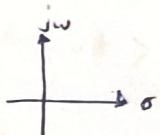


Introduction:-

» Laplace Transformation

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

complex variable $\sigma + j\omega$



$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds, \quad t > 0$$

fun. of time

not used ?!

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \text{converges}$$

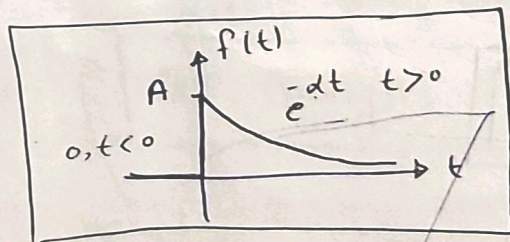
(improper integral)

- $\mathcal{L}[A f(t)] = A F(s)$
- $\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$

» Some frequently encountered time functions:-

1] Exponential function :

$$f(t) = \begin{cases} A e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



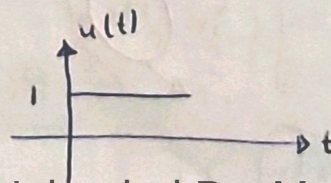
$$\mathcal{L}[f(t)] = \int_0^{\infty} A e^{-\alpha t} e^{-st} dt = A \int_0^{\infty} e^{-(\alpha+s)t} dt$$

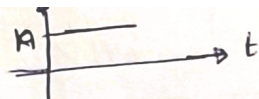
$$= \frac{A}{s + \alpha} \quad (\text{complex plane})$$

2] The step function

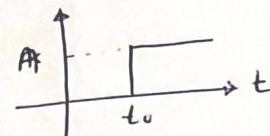
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Function

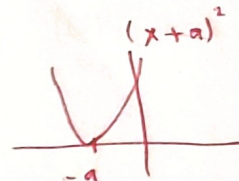
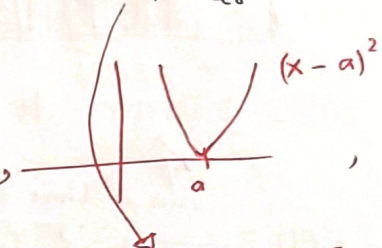
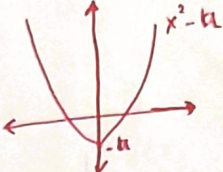
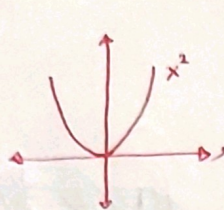
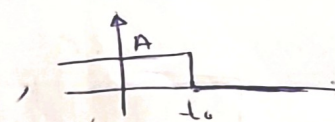


$$Au(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases}$$


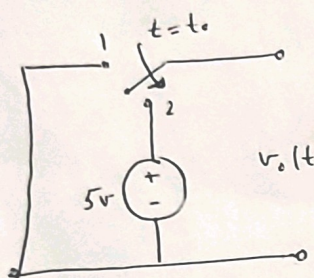
$A=1$ unit step function

$$Au(t-t_0) = \begin{cases} A & t > t_0 \\ 0 & t < t_0 \end{cases}$$


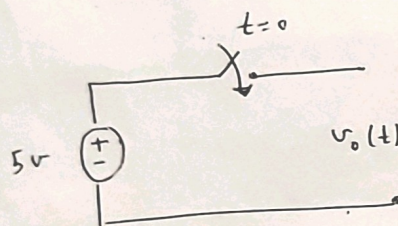
$Au(t_0-t), t > 0$



$$Au(t_0-t) = \begin{cases} A & t < t_0 \\ 0 & t > t_0 \end{cases}$$



$v_o(t) \Rightarrow v_o(t) = 5u(t-t_0)$



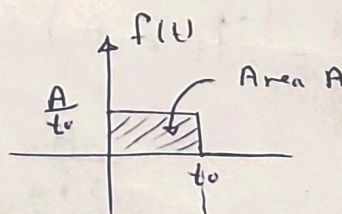
$v_o(t) \Rightarrow v_o(t) = 5u(t)$

$$\mathcal{L}[A] = \int_0^{\infty} A e^{-st} dt = \frac{A}{s}$$

$$\mathcal{L}[u(t)] = \boxed{\frac{1}{s}}$$

3) Pulse function

$$f(t) = \begin{cases} \frac{A}{t_0} & 0 < t < t_0 \\ 0 & t < 0, t > t_0 \end{cases}$$

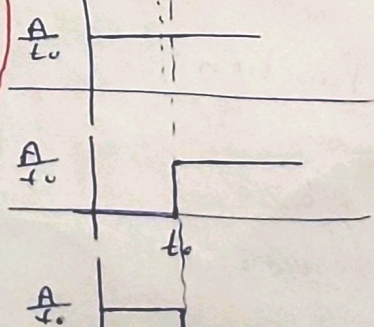


$$f(t) = \frac{A}{t_0} u(t) - \frac{A}{t_0} u(t-t_0)$$

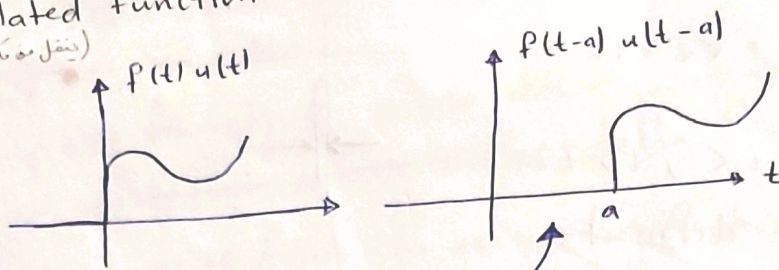
$$\mathcal{L}[f(t)] = \mathcal{L}\left[\frac{A}{t_0} u(t)\right] - \mathcal{L}\left[\frac{A}{t_0} u(t-t_0)\right]$$

$$= \frac{A}{t_0} \cdot \frac{1}{s} - \frac{A}{t_0} \cdot \frac{1}{s} e^{-st_0} \quad ?!$$

$$\begin{matrix} st \\ e^{st} \\ u(t-t_0) \\ e^{-st_0} \end{matrix}$$



* Translated function :-
(نقل و مکان)



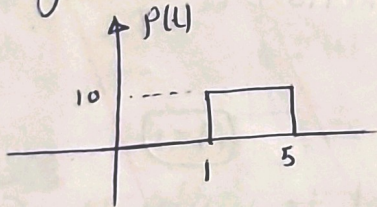
$$\tau = t - a$$

$$\begin{aligned} \mathcal{L}[f(t-a)u(t-a)] &= \int_0^{\infty} f(t-a)u(t-a)e^{-st} dt \\ &= \int_{-a}^{\infty} f(\tau)u(\tau)e^{-s(\tau+a)} d\tau \\ &= \int_0^{\infty} f(\tau)u(\tau)e^{-s(\tau+a)} d\tau \\ &= \int_0^{\infty} f(\tau)e^{-s\tau} \cdot e^{-as} d\tau = e^{-as} \int_0^{\infty} f(\tau)e^{-s\tau} d\tau \end{aligned}$$

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$$

* Example

rectangular pulse function



useful :-

CKT Analysis

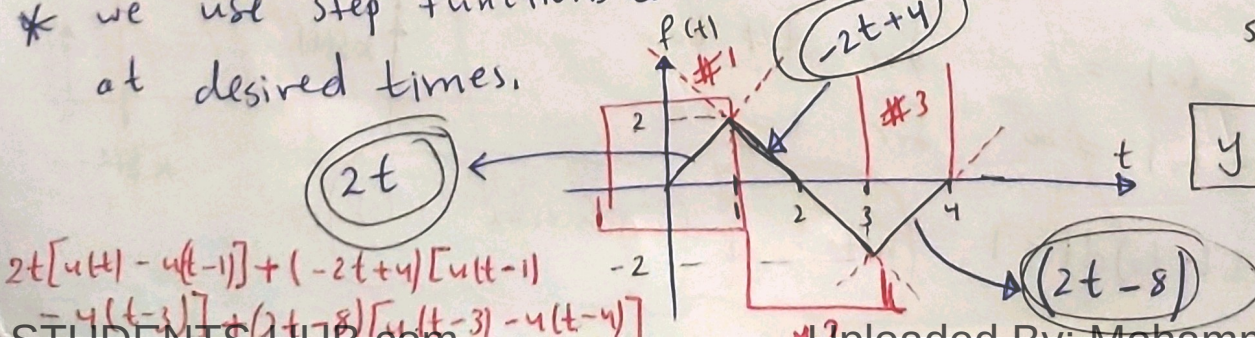
$$P(t) = 10u(t-1) - 10u(t-5)$$

find $\mathcal{L}[P(t)]$?!

* we use step functions to turn on and turn off linear function at desired times.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

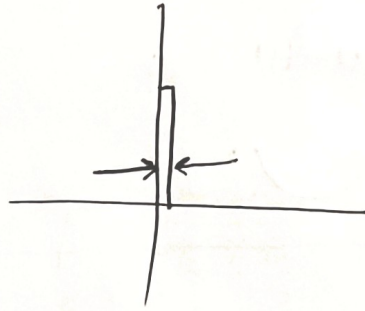
$$y = m(x - x_0) + y_0$$



$$2t[u(t) - u(t-1)] + (-2t+4)[u(t-1) - u(t-3)] + (2t-8)[u(t-3) - u(t-4)]$$

4) The Impulse Function

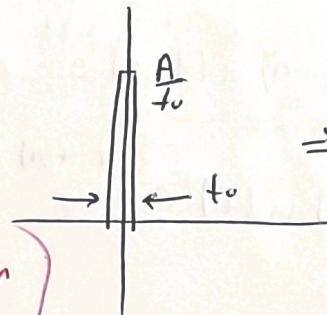
$$g(t) = \lim_{t_0 \rightarrow 0} \left(\frac{A}{t_0} \right) \text{ height} \quad 0 < t < t_0$$



The impulse function is defined:

$$\int_{-\infty}^{\infty} \delta(t) dt = A$$

$$\delta(t) = 0 \quad t \neq 0$$



$\Rightarrow \text{area} = A$

Mathematically, the impulse function is defined

$$\int_{-\infty}^{\infty} k \delta(t) dt = k$$

$$\delta(t) = 0, \quad t \neq 0$$

$$t_0 \rightarrow 0$$

$$\frac{A}{t_0} \rightarrow \infty$$

$$\boxed{\text{area} = A}$$

still

$$\mathcal{L}[g(t)] = \lim_{t_0 \rightarrow 0} \left[\frac{A}{t_0 s} (1 - e^{-st_0}) \right]$$

$$= \lim_{t_0 \rightarrow 0} \frac{\frac{d}{dt_0} [A(1 - e^{-st_0})]}{\frac{d}{dt_0} (t_0 s)} = \frac{A s}{s} = \boxed{A} \text{ Area}$$

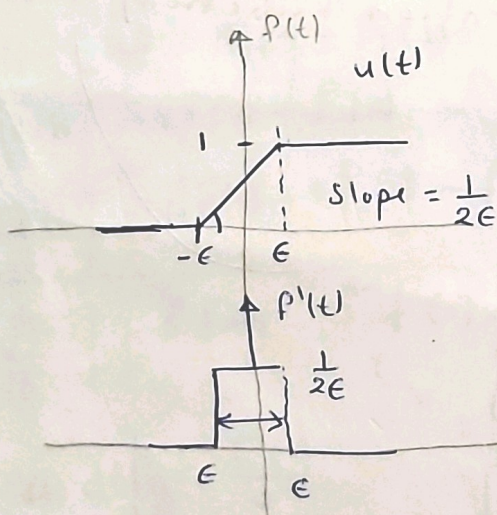
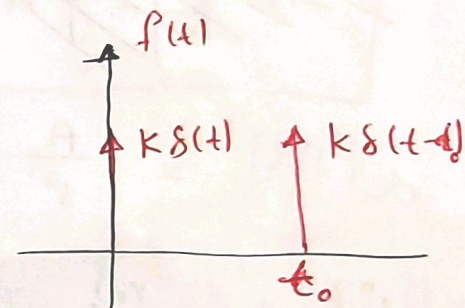
or $\mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1 \leftarrow \leftarrow \leftarrow$

4

- The Laplace transform of the impulse function is equal to the area under the impulse.
- The impulse function whose area is equal to unity is called unit-impulse function $\equiv$ Dirac delta function.
- The unit-impulse function occurring at $t = t_0$ is usually denoted by $\delta(t - t_0)$

$$\begin{cases} \delta(t - t_0) = 0 & \text{for } t \neq t_0 \\ \delta(t - t_0) = \infty & \text{for } t = t_0 \\ \int_{-\infty}^{\infty} \delta(t - t_0) dt = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \int_{-\infty}^{\infty} f(t) \delta(t) dt = f(0) \\ \int_{-\infty}^{\infty} f(t) \delta(t - a) dt = f(a) \end{cases}$$



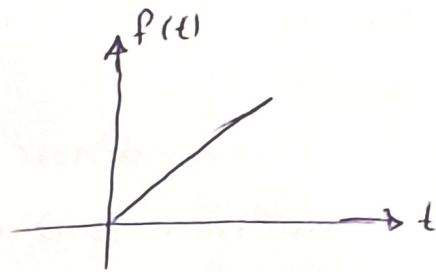
$$\begin{aligned} f(t) &\rightarrow u(t) \text{ as } \epsilon \rightarrow 0 \\ f'(t) &\rightarrow \delta(t) \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

$$\delta(t) = \frac{du(t)}{dt}$$

[5]

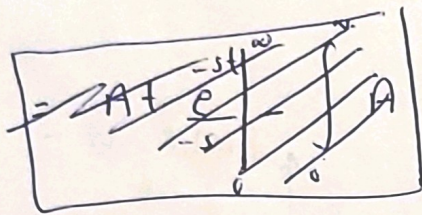
5 Ramp function

$$f(t) = \begin{cases} At & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



$$\mathcal{L}[At] = \int_0^{\infty} At e^{-st} dt$$

(by parts)



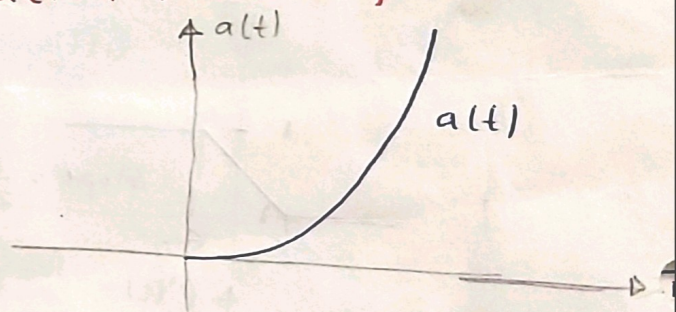
$$\mathcal{L}[At] = \frac{A}{s^2}$$



$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

6 unit-parabolic function (acceleration function)

$$a(t) = \begin{cases} \frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}\{a(t)\} = \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2} \mathcal{L}\{t^2\}$$

$$= \frac{1}{2} \left[\frac{2!}{s^3} \right]$$

$$\mathcal{L}\{a(t)\} = \frac{1}{s^3}$$

7 Sinusoidal function:

$$f(t) = \begin{cases} A \sin \omega t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\begin{aligned} \sin \omega t &= \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \\ \cos \omega t &= \frac{e^{j\omega t} + e^{-j\omega t}}{2} \end{aligned}$$

$$\mathcal{L}[A \sin \omega t] = \frac{A\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[A \cos \omega t] = \frac{As}{s^2 + \omega^2}$$

Complex numbers :-

- » Rectangular form. $\vec{A} = x + jy$
- » exponential form. $\vec{A} = |\vec{A}| e^{j\theta}$
- » polar form $\vec{A} = |\vec{A}| \angle \theta$

Euler's law

$$\vec{A} = A e^{j\theta} = A \cos \theta + j A \sin \theta$$

$$\begin{aligned} x &= A \cos \theta \\ y &= A \sin \theta \end{aligned} \Rightarrow \boxed{\begin{aligned} \sqrt{x^2 + y^2} &= A \\ \tan^{-1} \frac{y}{x} &= \theta \end{aligned}}$$

Properties of Laplace Transforms :-

- $\mathcal{L}\{k f(t)\} = k F(s)$
- $\mathcal{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s)$
- $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$
- $\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - s f(0) - f'(0)$
- $\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \frac{df(0)}{dt} - s^{n-3} \frac{d^2 f(0)}{dt^2} - \dots - \frac{d^{n-1} f(0)}{dt^{n-1}}$
 $\quad \quad \quad = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$
- $\mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$ where $f^{(k-1)}(t) = \frac{d^{k-1}}{dt^{k-1}} f(t)$

- $\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s), a \geq 0$ [Translation in the time domain]

- $\mathcal{L}[e^{-at} f(t)] = F(s+a), a \geq 0$ [Translation in the freq domain]

- $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right), a > 0$ [Scale changing]

- $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s) = f(\infty)$ [Final value theorem] ← The theorem is valid only if the poles of $F(s)$, except for a first-order pole at the origin, lie in the left half

- $f(0) = \lim_{s \rightarrow \infty} s F(s)$ [Initial value theorem] $= \lim_{t \rightarrow 0} f(t)$

$$\mathcal{L} \left[\frac{d}{dt} f(t) \right] = s F(s) - f(0)$$

$$\mathcal{L} [f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$\int_0^{\infty} \left[\frac{d}{dt} f(t) \right] e^{-st} dt = s F(s) - f(0)$$

$$\lim_{s \rightarrow 0} \int_0^{\infty} \left[\frac{d}{dt} f(t) \right] \underbrace{e^{-st}}_1 dt = \lim_{s \rightarrow 0} [s F(s) - f(0)]$$

$$= \int_0^{\infty} \left[\frac{d}{dt} f(t) \right] dt = f(t) \Big|_0^{\infty} = f(\infty) - f(0) = \lim_{s \rightarrow 0} [s F(s) - f(0)]$$

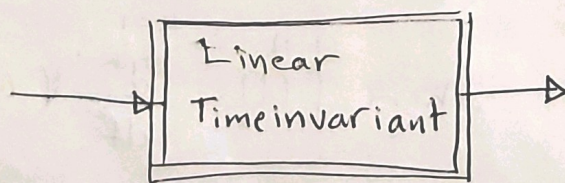
$$\Rightarrow \boxed{f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)}$$

Final value theorem

$$\int_0^{\infty} \left[\frac{d}{dt} f(t) \right] e^{-st} dt = s F(s) - f(0)$$

$$\lim_{s \rightarrow \infty} \int_0^{\infty} \left[\frac{d}{dt} f(t) \right] e^{-st} dt = \lim_{s \rightarrow \infty} [s F(s) - f(0)] = 0$$

$$\boxed{f(0) = \lim_{s \rightarrow \infty} s F(s)}$$

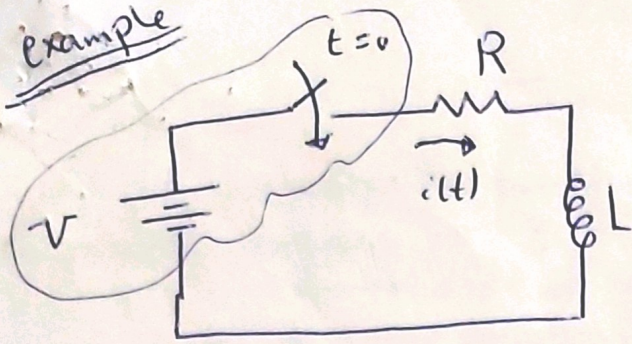


$$\begin{array}{l} x_1(t) \\ x_2(t) \\ x_1(t) + x_2(t) \end{array} \longrightarrow \begin{array}{l} y_1(t) \\ y_2(t) \\ y_1(t) + y_2(t) \end{array}$$

$$x(t) \longrightarrow y(t)$$

$$x(t-t_0) \longrightarrow y(t-t_0)$$

example



$$i_L(0) = 0$$

constant.

$$V u(t) = R i(t) + L \frac{di(t)}{dt}$$

$$V \cdot \frac{1}{s} = R I(s) + L (s I(s) - i_L(0))$$

$$\frac{V}{s} = (R + Ls) I(s)$$

$$I(s) = \frac{V}{s} \cdot \frac{1}{R + Ls}$$

$$I(s) = \frac{V/L}{s(s + \frac{R}{L})} = \frac{K_1}{s} + \frac{K_2}{s + \frac{R}{L}}$$

$$K_1 (s + \frac{R}{L}) + K_2 (s) = \frac{V}{L}$$

$$\textcircled{*} s = 0 \Rightarrow K_1 (\frac{R}{L}) = \frac{V}{L}$$

$$\boxed{K_1 = \frac{V}{R}}$$

$$\textcircled{*} s = -\frac{R}{L}$$

$$K_2 (-\frac{R}{L}) = \frac{V}{L}$$

$$\boxed{K_2 = -\frac{V}{R}}$$

$$i(t) = \frac{V}{R} u(t) - \frac{V}{R} e^{-\frac{R}{L}t} u(t)$$

$$i(t) = \frac{V}{R} [1 - e^{-\frac{R}{L}t}] u(t)$$

KCL

$$\sum i(t) = 0$$

$$\sum I(s) = 0$$

KVL

$$\sum v(t) = 0$$

$$\sum V(s) = 0$$

9

The Laplace Transform in Circuit Analysis

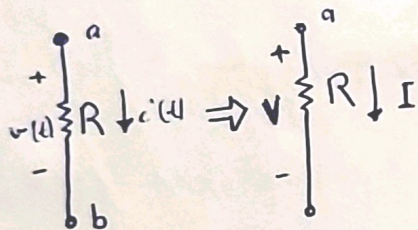
Chapter 13

3/5/2015

- » The Laplace transform has two characteristics that make it an attractive tool in circuit analysis.
- First, it transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations, which are easier to manipulate.
 - Second, it automatically introduces into the polynomial equations the initial values of the current and voltage variables. Thus, initial conditions are an inherent part of the transform process. (This contrasts with the classical approach to the solution of differential equations, in which initial conditions are considered when the unknown coefficients are evaluated.)

Circuit Elements in the s Domain :-

* Resistor in the s Domain



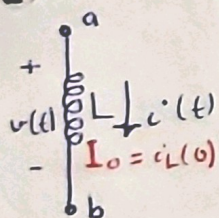
$$v = Ri$$

$$V = RI$$

$$V(s) = R \cdot I(s) \quad \text{constant}$$

$$\boxed{\frac{V(s)}{I(s)} = R}$$

* Inductor in the s Domain



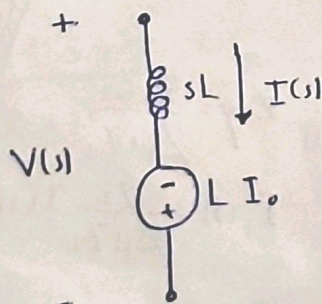
$$v = L \frac{di}{dt}$$

$$V(s) = L[sI(s) - i_L(0)]$$

$$V(s) = sL I(s) - L I_0 \Rightarrow Z_L(s) = \frac{V(s)}{I(s)}$$



$$\boxed{Z_L(s) = sL} \quad i_L(0) = 0$$



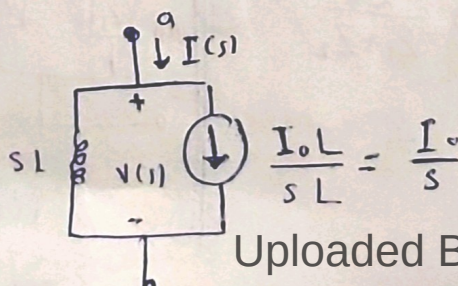
note that :-

$$* Z(s) \equiv \text{Impedance} = \frac{V(s)}{I(s)}$$

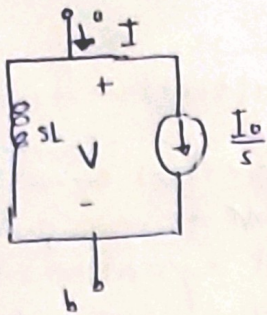
$$* Y(s) = \text{admittance} = \frac{I(s)}{V(s)} = \frac{1}{Z(s)}$$

From Source Transformation

From Current Inductor Law



10



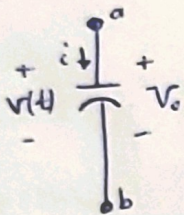
$$i_L(t) = i_L(0) + \frac{1}{L} \int_0^t v(t) dt$$

$$I(s) = \frac{I_0}{s} + \frac{1}{L} \left[\frac{V(s)}{s} \right]$$

$$= \frac{I_0}{s} + \frac{V(s)}{sL}$$

$$I(s) = I_1(s) + I_2(s)$$

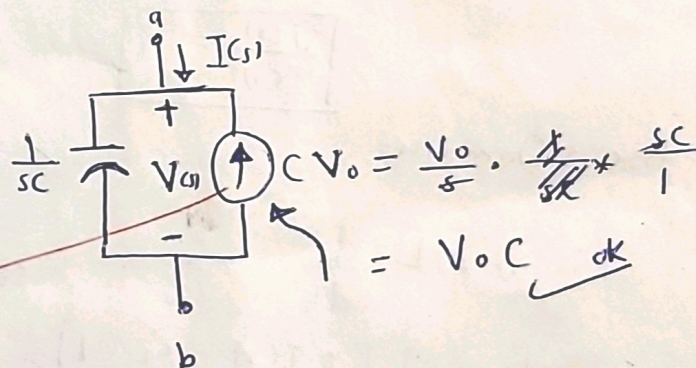
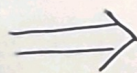
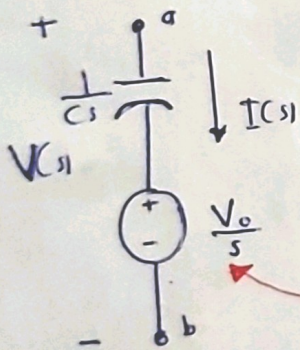
* Capacitor in the s Domain



$$v(t) = v_c(0) + \frac{1}{C} \int_0^t i(t) dt$$

$$V(s) = \frac{V_0}{s} + \frac{1}{C} \frac{I(s)}{s}$$

$$Z_C(s) = \frac{V(s)}{I(s)} \Big|_{V_0=0} = \frac{1}{Cs}$$

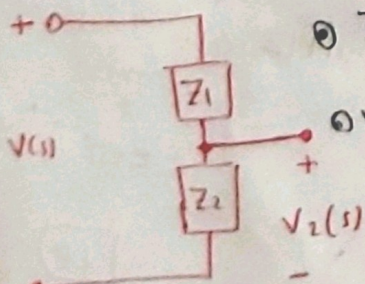


or

$$i(t) = C \frac{dv(t)}{dt}$$

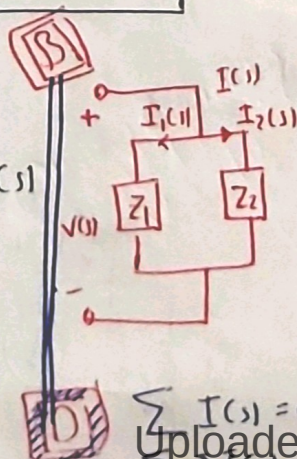
$$I(s) = C [sV(s) - V_0]$$

$$I(s) = CsV(s) - CV_0$$



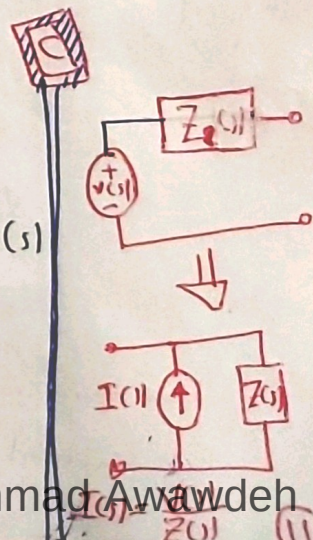
$$Z_{eq} = Z_1 + Z_2$$

$$V_2(s) = \frac{Z_2}{Z_1 + Z_2} V(s)$$



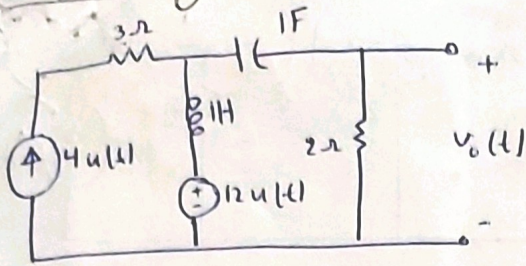
$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$I_1(s) = \frac{Z_2}{Z_1 + Z_2} I(s)$$



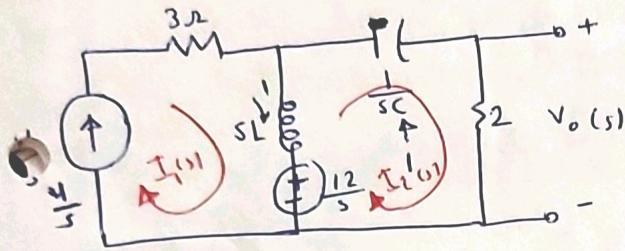
mesh analysis :-

Mesh Analysis :-



Find $v_o(t)$:-

① Transformed circuit to s-domain :-



mesh ① $I_1(s) = \frac{4}{s}$ (مصدر) Constraint eq

mesh ② $(s + \frac{1}{s} + 2) I_2(s) - s I_1(s) = \frac{12}{s}$

$$v_o(s) = 2 I_2(s)$$

we need to find $I_2(s)$

from eq ②

$$(s + \frac{1}{s} + 2) I_2(s) - s \left(\frac{4}{s} \right) = \frac{12}{s}$$

$$I_2(s) = \frac{4(3+s)}{(s+1)^2}$$

using partial fraction

but

$$v_o(s) = \frac{8(3+s)}{(s+1)^2} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2}$$

$$= \frac{8}{s+1} + \frac{16}{(s+1)^2}$$

$$k_1(s+1) + k_2 = 8(3+s)$$

$$[s = -1] \Rightarrow k_2 = 8(3-1) = 16$$

$$k_2 = 16$$

diff

$$k_1 = 8$$

or

$$k_1(s+1) + 16 = 8(3+s)$$

$$[s = 0]$$

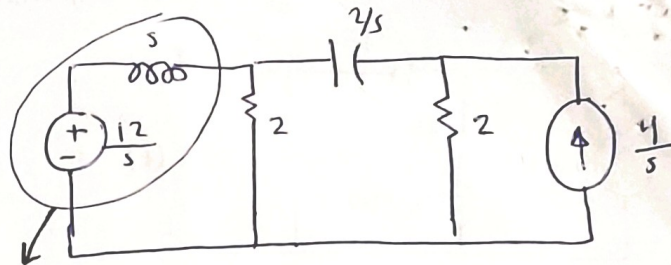
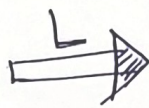
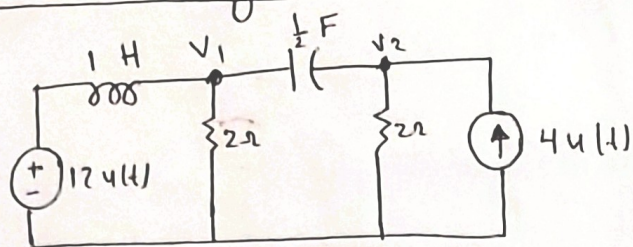
$$k_1 + 16 = 24$$

$$k_1 = 8$$

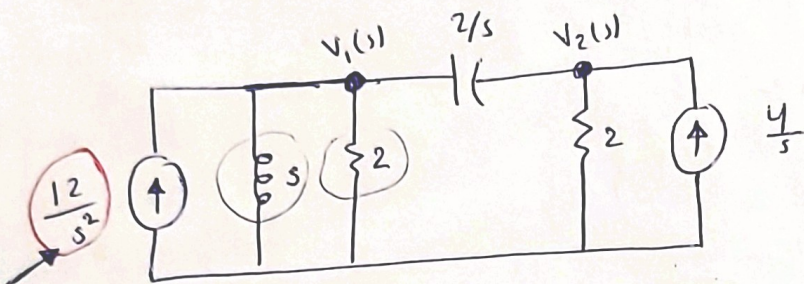
$$t^n e^{at} \rightarrow \frac{n!}{(s-a)^{n+1}} = (8e^{-t} + 16te^{-t}) u(t)$$

Nodal Analysis :-

((Admittances))



Source transformation.



at node ①

admittance.

$$\left(\frac{1}{s} + \frac{1}{2} + \frac{s}{2}\right)V_1(s) - \frac{s}{2}V_2(s) = \frac{12}{s^2} \quad \text{--- (1)}$$

at node ②

$$\left(\frac{1}{2} + \frac{s}{2}\right)V_2(s) - \frac{s}{2}V_1(s) = \frac{4}{s}$$

$$V = RI$$

$$I = \frac{1}{R}V$$

$$I = Y V(s) \quad \text{ok}$$

$$V_1(s) = \frac{4(s^2 + 3s + 3)}{s(s^2 + \frac{3}{2}s + 1)} = \frac{A}{s} + \frac{k + k^*}{(s + a)^2}$$

$$V_2(s) = \frac{4(s^2 + 3s + 2)}{s(s^2 + \frac{3}{2}s + 1)}$$

$$= 2|K|e^{-at} \cos(bt + \theta)$$

$$K = |K|e^{j\theta} = |K| \angle \theta$$

$$K^* = |K|e^{-j\theta} = |K| \angle -\theta$$

partial fraction coefficient.

$$V_1(t) = 12 + 8e^{-3/4 t} \cos\left[\left(\frac{\sqrt{7}}{4}\right)t + 180^\circ\right] \quad u(t)$$

$$\text{at } t = \infty \Rightarrow V_1(t) = 12$$

using Cramer's method to solve for V_1 and V_2

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos \theta$$

Review 2:-

$$\left(\frac{1}{s} + \frac{1}{2} + \frac{s}{2}\right)V_1(s) - \frac{s}{2}V_2(s) = \frac{12}{s^2}$$

$$-\frac{s}{2}V_1(s) + \left(\frac{1}{2} + \frac{s}{2}\right)V_2(s) = \frac{4}{s}$$

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$$A = \begin{vmatrix} \frac{2+s+s^2}{2s} & -\frac{s}{2} \\ -\frac{s}{2} & \frac{(1+s)}{2} \end{vmatrix}$$

$$N_1 = \begin{vmatrix} 12/s^2 & -s/2 \\ 4/s & (1+s)/2 \end{vmatrix}$$

$$N_2 = \begin{vmatrix} \frac{2+s+s^2}{2s} & 4/s \\ -s/2 & 12/s^2 \end{vmatrix}$$

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$$V_1 = \frac{N_1}{A}, V_2 = \frac{N_2}{A}$$

$$V_1(s) = \frac{4(s^2 + 3s + 3)}{s(s^2 + \frac{3}{2}s + 1)} = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}$$

* if $m > n$ proper rational function.

* if $m \leq n$ improper rational function.

$$\frac{N(s)}{D(s)} = \frac{K(s+z_1)(s+z_2)\dots(s+z_n)}{(s+p_1)(s+p_2)\dots(s+p_m)}$$

- Distinct Real Roots of $D(s)$.
- Distinct Complex Roots of $D(s)$.
- Repeated Real Roots of $D(s)$.
- Repeated Complex Roots of $D(s)$.

notes:-

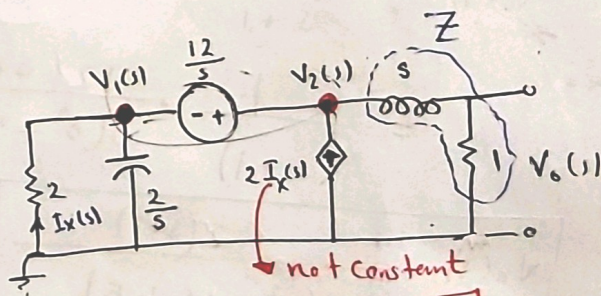
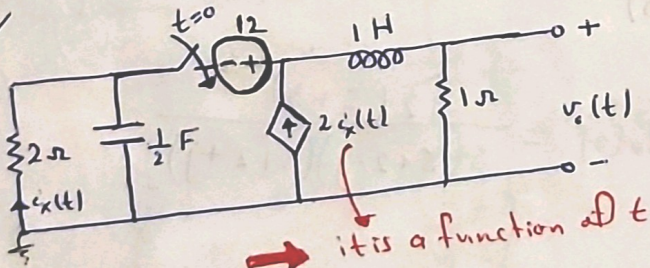
* Translation in time $\xrightarrow{\text{Time}} f(t-a)u(t-a) \rightarrow e^{-as} F(s)$
 * Translation in frequency $\xrightarrow{\text{Frequency}} e^{-at} f(t) \rightarrow F(s+a)$
 * exponential form $\vec{A} = |\vec{A}| e^{j\theta}$, $|\vec{A}| = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$, $\vec{A} = x + jy$
 * $\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

* $\frac{du(t)}{dt} = s(u(t))$, $s(t) \xrightarrow{L} 1$
 $\frac{ds(t)}{dt} \xrightarrow{L} s$

$\frac{d^2 s(t)}{dt^2} \xrightarrow{L} s^2$

$s(t-t_0) \xrightarrow{L} e^{-ts} \cdot \boxed{F(s)} = e^{-ts} \Rightarrow \text{Go to page } \boxed{17}$

Example
Super node



$i_x(t) \rightarrow I_X(s)$
 $f(t) \rightarrow F(s)$

$V_2(s) - V_1(s) = \frac{12}{s}$ (constraint eq)

$(\frac{s}{2} + \frac{1}{2})V_1(s) + \frac{1}{s+1}V_2(s) = 2I_X(s)$ (super-node equation)

but;

$I_X(s) = -\frac{V_1(s)}{2}$

and

$V_0(s) = \frac{1}{s+1} V_2(s)$

using voltage divider

$$\left(\frac{s}{2} + \frac{1}{2}\right) V_1(s) + \frac{1}{s+1} V_2(s) = 2 I_x(s)$$

$$V_1 = V_2(s) - \frac{12}{s}, \quad I_x = -\frac{V_1(s)}{2}$$

~~time~~

$$\left(\frac{s}{2} + \frac{1}{2}\right) V_1(s) + \frac{1}{s+1} V_2(s) = -2 \left(\frac{V_1(s)}{2}\right)$$

$$\frac{V_2(s)}{s+1} = \left[V_1(s) - \left(\frac{s}{2} + \frac{1}{2}\right) V_1(s) \right]$$

$$\frac{V_2(s)}{s+1} = -V_1(s) \left[\frac{s}{2} + \frac{s+1}{2} \right] = -\left(\frac{s+3}{2}\right) \left(\frac{sV_2}{2} - \frac{12}{s}\right)$$

$$\frac{V_2(s)}{s+1} + \left(\frac{s+3}{2}\right) V_2(s) = + \left(\frac{s+3}{2}\right) \left(\frac{12}{s}\right)$$

$$V_2(s) \left[\frac{1}{2(s+1)} + \frac{(s+3)}{(s+1)(2)} \right] = \left(\frac{s+3}{2}\right) \left(\frac{12}{s}\right) \left(\frac{2(s+1)}{(s+1)(s+3)+2}\right)$$

$$V_2(s) = \frac{12(s+3)(s+1)}{s(s^2+4s+5)}$$

$$V_o(s) = \frac{1}{s+1} \cdot V_2(s)$$

$$= \frac{1}{(s+1)} \cdot \frac{12(s+3)(s+1)}{s(s^2+4s+5)}$$

$$V_o(s) = \frac{12(s+3)}{s(s^2+4s+5)}$$

$$V_o(s) = \frac{12(s+3)}{s(s+2-j)(s+2+j)}$$

$$\frac{-4 \pm \sqrt{16-4(1)(5)}}{2(1)}$$

$$s_{1,2} = -2 \pm j$$

$$s_{1,2} = \frac{-4 \pm j2}{2} = (-2 \pm j)$$

$$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s+2-j} + \frac{K_2^*}{s+2+j}$$

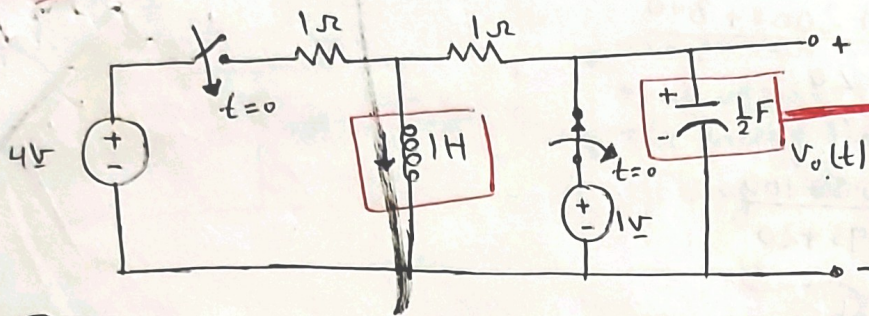
$$K_2 = 3.79 \angle 161.57^\circ$$

$$V_o(t) = 3.79 e^{-2t} \cos(t + 161.57^\circ)$$

$$K_2 = 3.79 \angle 161.57^\circ$$

$$V_o(t) = 3.79 e^{-2t} \cos(t + 161.57^\circ)$$

example elements with Initial Conditions:-



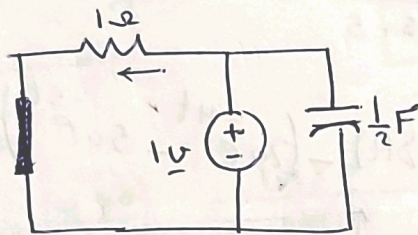
there is two models
For each one



when there is initial condition.

?! Find $v_o(t)$ for $t > 0$:-

for $t < 0$



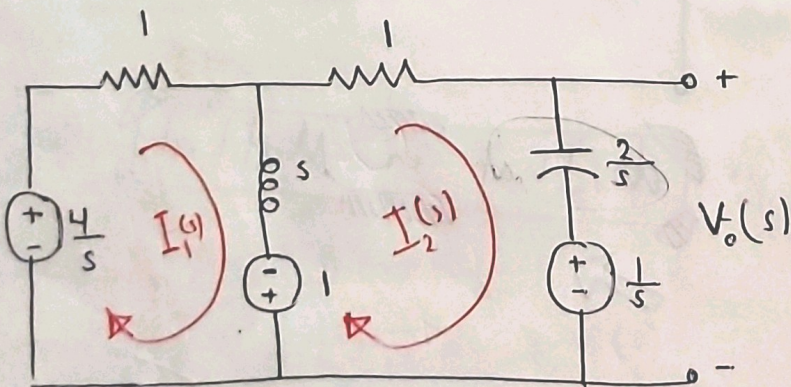
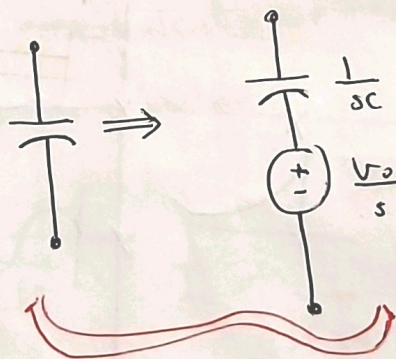
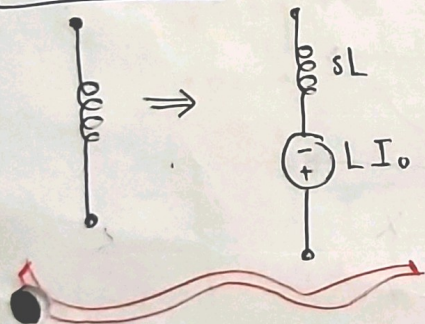
$$i_L(0) = 1A$$

$$v_C(0) = 1V$$

$$v = L \frac{di}{dt}$$

$v = 0 \Rightarrow$ short ckt

for $t > 0$



$$\Rightarrow \text{Find } I_2(s) \Rightarrow \boxed{V_o(s) = \frac{2}{s} I_2(s) + \frac{1}{s}}$$

16

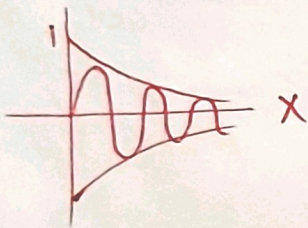
example

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

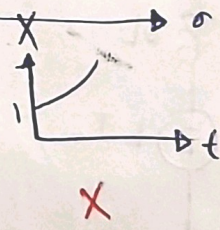
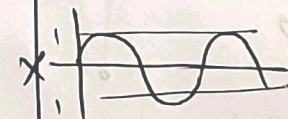
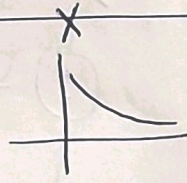
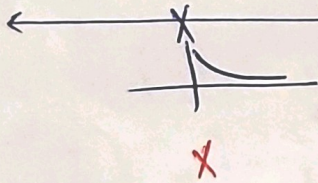
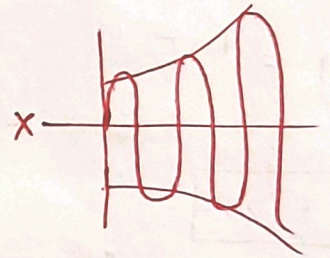
$$= \underbrace{s^2 + 4s + 10}_{\text{ساغى}} + \underbrace{\frac{30s + 100}{s^2 + 9s + 20}}_{\text{سايمانى}}$$

$$= s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

$$f(t) = \frac{d^2 \delta(t)}{dt^2} + 4 \frac{d\delta(t)}{dt} + 10\delta(t) - (20e^{-4t} - 50e^{-5t})u(t)$$



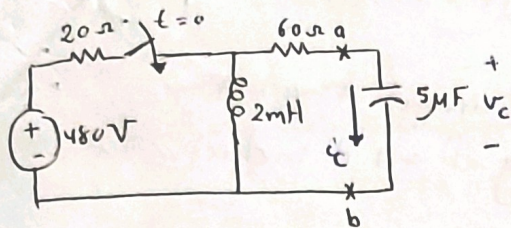
$j\omega$



$\frac{1}{s+a}$	$\rightarrow e^{-at}$
$\frac{w}{s^2 + w^2}$	$\rightarrow \sin wt$
$\frac{w}{(s+a)^2 + w^2}$	$\rightarrow e^{-at} \sin wt$

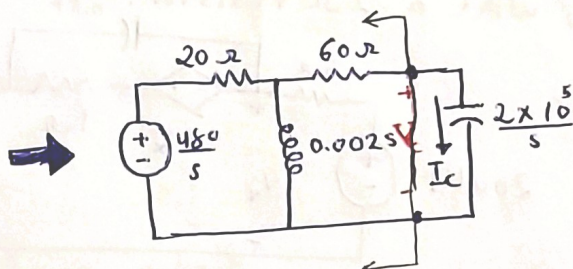
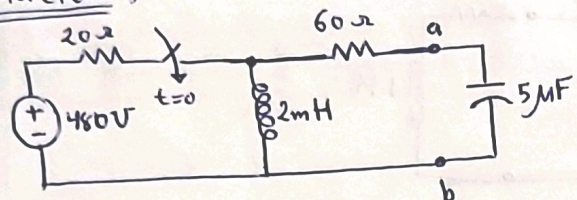
do that in MatLab

The Use of Thevenin's Equivalent:-



- The problem is to find the capacitor current that results from closing the switch.
- The energy stored in the circuit prior to closing is zero.

Solution:-



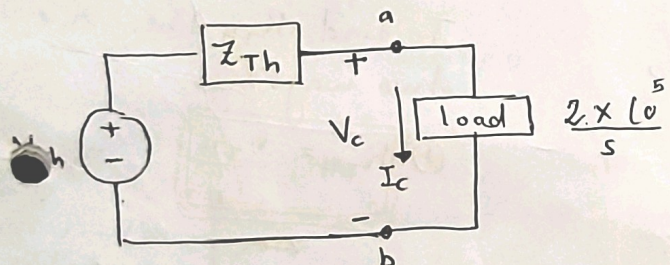
Voltage divider

$$V_{TH} = \frac{0.002s}{20 + 0.002s} \cdot \left(\frac{480}{s} \right) = \frac{480}{s + 10^4}$$

V_{TH} (load open ckt)
 Z_{TH} (open ckt + kill sources)

$$Z_{TH} = 60 + (0.002s \parallel 20) = 60 + \frac{0.002s(20)}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10^4}$$

Using the Thevenin equivalent, we reduce the circuit shown above to the one shown below:-



$$I_c = \frac{V_{TH}}{Z_{TH} + Z_{load}} = \frac{6s}{(s + 5000)^2}$$

$$I_c = \frac{-30,000}{(s + 5000)^2} + \frac{6}{s + 5000}$$

$$\mathcal{L}^{-1}[I_c] = i_c = (-30,000t e^{-5000t} + 6 e^{-5000t}) u(t) \text{ A.}$$

Two ways!

V_c ?!

① find $V_c \rightarrow V_c$

② $V_c = \frac{1}{C} \int_0^t i_c dt + V_c(0)$

$$V_c = \frac{1}{5C} I_c = \frac{2 \times 10^5}{s} \cdot \frac{6s}{(s + 5000)^2} = \frac{12 \times 10^5}{(s + 5000)^2} \quad \text{direct.}$$

$$V_c = 12 \times 10^5 t e^{-5000t} u(t).$$

$$V_c = 2 \times 10^5 \int_0^t (6 - 30,000t) e^{-5000t} dt$$

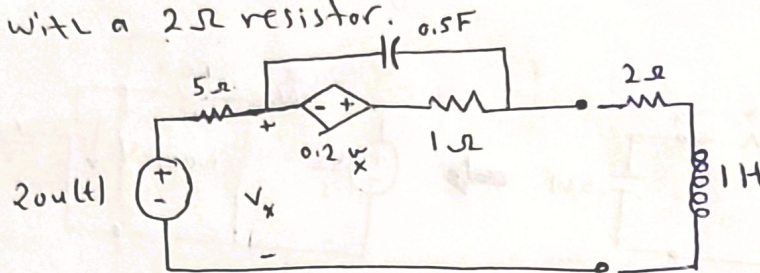
$V = ZI$

18

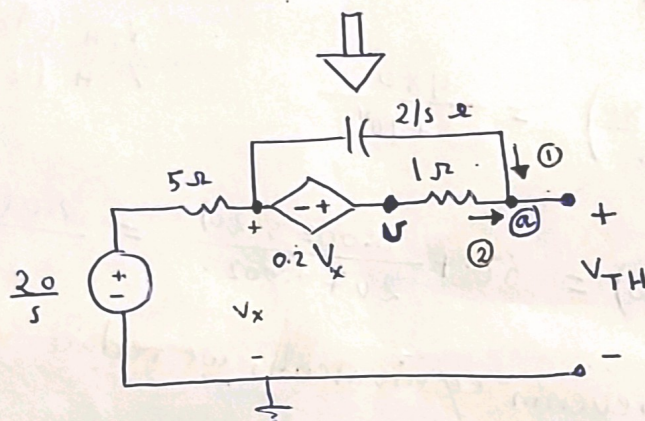
example

The initial charge on the capacitor in the circuit shown is zero.

- Find the s-domain Thevenin eq. ckt. with respect to terminals a and b.
- Find the s-domain expression for the current that the circuit delivers to a load consisting of a 1 H inductor in series with a 2 Ω resistor.



$$V = ZI$$



$$V = 0.2V_x + V_x = 1.2V_x$$

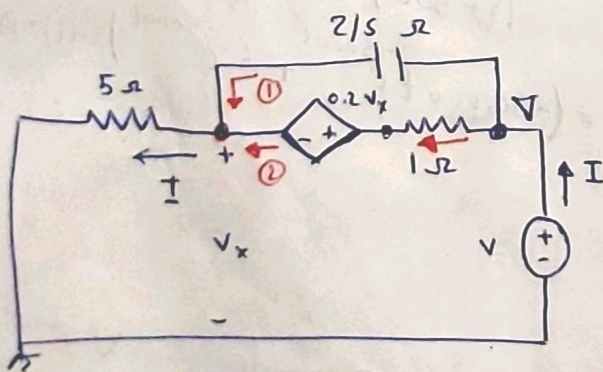
$$= 1.2 \left(\frac{20}{s} \right)$$

With no load across terminals a-b, $V_x = \frac{20}{s}$

KCL @ node (a)

$$\left(\frac{20}{s} - V_{TH} \right) \frac{s}{2} + \left(1.2 \left(\frac{20}{s} \right) - V_{TH} \right) 1 = 0$$

$$V_{TH} = \frac{20(s + 2.4)}{s(s + 2)}$$



$$V_x = 5I \text{ and } Z_{TH} = \frac{V}{I}$$

$$I = (V - V_x) \cdot \frac{s}{2} + (V - 1.2V_x)(1)$$

$$I = (V - 5I) \cdot \frac{s}{2} + V - 6I \Rightarrow Z_{TH} = \frac{5(s + 2.8)}{s + 2}$$

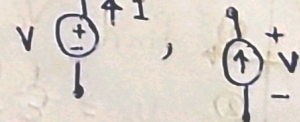
To find R_{TH} there are two ways:-

$$\textcircled{1} R_{TH} = \frac{V_{TH}}{I_N}$$

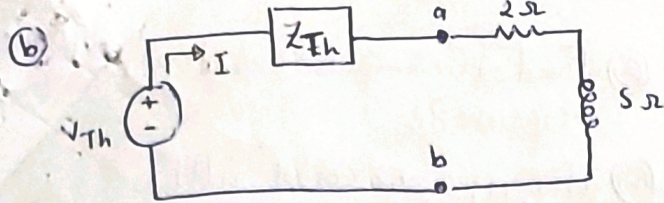
without killing the sources.

$\textcircled{2}$ (a) indep sources = 0

(b) applied source (1A)

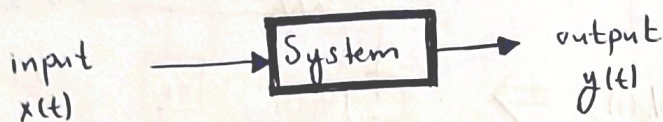


$$\frac{V}{I} = R_{TH}$$



$$I = \frac{V_{TH}}{Z_{TH} + 2 + s} = \frac{20(s+2.4)}{s(s+3)(s+6)}$$

The Transfer Function



The transfer function is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source).
In computing the transfer function, we restrict our attention to circuits where all initial conditions are zero.
If a circuit has multiple independent sources, we can find the transfer function for each source and use superposition to find the response to all sources.

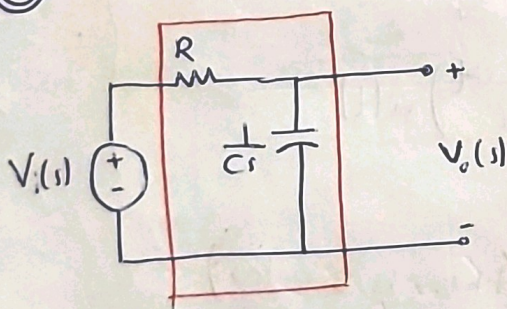
$$H(s) = \left. \frac{Y(s)}{X(s)} \right|_{\text{initial condits} = 0}$$

$$Y(s) = H(s) \cdot X(s)$$

output.
response.

- ① if $x(s) = \delta(t) \rightarrow$ impulse resp.
- ② if $x(s) = u(t) \rightarrow$ step resp.
- ③ if $x(s) = r(t) \rightarrow$ ramp resp.

Example ①



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$V_o(s) = \frac{1/Cs}{1/Cs + R} \cdot V_i(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{s+1} \quad \text{(order of the system)}$$

$$V_o(s) = \left(\frac{1}{s+1} \right) (1) = H(s)$$

the same of output since $V_i(s) = 1$

$$\textcircled{1} \mathcal{L}^{-1}[H(s)] = h(t) \equiv \text{impulse resp.}$$

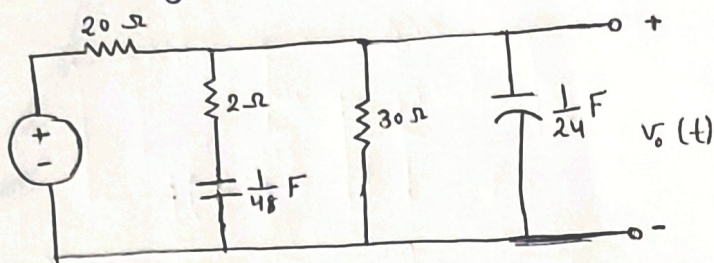
$$\textcircled{2} \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] = s(t) \equiv \text{unit step resp.}$$

MIMO System :-

$$\begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} \square \\ \square \end{bmatrix}$$

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Example (3) go to Example (2) →

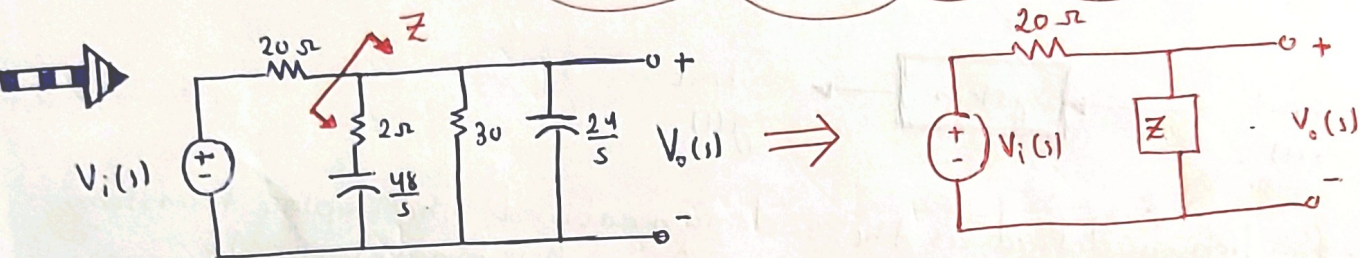


(a) Find the impulse and unit-step response?!

(b) If $v_i(t) = 50 \cos 2t u(t)$
Find $v_o(t)$?!

$$H(s) = \left. \frac{V_o(s)}{V_i(s)} \right|_{i.c=0}$$

Example (2) →
in the next page



$$Z = \frac{24}{s} \parallel 30 \parallel (2 + \frac{48}{s})$$

$$V_o(s) = \frac{Z}{Z+20} V_i(s) \Rightarrow H(s) = \frac{Z}{Z+20} = \frac{V_o(s)}{V_i(s)}$$

$$H(s) = \frac{1.2(s+2)}{(s+1)(s+4)} \quad \begin{array}{l} \leftarrow \text{Zeros.} \\ \leftarrow \text{poles.} \end{array}$$

if $v_i(t) = \delta(t)$

$$v_o(t) = h(t) = \mathcal{L}^{-1}[H(s)]$$

$$h(t) = (0.4 e^{-t} + 0.8 e^{-4t}) u(t)$$

if $v_i(t) = u(t)$

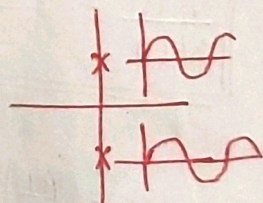
$$s(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] \quad \leftarrow \text{(there is 3 poles.)}$$

(b) $v_i(t) = 50 \cos 2t u(t)$

$$V_o(s) = H(s) \cdot V_i(s)$$

$$= \frac{1.2(s+2)}{(s+1)(s+4)} \cdot 50 \frac{s}{s^2+4}$$

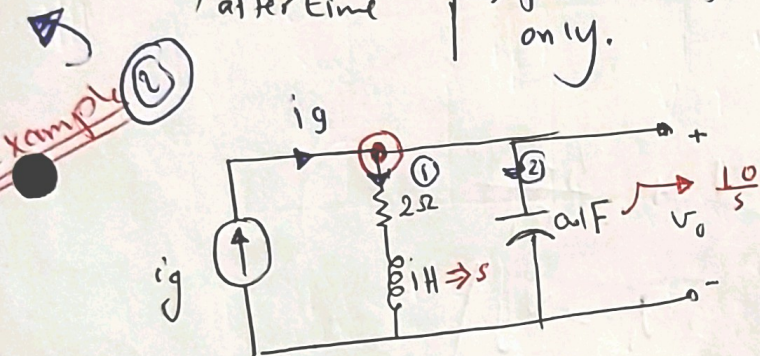
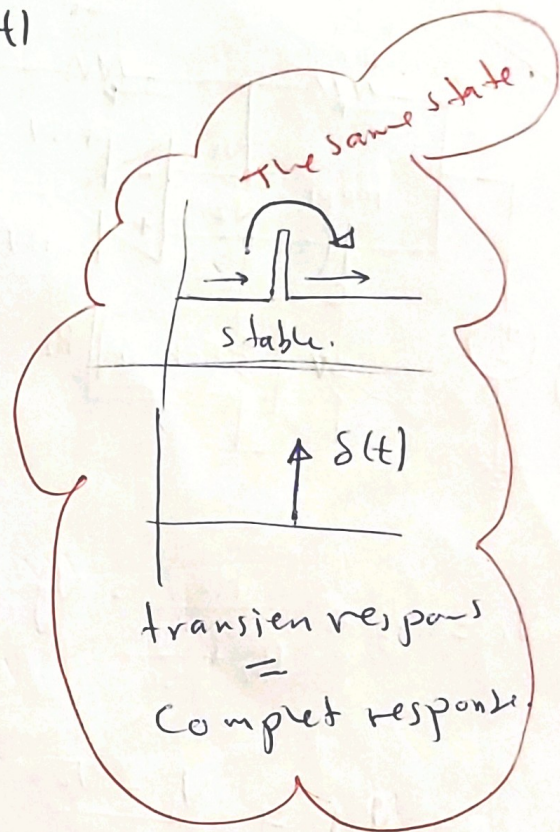
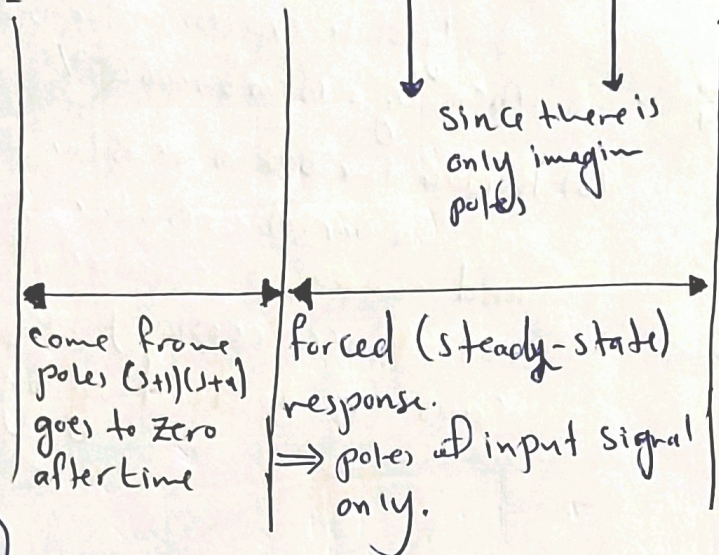
⇒ complex poles (sin, cosine)



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$$v_o(t) = [-4e^{-t} - 8e^{-4t} + (12\cos 2t + 12\sin 2t)] u(t)$$



- (a) Derive the numerical expression for the transfer function V_o/I_g for the circuit shown.
- (b) Give the numerical value of each pole and zero of $H(s)$.

$$[a] \quad I_g = \frac{V_o}{s+2} + \frac{V_o s}{10} \Rightarrow I_g = V_o \left(\frac{1}{s+2} + \frac{s}{10} \right)$$

$$= V_o \left(\frac{10 + s(s+2)}{(s+2)10} \right) \leftarrow \text{ok}$$

$$\boxed{\frac{V_o}{I_g} = H(s) = \frac{10(s+2)}{s^2 + 2s + 10}}$$

Zeros $s = -2$

poles $s^2 + 2s + 10 = 0$

$$(s+1+3j)(s+1-3j) = 0$$

Characteristic equation

$$\boxed{s_{1,2} = -1 \pm 3j}$$

stable

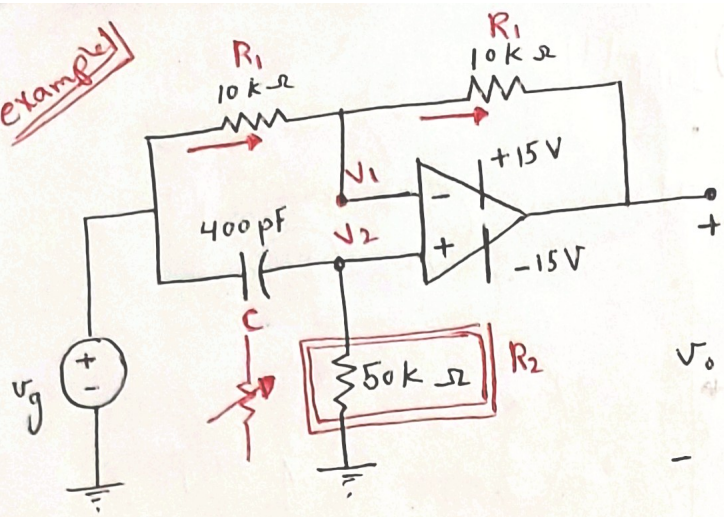
BIBO

$$= -1 \pm 3j$$

$$\frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2(1)} = \frac{-2 \pm \sqrt{-36}}{2(1)}$$

$$= \frac{-2 \pm j6}{2}$$

example



- (a) For the CKT shown, find the steady-state expression for v_o when $v_g = 10 \cos 50,000t$ V
- (b) Replace the $50k\Omega$ resistor with a variable resistor and compute the value of resistance necessary to cause v_o to lead v_g by 120° .

① $V_1 = V_2 = \frac{R_2}{R_2 + \frac{1}{sC}} \cdot V_g$

② $\frac{V_g - V_1}{R_1} = \frac{V_1 - V_o}{R_1} \Rightarrow \frac{V_1 - V_g}{R_1} + \frac{V_1 - V_o}{R_1} = 0$

$2V_1 - V_g - V_o = 0 \Rightarrow \boxed{V_o = 2V_1 - V_g}$

$V_o = 2V_1 - V_g$

$\Downarrow ? V_g \Rightarrow \frac{V_o}{V_g} = H(s)$

$H(s) = \frac{R_2 C s - 1}{R_2 C s + 1} = \frac{2 \times 10^{-5} s - 1}{2 \times 10^{-5} s + 1}$

$H(j\omega) = H(j50,000) = \frac{j-1}{j+1} = \frac{A e^{j(135^\circ)}}{A e^{45^\circ}}$

$\frac{j-1}{j+1} = \frac{1-j}{1+j}$

$= e^{j90^\circ} = 1 \angle 90^\circ$

or $= j = 1 \angle 90^\circ$

$y_{ss}(t) = A |H(j\omega)| \cos[\omega t + \phi + \phi(\omega)]$

$v_o = 10(1) \cos(50,000t + 0 + 90)$

$\boxed{v_o = 10 \cos(50,000t + 90^\circ) \text{ V}}$

