$\boxed{2.6}$ limits involving infinity; Asymptotes of Graph

Def $\boxed{1}$ $f(x)$ has the limit L as x approaches infinity and we write $\lim_{x\to\infty} f(x) = L$ if for every number $\epsilon > 0$, there exist a corresponding number M such that for all x

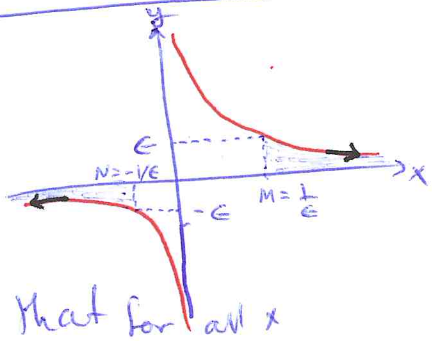
$$x > M \implies |f(x) - L| < \epsilon.$$

$\boxed{2}$ $f(x)$ has the limit L as x approaches minus infinity and we write $\lim_{x\to-\infty} f(x) = L$ if for every number $\epsilon > 0$, there exist a corresponding number N such that for all x

$$x < N \implies |f(x) - L| < \epsilon.$$

Example: $y = \frac{1}{x}$ show that

a) $\lim_{x\to\infty} \frac{1}{x} = 0$ (b) $\lim_{x\to-\infty} \frac{1}{x} = 0$



a) Let $\epsilon > 0$. We need to find a number M such that for all x

if $\boxed{x > M} \implies \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$ $\boxed{|x| > \frac{1}{\epsilon}}$ This is true if $M \geq \frac{1}{\epsilon}$

This proves that $\lim_{x\to\infty} \frac{1}{x} = 0$

b) let $\epsilon > 0$. We need to find a number N such that for all x

if $\boxed{x < N} \implies \left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \epsilon \implies |x| > \frac{1}{\epsilon}$ This is true if $N \leq \frac{-1}{\epsilon}$

$\boxed{x < -\frac{1}{\epsilon}}$

This proves that $\lim_{x\to-\infty} \frac{1}{x} = 0$

Note that $\lim_{x\to\pm\infty} \frac{1}{x} = 0$ and $\lim_{x\to\pm\infty} K = K$

limit at infinity of Rational functions

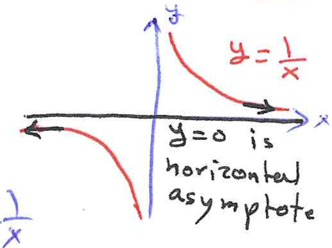
Example [1] $\lim_{x\to\infty} \dfrac{2x^2 - x + 4}{3x^2 + 5} = \lim_{x\to\infty} \dfrac{2 - \frac{1}{x} + \frac{4}{x^2}}{3 + \frac{5}{x^2}}$

$= \dfrac{2}{3}$

numerator بسط
denominator مقام

[2] $\lim_{x\to -\infty} \dfrac{8x - 6}{4x^3 + 1} = \lim_{x\to -\infty} \dfrac{\frac{8}{x^2} - \frac{6}{x^3}}{4 + \frac{1}{x^3}} = \dfrac{0+0}{4+0} = \dfrac{0}{4} = 0$

Horizontal Asymptotes

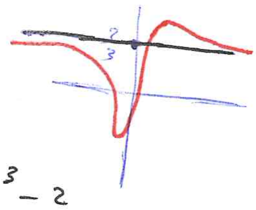


Example: $y = \frac{1}{x}$ the horizontal line $y=0$ "x-axis" is the horizontal asymptote of the graph $f(x) = \frac{1}{x}$

because $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to -\infty} \frac{1}{x} = 0$

Def A line $y=b$ is a horizontal asymptote of the graph $y=f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$

Example [1] $\lim_{x\to\pm\infty} \dfrac{2x^2 - x + 4}{3x^2 + 5} = \dfrac{2}{3}$

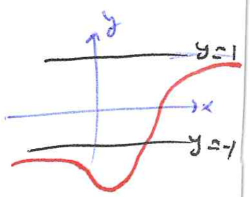


$\Rightarrow y = \frac{2}{3}$ is a horizontal asymptote

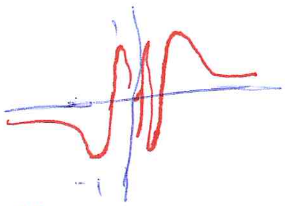
[2] Find the horizontal asymptote of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$

- For $x \geq 0 \Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \dfrac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = 1$
- For $x \leq 0 \Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to -\infty} \dfrac{x^3 - 2}{-x^3 + 1} = \lim_{x\to -\infty} \dfrac{1 - \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = -1$

The horizontal asymptotes are $y = -1$ and $y = 1$



Example: Find (a) $\lim_{x \to \infty} \sin\left(\frac{1}{x}\right)$

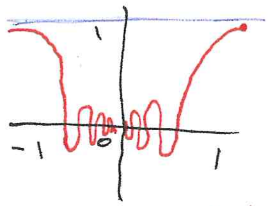


$\sin \frac{1}{x}$

take $t = \frac{1}{x}$ as $x \to \infty$, $t \to 0^+$

$$\lim_{x \to \infty} \sin\left(\frac{1}{x}\right) = \lim_{t \to 0^+} \sin t = 0$$

(b) $\lim_{x \to \pm\infty} x \sin\left(\frac{1}{x}\right)$



$x \sin \frac{1}{x}$

$$\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \to 0^+} \frac{\sin t}{t} = 1$$

$$\lim_{x \to -\infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \to 0^-} \frac{\sin t}{t} = 1$$

Example: Find $\lim_{x \to \infty} \left(x - \sqrt{x^2+4}\right) = \lim_{x \to \infty} \left(x - \sqrt{x^2+4}\right) \cdot \frac{x + \sqrt{x^2+4}}{x + \sqrt{x^2+4}}$

$$= \lim_{x \to \infty} \frac{x^2 - (x^2+4)}{x + \sqrt{x^2+4}} = \lim_{x \to \infty} \frac{-4}{x + \sqrt{x^2+4}} = \lim_{x \to \infty} \frac{\frac{-4}{x}}{1 + \sqrt{1 + \frac{4}{x^2}}} = 0$$

Oblique Asymptotes

* If the degree of the numerator (البسط) of a rational function is 1 greater than the degree of the denominator, then the graph has an oblique (منحرف) or slant (قاطع) line asymptote.

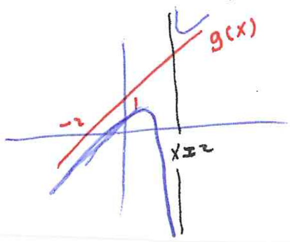
Example: Find the oblique asymptote of $f(x) = \frac{x^2 - 3}{2x - 4}$

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{g(x)} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder } r(x)}$$

$$\begin{array}{r|l} & \frac{x}{2} + 1 \\ \hline 2x-4 & x^2 - 3 \\ & -x^2 + 2x \\ \hline & 2x - 3 \\ & -2x + 4 \\ \hline & 1 \end{array}$$

oblique Asymptotes because

$$\lim_{x \to \infty} \frac{1}{2x - 4} = 0$$



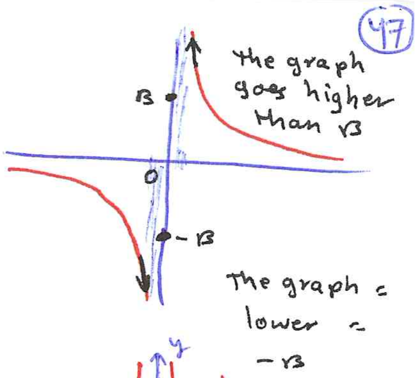
g(x) is dominante when x is Large

r(x) is dominante when x is near 2

Infinite limits

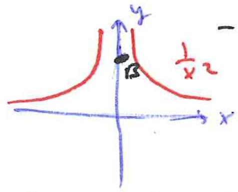
Example: Find

[a] $\lim_{x\to 0^+} \frac{1}{x} = \infty$ [b] $\lim_{x\to 0^-} \frac{1}{x} = -\infty$



[c] $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$ [d] $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

[e] $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$ [f] $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$



Note that in [a] and [c] ([b] and [d]) we are not saying the limit exists, nor that there is a real number ∞. We are saying $\lim_{x\to 0^+} \frac{1}{x}$ DNE because $\frac{1}{x}$ becomes arbitrary large and positive as $x \to 0^+$

Def [1] $f(x)$ approaches infinity as x approaches x_0 and we write $\lim_{x\to x_0} f(x) = \infty$ if for every positive real number B, there exist a corresponding $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow f(x) > B$.

[2] $f(x)$ approaches minus infinity as x approaches x_0, and we write $\lim_{x\to x_0} f(x) = -\infty$ if for every negative number $-B$, there exists a corresponding $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow f(x) < -B$

Example Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$

let $B > 0$, we need to find $\delta > 0$ such that

$0 < |x - 0| < \delta \Rightarrow \frac{1}{x^2} > B$

$x^2 < \frac{1}{B}$

$|x| < \frac{1}{\sqrt{B}}$

$\Rightarrow 0 < \delta \le \frac{1}{\sqrt{B}}$ Now

If $|x| < \delta$ then $\frac{1}{x^2} > \frac{1}{\delta^2} \ge B$

Therefore, by definition

$\lim_{x\to 0} \frac{1}{x^2} = \infty$

Vertical Asymptotes

Def A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

Example: Find vertical asymptote of $y = \frac{1}{x}$

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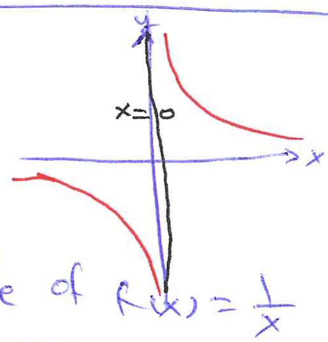
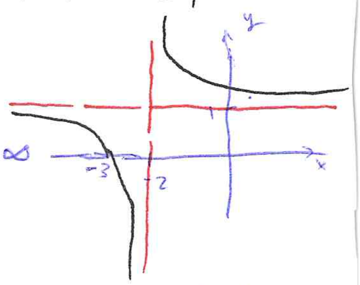


$\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\Rightarrow$ $(x=0)$ or (y-axis) is a vertical asymptote of $f(x) = \frac{1}{x}$

Example ① Find the horizontal and vertical asymptotes of

$$y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$$ قسمة طويلة



For Horizontal asymptotes: we check as $x \to \pm\infty$

For Vertical asymptotes: we check as $x \to -2$

$\lim_{x \to \pm\infty} \frac{x+3}{x+2} = \lim_{x \to \pm\infty} \frac{1 + \frac{3}{x}}{1 + \frac{2}{x}} = 1$ "horizontal asymptote" $y = 1$

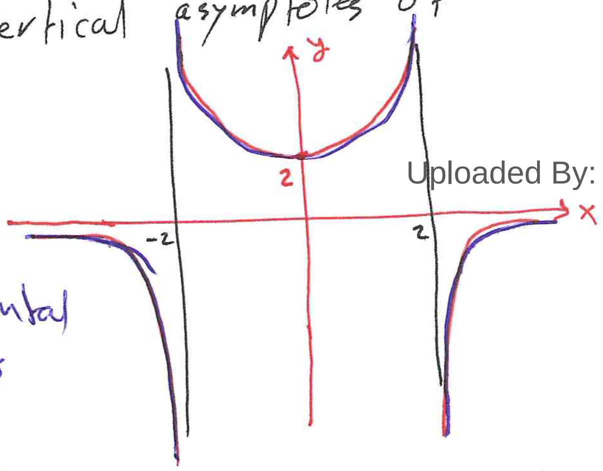
$\lim_{x \to 2^+} \frac{x+3}{x+2} = \frac{1}{\text{small positive}} = \infty$

$\lim_{x \to -2^-} \frac{x+3}{x+2} = \frac{1}{\text{small negative}} = -\infty$

$\Rightarrow$ $x = -2$ is vertical asymptote

② Find the horizontal and vertical asymptotes of

$$y = -\frac{8}{x^2 - 4}$$



- $\lim_{x \to -\infty} f(x) = 0$
  $\lim_{x \to +\infty} f(x) = 0$
  $y = 0$ is horizontal asymptotes

- $\lim_{x \to 2^+} f(x) = -\infty$
  $\lim_{x \to 2^-} f(x) = \infty$
  $x = 2$ is vertical asymptotes
  also $x = -2$ " " " .

③ $y = \tan x$ odd multiple of $\frac{\pi}{2}$