1 State transition Matrix

Tuesday, April 27, 2021 11:35 AM

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\} \text{ — } (*)$$

Homogeneous Solution (Zero-input Solution)

IC = x(0) exist (u=0)

$\boxed{\dot{x}(t) = A\,x(t)}$, $y(t) = Cx$ ← IC, (u=0)

take laplace transformation

$$s\,X(s) - x(0) = A\,X(s)$$

$$[sI - A]\,X(s) = x(0) \Rightarrow X(s) = (sI - A)^{-1}\,x(0)$$

s-domain

based on that, let's take laplace inverse

$$x(t) = \boxed{\mathcal{L}^{-1}(sI - A)^{-1}}\,x(0) \text{ — } (1)$$

→ State trans. matrix

$$\mathcal{L}^{-1}(sI - A)^{-1} = \phi(t) \triangleq e^{At}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \overbrace{\frac{f_0}{m}\sin(\omega t)}^{F(t)=}$$

Forced vib + damped

Complementary — steady state solution (Particular)

$$\ddot{x}_c + \frac{c}{m}\dot{x}_c + \frac{k}{m}x = 0 \Leftarrow \text{free + damped sys}$$

IC x(t)

$z_1 = x$

$z_2 = \dot{x}$

$\dot{z}_1 = z_2$ — (1)

$\dot{z}_2 = -\frac{c}{m}z_2 - \frac{k}{m}z_1$ — (2)

Tuesday, April 27, 2021 11:54 AM

Ex: Compute the homogenous solution of $x(t)$ for the sys shown below

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \quad \text{if } x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$t=0$

$$x(t) = \mathcal{L}^{-1} (SI-A)^{-1} x(0)$$

$$[SI-A] = S \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ -2 & S-3 \end{bmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} S-3 & 1 \\ 2 & S \end{bmatrix} \frac{1}{S^2-3S-2}$$

$$\phi(s) = \mathcal{L}^{-1} (SI-A)^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{S-3}{S^2-3S-2} & \frac{1}{S^2-3S-2} \\ \frac{1}{S^2-3S-2} & \frac{S}{S^2-3S-2} \end{bmatrix}$$

$$\phi(t) = \text{ilaplace}(\phi(s))$$

$$x(t) = \begin{bmatrix} \phi(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

when $t=0$

Tuesday, April 27, 2021 12:10 PM

Non-Homogeneous Solution ↯

$\dot{x}(t) = Ax + Bu$
$y(t) = Cx + Du$

solve the problem under the effect of input force

IC = 0

take laplace

$sX(s) - \cancel{x(0)} = A\,x(s) + B\,u(s)$

$(sI-A)^{-1}\,x(s) = B\,u(s)$

$X(s) = (sI-A)^{-1}\,B\,u(s)$

Particular Solution

take the inverse of laplace

$x(t) = \mathcal{L}^{-1}(sI-A)^{-1}\,B\,u(s)$

Total Solution IC is given $\neq 0$ also $u \neq 0$

$\dot{x}(t) = A\,x(t) + B\,u(t)$

take laplace

$s\,X(s) - X(0) = A\,x(s) + B\,u(s)$

$(sI-A)\,x(s) = X(0) + Bu(s)$

$x(s) = (sI-A)^{-1}\,x(0) + (sI-A)^{-1}\,Bu(s)$

to find the solution in time domain lets take laplace inverse

$x(t) = \mathcal{L}^{-1}[(sI-A)^{-1}]\,x(0) + \mathcal{L}^{-1}[(sI-A)^{-1}\,B\,u(s)]^{-1}$

$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,B\,u(\tau)\,d\tau$

Tuesday, April 27, 2021 12:23 PM

if you start from $t=0$

$$x(t) = \phi(t)\, x(0) + \int_0^t \phi(t-\tau)\, B\, u(\tau)\, d\tau \quad \text{—} (*)$$

$$x(t) = \phi(t-t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t-\tau)\, B\, u(\tau)\, d\tau \quad \text{—} (**)$$

if you start from $t=t_0$

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$\boxed{y(t) = C\,x(t) + D\,u(t)}$$

SSR

lets sub Eq (*) & (**) into the Second Eq of SSR

$$y(t) = C\left[\phi(t-t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t-t_0)\, B\, u(\tau)\, d\tau\right] + D\,u(t)$$

$$= \underbrace{C\,\phi(t-t_0)\, x(t_0)}_{\text{Zero input responce}} + \underbrace{\int_{t_0}^{t} C\,\phi(t-t_0)\, B\, u(\tau)\, d\tau + D\,u(t)}_{\text{Zero state responce}}$$

Zero input responce

= "free Vibrations"

the motion under the effect of IC $(u=0)$

Zero state responce

= "forced Vib."

= "Particular" solution

the motion under the effect of input force $u(t)$

$IC = 0$

Tuesday, April 27, 2021 12:33 PM

Example find the Zero input response for the following Sys

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{if } x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

for the Zero_input respone $u(t) = 0$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1} (sI - A)^{-1} x(0)$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{s^2 + 3s + 2}$$

$$\mathcal{L}^{-1} (sI - A)^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

$$x(t) = \underbrace{\qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = R$$

$$y(t) = C\,x(t)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} * R \leftarrow \text{Zero input response}$$

Consider the system which is shown below :-

$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

find the responce & the system x(t) if the input force is unit step $(u(t) = 1)$.

$$x(t) = \underbrace{\phi(t)\, x(0)}_{\substack{\text{Free responce} \\ \text{IC}}} + \underbrace{\int_0^t \phi(t-T)\, B\, u(T)\, dT}_{\substack{\text{Forced responce} \\ u(t)}}$$

$\phi(t) = \mathcal{L}^{-1}(SI-A)^{-1} = e^{(At)}$ " transition matrix

$$(SI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}$$

$$\phi(s) = (SI-A)^{-1} = \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \frac{1}{s^2+6s+8}$$

$$\phi(t) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{s+6}{s^2+6s+8} & \frac{1}{s^2+6s+8} \\ \frac{-8}{s^2+6s+8} & \frac{s}{s^2+6s+8} \end{bmatrix}\right)$$

syms t

Phi = expm(A * t) " time domain "

$$\phi(t) = \begin{bmatrix} 2e^{-2t} - e^{-4t} & \frac{e^{-2t}}{2} - \frac{e^{-4t}}{2} \\ 4e^{-4t} - 4e^{-2t} & 2e^{-4t} - e^{-2t} \end{bmatrix}$$

Thursday, April 29, 2021 11:29 AM

$$\phi(t)\,x(0)=\begin{bmatrix} 2e^{-2t}-e^{-4t} & , & \frac{e^{-2t}}{2}-\frac{e^{-4t}}{2} \\ 4e^{-4t}-4e^{-2t} & , & 2e^{-4t}-e^{-2t} \end{bmatrix}_{2\times2}\begin{bmatrix}1\\0\end{bmatrix}_{2\times1}$$

$$\phi(t)\,x(0)=\begin{bmatrix} 2e^{-2t}-e^{-4t} \\ 4e^{-4t}-4e^{-2t} \end{bmatrix}$$

$$\phi(t-\tau)\begin{bmatrix} 2e^{-2(t-\tau)}-e^{-4(t-\tau)} & , & \frac{e^{-2(t-\tau)}}{2}-\frac{e^{-4(t-\tau)}}{2} \\ -4e^{-2(t-\tau)} & , & 2e^{-4(t-\tau)}-e^{-2(t-\tau)} \end{bmatrix}$$

$$\phi(t-\tau)*B=\begin{bmatrix} 2e^{-2(t-\tau)}-e^{-4(t-\tau)} & , & \frac{e^{-2(t-\tau)}}{2}-\frac{e^{-4(t-\tau)}}{2} \\ 4e^{-4(t-\tau)}-4e^{-2(t-\tau)} & , & 2e^{-4(t-\tau)}-e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\phi(t-\tau)B=\begin{bmatrix} \frac{1}{2}e^{-2(t-\tau)}-\frac{1}{2}e^{-4(t-\tau)} \\ -e^{-2(t-\tau)}+2e^{-4(t-\tau)} \end{bmatrix}$$

$$\int_0^t \phi(t-\tau)Bu(\tau)\,d\tau=\begin{bmatrix} \frac{1}{2}e^{-2t}\int_0^t e^{2\tau}d\tau-\frac{1}{2}e^{-4t}\int_0^t e^{4\tau}d\tau \\ -e^{-2t}\int_0^t e^{2\tau}d\tau+2e^{-4t}\int_0^t e^{4\tau}d\tau \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{8}-\frac{1}{4}e^{-2t}+\frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t}-\frac{1}{2}e^{-4t} \end{bmatrix}$$

Thursday, April 29, 2021 11:29 AM

$$x(t) = \phi(t)\, x(0) + \int_0^t \phi(t-\tau) B\, u(\tau)\, d\tau$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ 4e^{-4t} - 4e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{8}e^{-4t} \end{bmatrix}$$