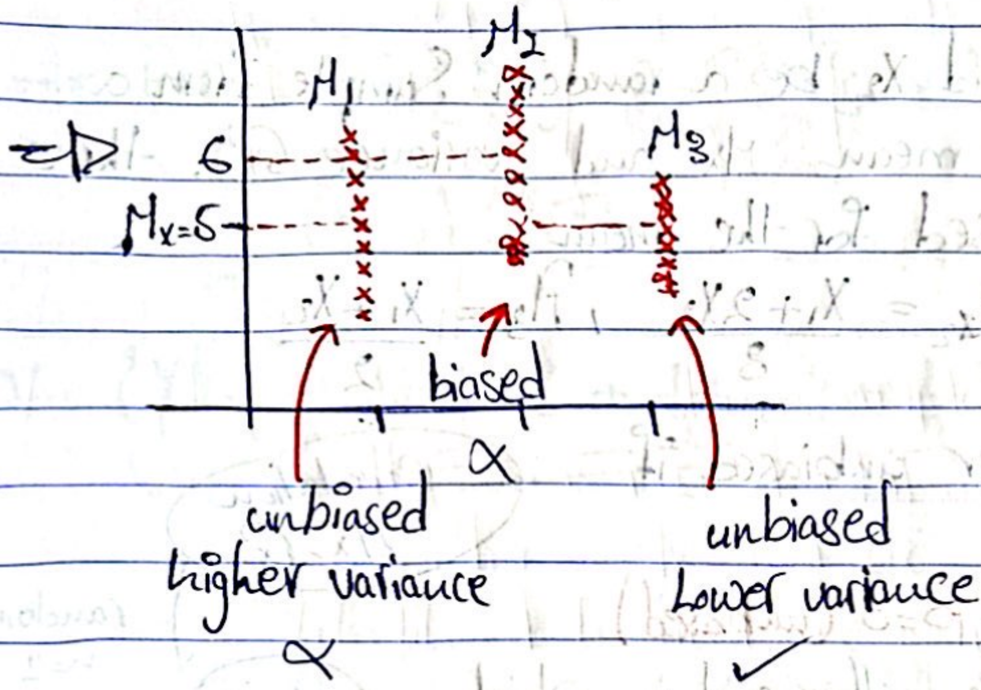
(2) For an unbiased estimator, we prefer the one with minimum variance (more efficient)

$$\Rightarrow \mathrm{Var}\{\hat{\mu}_1\} = \frac{1}{9}\sigma_X^2 + \frac{4}{9}\sigma_X^2 = \frac{5}{9}\sigma_X^2$$

$$\mathrm{Var}\{\hat{\mu}_3\} = \frac{1}{4}\sigma_X^2 + \frac{1}{4}\sigma_X^2 = \frac{1}{2}\sigma_X^2$$

So μ_3 is more efficient than μ_1 because it has loowe variance

(3) $MSE = E\{(\hat{\mu}_x - \mu_x)^2\} = Var\{\hat{\mu}_x\} + B$



Example 6-1

① p(H) = 0.1   ② p(H) = 0.9

randomly

? tossed 10 times

a) and 3 heads are observed

a) → Binomial distribution :-

$\rightarrow p(X=3 ; p(H)=0.1) = \binom{10}{3} p(H)^3 (1-p)$

$= 0.05739$

$\rightarrow p(X=3 ; p(H)=0.9) = \binom{10}{3} p(H)^3 (1-p)$

$= 8.748 \times 10^{-6}$

→ الجواب هي الأولى (0.05739)

b) Assume the coin is flipped 10 times and head is observed in 6 trials . what is the likelihood of the valu of P(H)?

$\rightarrow p(X=6 ; p(H)=0.1) = \binom{10}{6} (0.1)^6 (0.9)^4 = 1.3778 \times 10^{-4}$

$p(X=6 ; p(H)=0.9) = \binom{10}{6} (0.9)^6 (0.1)^4 = 0.011160$ ✓

Example → tossed 10 times and head is observed in 3 trials what is the likelihood of the value of p(H)?

max بشتق وبساوي بالصفر عشان الاقي ال ←

$p(X=3, p) = \binom{10}{3} p^3 (1-p)^7 = 120 p^3 (1-p)^7$

$\Rightarrow \frac{d\, P(X=3, p)}{dp} = 120 \left[p^3 \times 7 (1-p)^6 (-1) + (1-p)^7 3p^2 \right] = 0$

① **Case 1:-** $L(\hat{\mu}_x) = \frac{-n}{2} \ln(2\pi\sigma_x^2) - \frac{1}{2\sigma_x^2} \sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2$

$$\frac{\partial L(\hat{\mu}_x)}{\partial \hat{\mu}_x} = \underset{\text{known } \sigma_x^2 \text{ لأن}}{0} - \frac{1}{2\sigma_x^2} \sum_{i=1}^{n} 2(x_i - \hat{\mu}_x)(-1) = 0$$

(بساويه بالصفر)

$$\rightarrow \frac{\cancel{-1}}{\cancel{2\sigma_x^2}} \sum_{i=1}^{n} \cancel{2}(x_i - \mu_x)(\cancel{-1}) = 0$$

$$= \sum_{i=1}^{n} (x_i - \hat{\mu}_x) = 0$$

$$= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \hat{\mu}_x = 0$$

$$= \sum_{i=1}^{n} x_i - n\hat{\mu}_x = 0 \longrightarrow \hat{\mu}_x = \frac{1}{n} \sum_{i=1}^{n} x_i$$

ⓑ notes

③ Case 3: for the variance if the mean is ~~un~~known

$$L(\hat{\sigma}_x^2) = -\frac{n}{2} \ln(2\pi\hat{\sigma}_x^2) - \frac{1}{2\hat{\sigma}_x^2} \sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2$$

$$\frac{\partial L(\hat{\sigma}_x^2)}{\partial \hat{\sigma}_x^2} = \frac{-n}{2} \cdot \frac{2\pi}{2\pi\hat{\sigma}_x^2} - \frac{1}{2}\left(\frac{-1}{(\hat{\sigma}_x^2)^2}\right) \sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2 = 0$$

$$\frac{n}{2} \frac{1}{\hat{\sigma}_x^2} = \frac{1}{2(\hat{\sigma}_x^2)^2} \sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2$$

$$\hat{\sigma}_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu}_x)^2$$

ⓒ نفس b → Mean is unknown.

بس $\hat{\mu}_x$

$\Rightarrow 7p^{\cancel{8}} \cdot \cancel{(1-p)^{6}} = 3\cancel{p^{7}}(1-p)^{7}$

$7p = 3(1-p) \Rightarrow 3 - 3p = 7p$

max لما نساوي بال ← $\boxed{p = \frac{3}{10}}$

لما نجربه
كل التوزيعات عنده
قيم معروفة
أو بنشتقه

Example Given a random sample from a population with Gaussian Distribution with parameters μ_x and σ_x^2. Use the Maximum Likelihood to find an estimator for each of the following cases:-

[1] for the mean if the variance is known.

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x_1-\mu_x)^2}{2\sigma_x^2}}$$

Population
μ_x σ_x^2
unknown known

random size=n

$X_1, X_2 \dots X_n$

$$f_{X_2}(x_2) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x_2-\mu_x)^2}{2\sigma_x^2}}$$

$\vdots$

$$f_{X_n}(x_n) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x_n-\mu_x)^2}{2\sigma_x^2}}$$

μ_x ?

$$\Rightarrow f_{X_1, X_2 \dots X_n}(x_1, x_2 \dots x_n) = \left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right)^n e^{\frac{-1}{2\sigma_x^2}\sum_{i=1}^{n}(x_i-\mu_x)^2}$$

ln ln

$$= \frac{-n}{2}\ln(2\pi\sigma_x^2) - \frac{1}{2\sigma_x^2}\sum_{i=1}^{n}(x_i-\mu_x)^2$$

<u>Recall:</u>

① $\sigma_x^2 = E\{(x-\mu_x)^2\}$

$\sigma_x^2 = E\{x^2\} - (E\{x\})^2$ / $\sigma_x^2 = E\{x^2\} - \mu_x^2$

② $E\{\hat{\mu}_x\} = \mu_x$

③ $Var\{\hat{\mu}_x\} = \frac{\sigma_x^2}{n}$

For the mean

ⓐ variance is known

$$\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i$$

variance is unknown

$$\hat{\mu}_x = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$E\{\hat{\mu}_x\} = E\left\{\frac{1}{n}\sum_{i=1}^{n} x_i\right\} =$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\{x_i\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu_x = \frac{1}{\cancel{n}} * \cancel{n}\mu_x = \mu_x$$

So, $\hat{\mu}_x = \frac{1}{n}\sum_{i=1}^{n} x_i$ are unbiased estimator for the mean.

For the variance

ⓑ mean known

$$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_x)^2$$

ⓒ mean unknown

$$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_x$$

تحت

$$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_x)^2$$

$$E\{\hat{\sigma}_x^2\} = E\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_x)^2\right\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\{(x_i - \mu_x)^2\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \sigma_x^2 = \frac{1}{\cancel{n}} * \cancel{n}\sigma_x^2 = \sigma_x^2$$

So, $\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_x)^2$ is an unbiased estimator for the variance when the mean is known.

ⓒ mean unknown

$$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_x)^2$$

$$E\{\hat{\sigma}_x^2\} = E\left\{\frac{1}{n}\sum(x_i - \hat{\mu}_x)^2\right\}$$

$$= E\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i^2 - 2x_i\hat{\mu}_x + \hat{\mu}_x^2)\right\}$$

$$= E\left\{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{2}{n}\sum_{i=1}^{n} x_i \hat{\mu}_x + \frac{1}{n}\sum_{i=1}^{n} \hat{\mu}_x^2\right\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\{x_i^2\} - 2E\{\hat{\mu}_x^2\} + E\{\hat{\mu}_x^2\}$$

$$= \frac{1}{\cancel{n}} * \cancel{n}\, E\{x_i^2\} - E\{\hat{\mu}_x^2\}$$

$$= \sigma_x^2 + \mu_x^2 - \frac{\sigma_x^2}{n} - \mu_x^2 = \sigma_x^2 - \frac{\sigma_x^2}{n} = \frac{n-1}{n}\sigma_x^2$$

$\Rightarrow$ note:- $\sigma_x^2 = E\{x^2\} - (E\{x\})^2$

$= E\{\hat{\mu}_x^2\} - (E\{\hat{\mu}_x\})^2$

$\frac{\sigma_x^2}{n} = E\{\hat{\mu}_x^2\} - \mu_x^2$

Last Lecture :-

Population
$\mu_x, \sigma_x^2, P, \lambda$

random sample

$x_1, x_2, \ldots, x_n$ size of the sample $= n$

$\hat{\mu}_x$ → point estimator of the true mean "mean"

⇒ Interval estimator
θ = any parameter missing
$= \mu_x, \sigma_x^2, P, \lambda, b$

مثلاً → $P(\underset{\text{Lower}}{\theta_1} \leq \underset{\text{true mean}}{\mu_x} \leq \underset{\text{upper}}{\theta_2})$

- Confidence التأكد بنسبة معينة

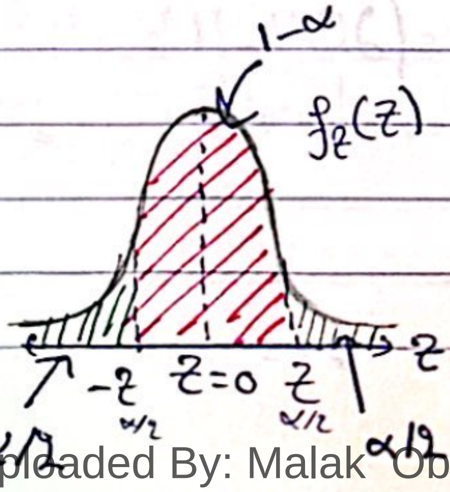
⇒ $P(\theta_1 \leq \theta \leq \theta_2) = 1-\alpha$

where :- θ → unknown parameter
$(1-\alpha)$ → confidence coefficient
α → Is called the confidence level
θ_1, θ_2 → are the lower and the upper con. Limit

⇒ Confidence Interval on the mean (variance known)
(Gaussian distribution)

standard gaussian random variable ← $Z = \dfrac{\hat{\mu}_x - E\{\hat{\mu}_x\}}{\sqrt{\text{var}\{\hat{\mu}_x\}}}$

⇒ $\mu_Z = 0, \sigma_Z^2 = 1$



$$P(-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}) = 1-\alpha \quad , \quad P(\underset{\text{S.G.R}}{Z} \leq Z_{\alpha/2}) = 1-\alpha+\frac{\alpha}{2}$$

$$P\left(-Z_{\alpha/2} \leq \frac{\hat{\mu}_x - \mu_x}{\sqrt{\frac{\sigma_x^2}{n}}} \leq Z_{\alpha/2}\right) = 1-\alpha \quad \boxed{\phi(Z_{\frac{\alpha}{2}}) = 1-\frac{\alpha}{2}}$$

$$P\left(-Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}} \leq \hat{\mu}_x - \mu_x \leq Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}}\right) = 1-\alpha$$

$$P\left(-\hat{\mu}_x - Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}} \leq -\mu_x \leq -\hat{\mu}_x + Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}}\right) = 1-\alpha$$

$$P\left(\underbrace{\hat{\mu}_x - Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}}}_{\Theta_1} \leq \mu_x \leq \underbrace{\hat{\mu}_x + Z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n}}}_{\Theta_2}\right) = 1-\alpha$$

$$P(\Theta_1 \leq \mu_x \leq \Theta_2) = 1-\alpha$$

confidence limits

Example 1 The following data is obtained from a gaussian population.

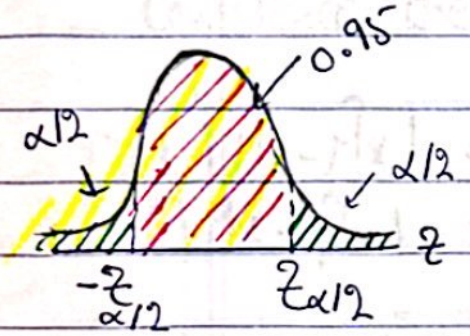
7.31, 10.8, 11.27, 11.91, 5.51, 8.00, 9.03
14.42, 10.24, 10.91

a) Find the 95% confendence interval for the mean if the true variance of the population is 4.

$\sigma_x^2 = 4$ (variance known → case 1)

confedence interval of the mean → $P(\Theta_1 \leq \mu_x \leq \Theta_2) = 0.95$

$0.95 = 1-\alpha$



$\Rightarrow 1-\alpha = 0.95$

$\alpha = 0.05$

$\Rightarrow \Phi(z_{\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2} = 1 - 0.025 = \boxed{0.975}$

برجع للجدول عشان الاقي $z_{\alpha/2}$

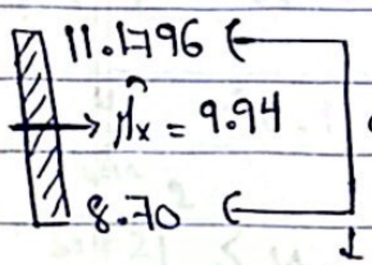
$z_{\alpha/2} = \underline{1.96}$

$\rightarrow P\left(\hat{\mu}_x - z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}} \leq \mu_x \leq \hat{\mu}_x + z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}\right) = 0.95$

$n = 10$, $\sigma_x = \sqrt{4} = 2$, $\hat{\mu}_x = \frac{1}{n}\sum_{i=1}^{n} x_i = 9.94$

بعد التعويض ← $P(8.70 \leq \mu_x \leq 11.1796) = 0.95$

confendence interval



في هاد ال interval

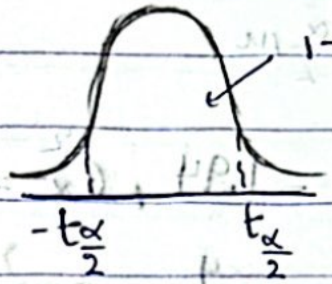
→ confidence interval for the error

$$P\left(-1.96 \cdot \sqrt{\frac{4}{10}} \le \mu_x - \hat{\mu}_x \le 1.96 \cdot \sqrt{\frac{4}{10}}\right) = 0.95$$

→ note 2 absolute value for the error

$$P\left(|\mu_x - \hat{\mu}_x| \le 1.96 \cdot \sqrt{\frac{4}{10}}\right) = 95\%$$

Same example (b) Find the 95% confidence interval for the mean?

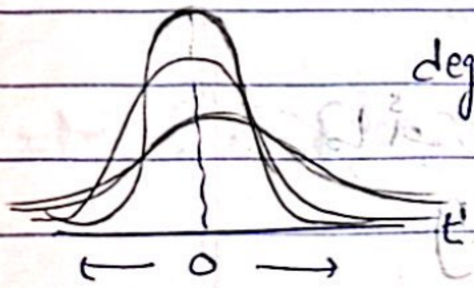


→ الاشتقاق في ال notes

⇒ T – distribution

$$P\left(\overbrace{\hat{\mu}_x - t_{\frac{\alpha}{2}, n-1}\frac{\hat{\sigma}_x}{\sqrt{n}}}^{\Theta_1} \leq \mu_x \leq \overbrace{\hat{\mu}_x + t_{\frac{\alpha}{2}, n-1}\frac{\hat{\sigma}_x}{\sqrt{n}}}^{\Theta_2}\right) = 1-\alpha$$

$n = 10$, $\hat{\mu}_x = 9.94 = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{10}(x_i - \hat{\mu}_x) = 6.51$



degree of freedom = $n-1$
$= 10 - 1 = 9$

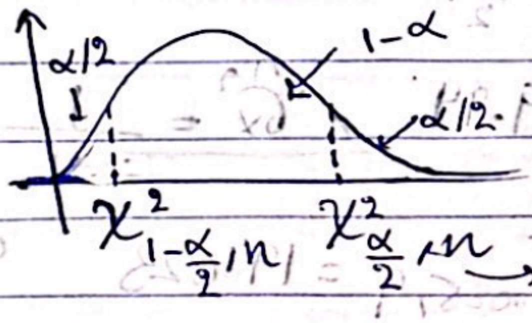
$\alpha/2 = 0.025 \longrightarrow t_{\frac{\alpha}{2}, 9} = 2.262$

from T – distibution table

$P(8.11 \leq \mu_x \leq 11.77) = 0.95$

(c) Find the 95% confidence interval for the variance if the true mean of the population is 10

χ^2 chi – distribution
square



→ degree of freedom

Confidence Interval on a Binomial proportion :-

Population Binomial
p , probability of success

(notes) احفظها ←

n

X / n-X

number of item with success

$\hat{p} = \frac{x}{n}$

$$P\left\{\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right\} = 1-\alpha$$

See Example (6-9)