Birzeit University | Faculty of Engineering | Dep. of Electrical Engineering

Probability ENEE 331 | Problem Set (2) | Single Random Variables

P1) Let X be a random variable with mean $E\{X\}=11/12$, and probability density function *Where a,b constants*

a) What is the value of a,b?
b) What is the cumulative distribution function of X?

$$f(x)=\begin{cases} ax^2+b & for\, 0\le x\le 1 \\ 0 & otherwise \end{cases}$$

P2) If you reach bus stop at 11 o'clock, knowing that the bus will arrive at some time uniformly distributed between 11 and 11:30.

a) What is the probability that you will have to wait longer than 5 minutes?
b) If at 11:15 the bus has not yet arrived, what is the probability that you will have to what at least an additional 10 minutes?

P3) Let X be a continuous random variable that has the following probability density function

$$f_X(x)=\begin{cases} 2x & 0<x<1 \\ 0 & \text{Elsewhere} \end{cases}$$

a) Find the mean and variance of X.
b) Find and plot the cumulative distribution function of X.
c) What is P(0.3 < X < 0.6)?
d) Let Y = 1/X compute E(1/X).

P4) Given a random variable X having a normal distribution with $\mu_x = 50$ and $\sigma_x = 10$. A new Random Variable $Y = X^2$

a) find the probability that X is less than 50
b) find the probability that X is between 45 and 62
c) find the mean of Y
d) find the probaility density function of Y.

P5) Bits are sent over a communications channel in packets of 12. If the probability of a bit being corrupted over this channel is 0.1 and such errors are independent, what is the probability that no more than 2 bits in a packet are corrupted?

a) If 6 packets are sent over the channel, what is the probability that at least one packet will contain 3 or more corrupted bits?
b) Let X denote the number of packets containing 3 or more corrupted bits. What is the probability that X will exceed its mean by more than 2 standard deviations?

P6 P5) Suppose that the lifetime X of a power transmission tower, measured in years, is described by an exponential distribution with mean equals to 25 years

$$f_x(x)=\begin{cases} \frac{1}{25}e^{-x/25} & x\ge 0 \\ 0 & otherwise \end{cases}$$

If three towers, operated independently, were erected at the same time, what is the probability that at least two will still stand after 35 years.

Ahmad Alyan

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a) What is the value of a,b?
b) What is the cumulative distribution function of X?

$$f(x)=\begin{cases} ax^2+b & for\, 0\le x\le 1 \\ 0 & otherwise \end{cases}$$

$$1=\int_0^1 f(x)dx=\int_0^1 (ax^2+b)dx=\frac{ax^3}{3}+bx\Big]_0^1=\frac{a}{3}+b-0=\frac{a}{3}+b$$

$$a=3-3b$$

$$\frac{11}{12}=\int_0^1 xf(x)dx=\int_0^1 x(ax^2+b)dx=\frac{ax^4}{4}+\frac{bx^2}{2}\Big]_0^1=\frac{a}{4}+\frac{b}{2}$$

$$a=-2b+\frac{11}{3}$$

$$b=\frac{-2}{3},a=5$$

$$F_x(X\le x)=\int_0^x\left(5x^2-\frac{2}{3}\right)dx=\frac{5x^3}{3}-\frac{x}{3}\Big]_0^x=\frac{5x^3-2x}{3}$$

P2) If you reach bus stop at 11 o'clock, knowing that the bus will arrive at some time uniformly distributed between 11 and 11:30.

a) What is the probability that you will have to wait longer than 5 minutes?
b) If at 11:15 the bus has not yet arrived, what is the probability that you will have to what at least an additional 10 minutes?

Uniformly distributed a)

$$P(X\le x)=\frac{x}{b-a}$$

$$P(X>5)=1-P(X\le 5)=1-\frac{5}{30-0}=\frac{5}{6}$$

b) If at 11:15 the bus has not yet arrived, what is the probability that you will have to what at least an additional 10 minutes

$$P(X\ge(15+10)\big|X\ge 15)=\frac{P(X\ge 25\cap X\ge 15)}{P(X\ge 15)}=\frac{P(X\ge 25)}{P(X\ge 15)}=\frac{1-\frac{25}{30}}{1-\frac{15}{30}}=\frac{1}{3}$$

P3) Let X be a continuous random variable that has the following probability density function

$$f_X(x)=\begin{cases} 2x & 0<\mathrm{x}<1 \\ 0 & \text{Elsewhere} \end{cases}$$

a) Find the mean and variance of X.
b) Find and plot the cumulative distribution function of X.
c) What is P(0.3 < X < 0.6)?

d) Let Y = 1/X compute E(1/X).

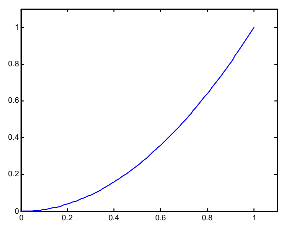
a) Find the mean and variance of X.

$$\mu_x = E(x) = \int_{-\infty}^{\infty} x f_x(x)dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)dx = \int_0^1 2x\left(x-\frac{2}{3}\right)^2 dx = \frac{2x^4}{4} - \frac{8}{9}x^3 + \frac{4}{9}x^2 \Big|_0^1 = \frac{1}{18}$$

b) Find and plot the cumulative distribution function of X.

$$F_x(X \le x) = \int_0^x 2x dx = x^2 \Big]_0^x = x^2$$



c) What is P(0.3 < X < 0.6)?

$$P(0.3 < X < 0.6) = F_x(0.6) - F_x(0.3) = 0.6^2 - 0.3^2 = 0.27$$

$$or \int_{0.3}^{0.6} 2x dx = x^2 \Big]_{0.3}^{0.6}$$

d) Let Y = 1/X compute E(1/X).

$$E(\frac{1}{x}) = \int_{-\infty}^{\infty} \frac{1}{x} f_x(x)dx = \int_0^1 \frac{1}{x}(2x)dx = 2x \Big|_0^1 = 2$$

P4) Given a random variable X having a normal distribution with $\mu_x = 50$ and $\sigma_x = 10$. A new Random Variable $Y = X^2$

a) find the probability that X is less than 50

$$P(X < 50) = \Phi(\frac{50-50}{10}) = 0.5$$

b) find the probability that X is between 45 and 62

$$P(45 < X < 62) = \Phi(\frac{62-50}{10}) - \Phi(\frac{45-50}{10}) = 0.7580 - 0.3085 = 0.4495$$

c) find the mean of Y

$$\sigma_x^2 = 100 = E(x^2) - \mu_x^2 = E(x^2) - 50^2$$

$$E(x^2) = 2600$$

d) find the probaility density function of Y.

$$f_y(y) = 2\frac{f_x(x)}{|dx/dy|} = \frac{2}{2x}\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} = \frac{e^{\frac{-y}{2}}}{\sqrt{2\pi y}}; y \ge 0$$

b. Find the probability that at least two packets arrive in two seconds.

$$P(X \geq 2) = 1 - P(X < 2) = 1 - (5*2)^0 \frac{e^{-5*2}}{0!} - (5*2)^1 \frac{e^{-5*2}}{1!} = 1 - 11e^{-10} = 0.9995$$

c. Find the mean value of packets that arrive in two seconds.

$$\mu_x = E\{X\} = \lambda T = 2*5 = 10$$

P9) The monthly income of Ahmed is a Gaussian random variable X with mean NIS 3000 and standard deviation NIS 400 and is independent from month to month

a. Find the probability that Ahmad's income in a given month is greater than NIS 3500

$$P(X>3500)=1-P(X \leq 3500) = 1 - \Phi\left(\frac{3500-3000}{400}\right) = 0.1056$$

b. Find the probability that his income in a given month is between NIS 2600 and 3400

$$P(2600<X<3400)=\Phi\left(\frac{3400-3000}{400}\right) - \Phi\left(\frac{2600-3000}{400}\right)$$

$$2\Phi(1) - 1 = 2*0.8413 = 0.6826$$

c. Find the probability that his income in two consecutive months is greater than NIS 3000 in each one of them.

Each month greater than NIS 3000

$$P(X>3000)=1-P(X \leq 3000) = 1 - \Phi\left(\frac{3000-3000}{400}\right) = 0.5$$

Two consecutive months is greater than NIS 3000 in each one of them.

$P(X>3000) \cap P(X>3000)=P(X>3000)P(X>3000)=0.5*0.5=0.25$

Since all months are independent.

Probability and Statistical Engineering, ENEE2307

Class work

Q1. Classify the following random variables as discrete or continuous:
X: the number of automobile accidents per year in USA.
Y: the length of time to play 18 holes of golf.
M: the amount of milk produced yearly by a particular cow.
N: the number of eggs laid each month by a hen.
P: the number of building permits issued each month in a certain city.
Q: the weight of grain produced per acre.

Solution:
X: Discrete
Y: Continuous
M: Continuous
N: Discrete
P: Discrete
Q: Continuous

Q2.Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each sample point assign a value w of W.

W=(H-T)=

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$W = \{3, 1, -1, -3\}$

Space	w
HHH	3
HHT,HTH,THH	1
HTT,THT,TTH	-1
TTT	-3

Q3.Determine the value c so that the following functions can serve as a probability distribution of the discrete random variable X:

$f(x) = c(x^2 + 4)$, for $x = 0, 1, 2, 3$;

$$\sum_{-\infty}^{\infty} f_x(x) = 1 = \sum_{0}^{3} c\left(x^2 + 4\right) = c[0^2 + 4 + 1^2 + 4 + 2^2 + 4 + 3^2 + 4] = 30c$$

$$c = \frac{1}{30}$$

Q4. A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If x is the number of defective sets purchased by the hotel,

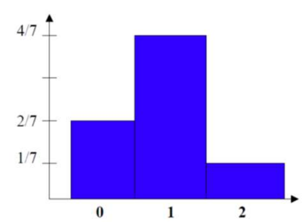
- find the probability distribution of X. Express the results graphically as a probability histogram.
- Find the cumulative distribution function of the random variable X Then using $F(x)$, find (a) $P(X = 1)$; (b) $P(0 < X \leq 2)$.

$$P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad P(X = 0) = \frac{\binom{2}{0}\binom{7-2}{3-0}}{\binom{7}{3}} = \frac{2}{7}$$

$$P(X = 1) = \frac{\binom{2}{1}\binom{7-2}{3-1}}{\binom{7}{3}} = \frac{4}{7} \qquad P(X = 2) = \frac{\binom{2}{2}\binom{7-2}{3-2}}{\binom{7}{3}} = \frac{1}{7}$$

The probability distribution of x is

x	0	1	2
P(X=x)	2/7	4/7	1/7



Probability histogram

Cumulative distribution function of the random variable X

x	0	1	2
F(x)	2/7	6/7	7/7

$$P(X = 1) = F(1) - F(0) = \frac{6}{7} - \frac{2}{7} = \frac{4}{7}$$

$$P(0 < X \leq 2) = F(2) - F(0) = 1 - \frac{2}{7} = \frac{5}{7}$$

Q5.An important factor in solid missile fuel is the particle size distribution. Significant problems occur if the particle sizes are too large. From production data in the past, it has been determined that the particle size (in micrometers) distribution is characterized by

$$f(x) = \begin{cases} 3x^{-4}; x > 0 \\ 0; otherwise \end{cases}$$

(a) Verify that this is a valid density function.
(b) Evaluate $F(x)$.
(c) What is the probability that a random particle from the manufactured fuel exceeds 4 micrometers?

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{1}^{\infty} \frac{3}{x^4} dx =$$

$$\left. \frac{-3}{3x^3} \right|_1^{\infty} = 1$$

$$F(X = x) = \int_{-\infty}^{x} f(x)dx = \int_{1}^{x} \frac{3}{x^4} dx =$$

$$\left. \frac{-3}{3x^3} \right|_1^{x} = \frac{-3}{3x^3} - \frac{-3}{3} = 1 - x^{-3}$$

$$F(X \geq 4) = 1 - F(4) = 1 - \left(1 - 4^{-3}\right) = 0.015$$

Q6.The hospitalization period, in days, for patients following treatment for a certain type of kidney disorder is a random variable $Y = X + 4$, where X has the density function

$$f(x) = \begin{cases} \dfrac{32}{(x+4)^{-3}}; x > 0 \\ 0; otherwise \end{cases}$$

a- Find the average number of days that a person is hospitalized following treatment for this disorder.
b- Find the variance of Y.

$$\mu_y = E(y) = E(x+4) = \int_{-\infty}^{\infty} (x+4) f_x(x)dx = \int_0^{\infty} \frac{32(x+4)}{(x+4)^3} dx = \int_0^{\infty} \frac{32}{(x+4)^2} dx$$

$$let\_u = x+4; du = 1$$

$$\int_4^{\infty} \frac{32}{u^2} du = \frac{32}{u^1}\Big|_4^{\infty} = 8$$

$$\sigma_y^2 = \int_{-\infty}^{\infty} (x+4-8)^2 f_x(x)dx = \int_0^{\infty} \frac{32(x-4)^2}{(x+4)^3} dx = 32\ln|x+4| \,|_0^{\infty} = \infty$$

Q7. According to *Chemical Engineering Progress* (November 1990), approximately 30% of all pipework failures in chemical plants are caused by operator error.

(a) What is the probability that out of the next 20 pipework failures at least 10 are due to operator error?

(b) What is the probability that no more than 4 out of 20 such failures are due to operator error?

(c) Suppose, for a particular plant, that out of the random sample of 20 such failures, exactly 5 are due to operator error. Do you feel that the 30% figure stated above applies to this plant? Comment.

Solution:

(a) This problem requires the binomial PDF because it satisfies the three criteria for the binomial distribution, namely (1) each experiment is the same and independent, (2) there are 2 outcomes, and (3) the probability of success, p , is the same for each experiment. p = 0.30 = probability that operator error caused pipework failure. n=20. x = number of successes with a range from 0 to 20.

$$f_x(x) = P(X = x) = \binom{20}{x} 0.3^x (1-0.3)^{20-x}$$

$$P(X \geq 10) = 1 - \sum_{0}^{9} \binom{20}{x_i} 0.3^x (1-0.3)^{20-x} = 0.047962$$

By approximation

$$P(X \geq 10) = 1 - \Phi\left(\frac{9 - 20*0.3}{\sqrt{20*0.3*0.7}}\right) = 1 - 0.9520 = 0.048$$

(b) The probability that no more than 4 out of 20 such failures are due to operator error

(c) Suppose, for a particular plant, that out of the random sample of 20 such failures, exactly 5 are due to operator error. Do you feel that the 30% figure

$$P(X \le 4) = \sum_{0}^{4} \binom{20}{x_i} 0.3^{x} (1-0.3)^{20-x} = 0.237508$$

By approximation $$P(X \le 4) = \Phi\left(\frac{4 - 20*0.3}{\sqrt{20*0.3*0.7}}\right) = 0.2375$$

Suppose, for a particular plant, that out of the random sample of 20 such failures, exactly 5 are due to operator error. Do you feel that the 30% figure stated above applies to this plant

$$P(X = 5) = \binom{20}{5} 0.3^{5} (0.7)^{15} = 0.1789$$

Q8. From a lot of 10 missiles, 4 are selected at random and fired. If the lot contains 3 defective missiles that will not fire, what is the probability that (a) all 4 will fire? (b) at most 2 will not fire?

Solution: Hyper Geometric

a) all 4 will fire

$$P(X = 0) = \frac{\binom{3}{0}\binom{10-3}{4-0}}{\binom{10}{4}} = \frac{1}{6}$$

(b) at most 2 will not fire

$$P(X \le 2) = \sum_{0}^{2} \frac{\binom{3}{x_i}\binom{10-3}{4-x_i}}{\binom{10}{4}} = \frac{29}{30}$$

Q9.On average, 3 traffic accidents per month occur at a certain intersection. What is the probability that in any given month at this intersection

(a) exactly 5 accidents will occur?

(b) fewer than 3 accidents will occur?

(c) at least 2 accidents will occur?

Poisson distribution $\mu_x = E\{X\} = b = 3$

$$P(X = x) = \frac{e^{-b} b^x}{x!}$$

$$a) P(X = 5) = \frac{e^{-3} 3^5}{5!} = 0.10082$$

$$b) P(X < 3) = \sum_{0}^{2} \frac{e^{-3} 3^x}{x!} = 0.4232$$

$$c) P(X \geq 2) = 1 - P(X < 2) = 1 - \sum_{0}^{1} \frac{e^{-3} 3^x}{x!} = 0.8009$$

Q10. A research scientist reports that mice will live an average of 40 months when their diets are sharply restricted and then enriched with vitamins and proteins. Assuming that the lifetimes of such mice are normally distributed with a standard deviation of 6.3 months, find the probability that a given mouse will live

(a) more than 32 months; (b) less than 28 months; (c) between 37 and 49 months.

Solution

$$\mu_x = E\{X\} = b = 40; \sigma = 6.3$$

a) more than 32 months

$$P(X \geq 32) = 1 - \Phi(\frac{32 - 40}{6.3}) = 1 - 0.1020 = 0.898$$

(b) less than 28 months

$$P(X \leq 28) = \Phi(\frac{28 - 40}{6.3}) = 0.0281$$

b) between 37 and 49 months.

$$P(37 \leq X \leq 49) = \Phi(\frac{49 - 40}{6.3}) - \Phi(\frac{37 - 40}{6.3}) =$$

$$\Phi(1.429) - \Phi(0.4762) = 0.9236 - 0.3156 = 0.608$$