• Causality and stability of Discrete-Time LTI Systems

• Causality

Recall that a system is causal if its output $y[n_0]$ at time n_0 depends only on the input $x[n]$ for values of $n \leq n_0$. Consider then the characterization of the system by the convolution sum:

$$y[n] = x[n] * h[n] = \sum_{m=\infty}^{\infty} x[m]\, h[n-m]$$

We replace the time variable (n) by a fixed value n_0 :

$$y[n_0] = \sum_{m=\infty}^{\infty} x[m]\, h[n_0 - m]$$

For this input to depend only on values of time $m \leq n_0$, we must have its contribution to the integral being nulled out by requiring the value of the impulse response $h[n_0-m] = 0$ for values of $m > n_0$. By making a change of variables $n = n_0 - m$ (so that $m = n_0 - n$), this means that we require $h[n] = 0$ for $n_0 - n > n_0$; this means that we require $h[n] = 0$ for $n < 0$.

Thus, we obtain that an LTI system with impulse response $h[n]$ is causal if and only if $h[n] = 0$ for all $n < 0$

• BIBO Stability

Recall that a system is BIBO stable if whenever the input $x[n]$ is bounded (that is, if there exists $M < \infty$ such that $|x[n]| < M$ for all n) then we have that the system output is also bounded (that is, if there exists $N < \infty$ such that $|y[n]| < N$ for all n). As before: consider then the characterization of the system by the convolution integral:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \right| \leq \sum_{m=-\infty}^{\infty} |x[m]\, h[n-m]|$$

$$\leq M \sum_{m=-\infty}^{\infty} |h[n-m]|$$

where $\displaystyle\sum_{m=-\infty}^{\infty} |h[n-m]| = \sum_{m=-\infty}^{\infty} |h[n]|$ for LTI system

Thus, we obtain that an LTI system with impulse response $h[n]$ is stable if and only if $\displaystyle\sum_{m=-\infty}^{\infty} |h[n]|$ is finite

- Interconnection Schemes of LTI systems

  - Cascade Connection

$$\longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow$$

$$\equiv$$

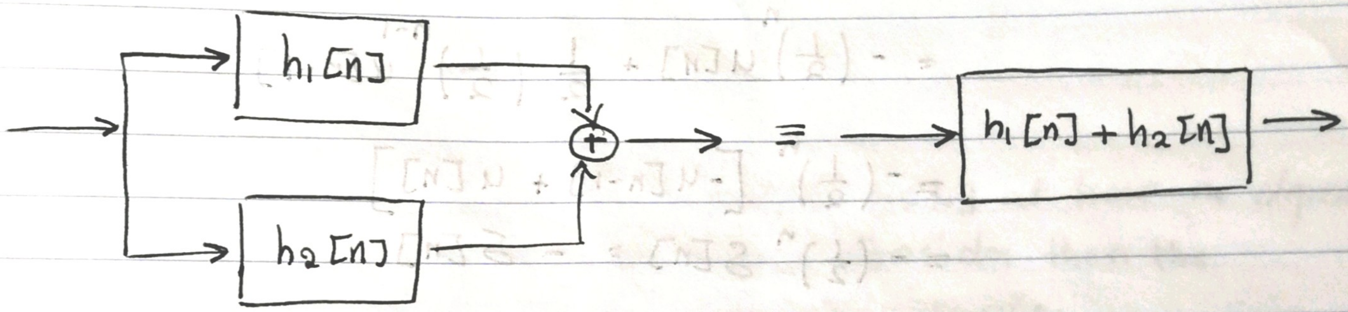
$$\longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow$$

$$\equiv$$

$$\longrightarrow \boxed{h_1[n] * h_2[n]} \longrightarrow$$

Note:
- Cascade of stable system is stable.
- The ordering of the Systems in the cascade has no effect on the overall impulse response because of the commulative property of convolution.

• Parallel Connection :



Note: Parallel of stable system is also stable.

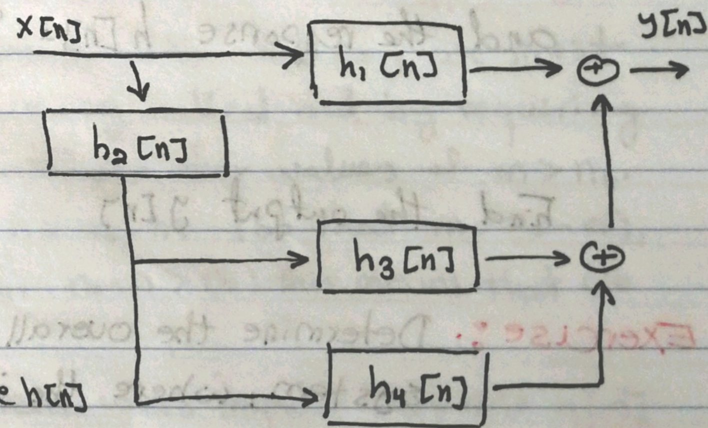
Example: Consider a Discrete-Time system where,

$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$

$h_3[n] = 2\delta[n]$

$h_4[n] = -2\left(\frac{1}{2}\right)^n u[n]$



Evaluate the overall impulse response h[n]

Ans:

The overall impulse response h[n] is given:

$$h[n] = h_1[n] + h_2[n] * (h_3[n] + h_4[n])$$
$$= h_1[n] + h_2[n] * h_3[n] + h_2[n] * h_4[n]$$
$$= \delta[n] + \frac{1}{2}\delta[n-1] + h_2[n] * h_3[n] + h_2[n] * h_4[n]$$

where

$$h_2[n] * h_3[n] = \left(\frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]\right) * 2\delta[n]$$
$$= \delta[n] - \frac{1}{2}\delta[n-1]$$

and $h_2[n] * h_4[n] = \left(\frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]\right) * -2\left(\frac{1}{2}\right)^n u[n]$

$$= -\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= -\left(\frac{1}{2}\right)^n \left[-u[n-1] + u[n]\right]$$

$$= -\left(\frac{1}{2}\right)^n \delta[n] = -\delta[n]$$

$\Rightarrow$

$$h[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \delta[n] - \frac{1}{2}\delta[n-1] - \delta[n]$$
$$= \delta[n]$$

Exercise: Consider LTI system has the input $x[n] = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{o.w} \end{cases}$

and the response $h[n] = \begin{cases} 1.8 - 0.3n & 0 \le n \le 5 \\ 0 & \text{o.w} \end{cases}$

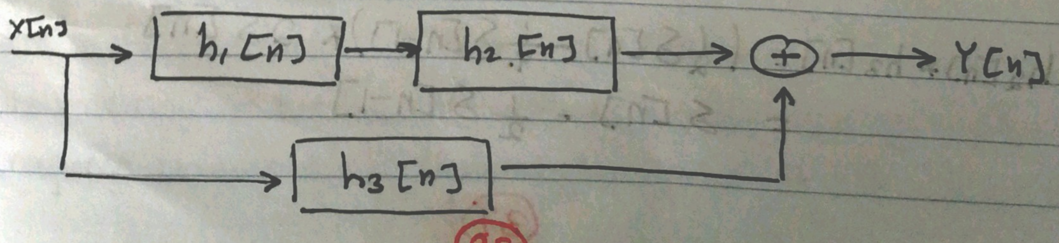
Find the output $y[n]$

Exercise: Determine the overall impulse response of the following system, where the impulse responses of the component system are:

$$h_1[n] = \{-3, \underline{0}, 0, 2\}$$

$$h_2[n] = \{2, 0, \underline{0}, 0, 1\}$$

$$h_3[n] = \{3, \underline{-1}, 2, 0, 7, 0, 5\}$$



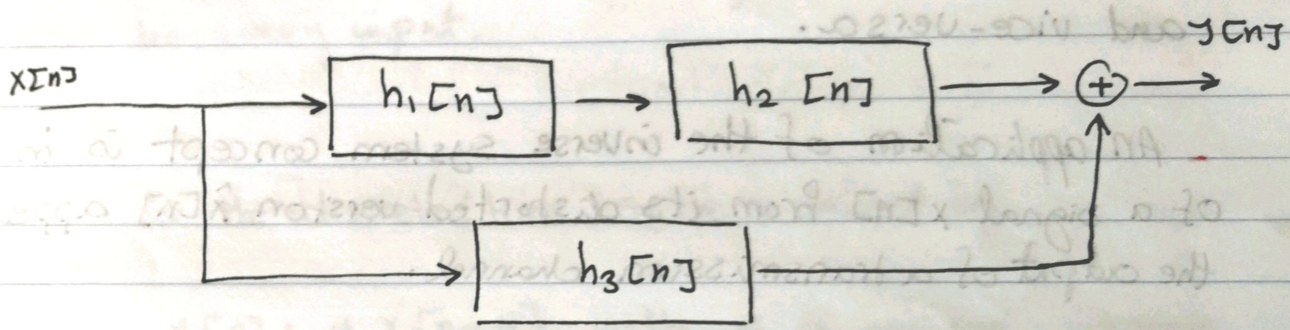
Exercise: Determine the overall impulse response of the following system, where the impulse responses of the components systems are:

$$h_1[n] = 2\delta[n-2] - 3\delta[n+1]$$

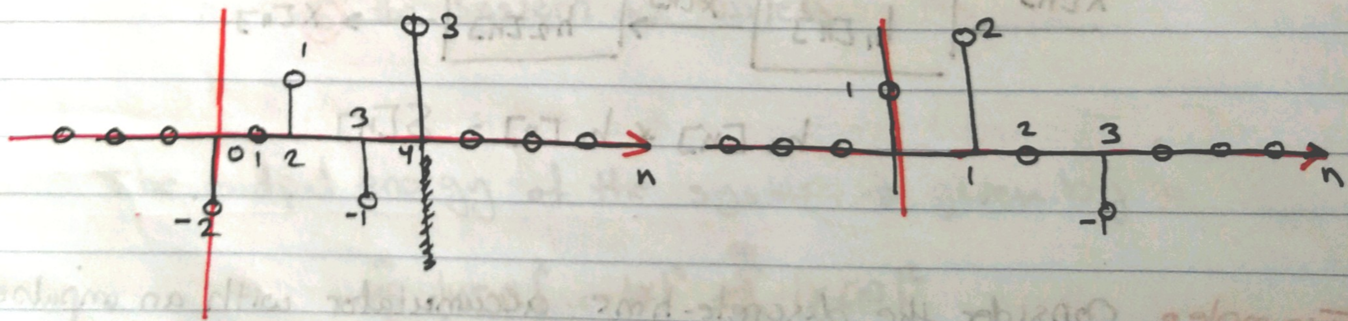
$$h_2[n] = \delta[n-1] + 2\delta[n+2]$$

and

$$h_3[n] = 5\delta[n-5] + 7\delta[n-3] + 2\delta[n-1] - \delta[n] + 3\delta[n+1]$$



Exercise: Develop the sequence y[n] generated by the convolution of the sequences x[n] and h[n] shown below



- **Convolution Using Matlab**

The M-file **conv** implements the convolution sum of two finite-length sequences

If $a = [-2 \quad 0 \quad 1 \quad -1 \quad 3]$

$b = [1 \quad 2 \quad 0 \quad -1]$

Then conv(a,b) yields

[-2 -4 1 3 5 1 -3]

• Inverse System

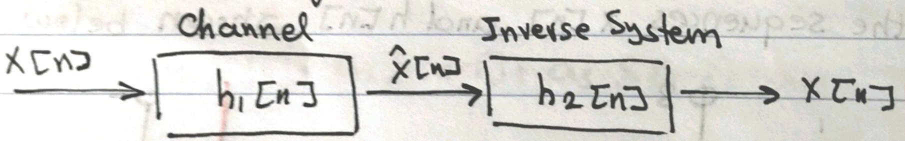
- The LTI system $h_1[n]$ is said to be inverse of $h_2[n]$ if

$$h_1[n] * h_2[n] = \delta[n]$$

and vice-versa.

- An application of the inverse system concept is in the recovery of a signal $x[n]$ from its distorted version $\hat{x}[n]$ appearing at the output of a transmission channel.

- If the impulse response of the channel is known, then $x[n]$ can be recovered by designing an inverse system of the channel



$$h_1[n] * h_2[n] = \delta[n]$$

Example: Consider the discrete-time accumulator with an impulse response. Determine the value of $h_2[n]$ that satisfy

$$u[n] * h_2[n] = \delta[n] \longrightarrow (*)$$

Ans: To satisfy (*), $h_2[n]$ should be

$$h_2[n] = \delta[n] - \delta[n-1]$$

• Passive and Lossless Systems

- A discrete-time system is defined to be **Passive** if, for every finite-energy input $x[n]$, the output $y[n]$ has, at most, the same energy, i.e.,

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- For a **lossless** system, the above inequality is satisfied with an equal sign for every input.

Example: Consider the discrete-time system defined by

$$y[n] = \alpha\, x[n-N] \quad \text{with } N \text{ a positive integer}$$

check if

(a) the system is passive.

(b) the system is lossless

Ans: The output energy of the system is given by

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n]|^2$$

(a) The system will be passive if $|\alpha| < 1$

(b) The system will be lossless if $|\alpha| = 1$

• Impulse and Step Responses •

• The response of a discrete-time system to a unit sample sequence $\delta[n]$ is called the unit sample response or, simply, the impulse response and is denoted by $h[n]$

• The response of a discrete-time system to a unit step sequence $u[n]$ is called the unit step response or simply, the step response, and is denoted by $s[n]$.

Example: The impulse response of the system

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3] \text{ is ?}$$

Ans:

• The impulse response of the system is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

• The impulse response is thus a finite-length sequence of length 4 given by

$$h[n] = \{\underline{a_1}, a_2, a_3, a_4\}$$

Example: The impulse response of the discrete-time accumulator $y[n] = \sum_{\ell=\infty}^{n} x[\ell]$ is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = \sum_{\ell=\infty}^{n} \delta[\ell] = u[n]$$

- Correlation of Discrete-Time Signals

A signal operation similar to signal convolution, but with completely different physical meaning, is signal correlation. The signal correlation operation can be performed either with one signal (auto-correlation) or between two different signals (cross-correlation). Physically, signal autocorrelation indicates how the signal energy (power) is distributed within the signal, and as such is used to measure the signal power. Typical applications of signal autocorrelation are in radar, sonar, satellite, and wireless communication systems. Devices that measure signal power using signal correlation are known as signal correlators. There are also many applications of signal crosscorrelation in signal processing systems, especially when the signal is corrupted by another undesirable signal (noise) so that the signal estimation (detection) from a noisy signal has to performed. Signal cross-correlation can be also considered as a measure of similarity of two signals.

- Given two-discrete-time real signals (sequences) $x[k]$ and $y[k]$. The autocorrelation and cross-correlation functions are respectively defined by

$$R_{xx}[k] = \sum_{m=-\infty}^{\infty} x[m]\,x[m-k] \quad , \quad R_{yy}[k] = \sum_{m=-\infty}^{\infty} y[m]\,y[m-k]$$

$$R_{xy}[k] = \sum_{m=-\infty}^{\infty} x[m]\,y[m-k] \quad , \quad R_{yx}[k] = \sum_{m=-\infty}^{\infty} y[m]\,x[m-k]$$

where the parameter k is any integer, $-\infty \le k \le \infty$

- Using the definition for the total discrete-time signal energy, we see that for $k=0$, the autocorrelation function represents the total signal energy. that is

$$R_{xx}[0] = E_\infty^x \quad , \quad R_{yy}[0] = E_\infty^y$$

- Naturally, the autocorrelation and crosscorrelation sums are convergent under assumptions that the signals $x[k]$ and $y[k]$ have finite total energy. It can be observed that $R_{xx}[k] \leq R_{xx}[0] = E_\infty^x$.
In addition, it is easy to show that the autocorrelation function is an even function, that is

$$R_{xx}[k] = R_{xx}[-k]$$

Hence, the autocorrelation function is symmetric with respect to the vertical axis.

Proof:

$$R_{xx}[k] = \sum_{m=-\infty}^{\infty} x[m]\, x[m-k]$$

By using the change of variables as $m = n+k$

$$\Rightarrow R_{xx}[k] = \sum_{n=-\infty}^{\infty} x[n]\, x[n+k] = R_{xx}[-k]$$

Example: show that

$$R_{xy}[k] = R_{yx}[-k]$$

Proof:

$$R_{xy}[k] = \sum_{m=-\infty}^{\infty} x[m]\, y[m-k]$$

By using change of variables as $m = n+k$

$$R_{xy}[k] = \sum_{n=-\infty}^{\infty} x[n+k]\, y[n+k-k]$$

$$= \sum_{n=-\infty}^{\infty} y[n]\, x[n+k] = R_{yx}[-k]$$

Example: Show that

$$R_{xx}[k] = x[k] * x[-k]$$

and

$$R_{xy}[k] = x[k] * y[-k]$$

Proof:

$$R_{xx}[k] = \sum_{m=-\infty}^{\infty} x[m]\, x[m-k]$$

By assuming $m = k-n$ in the definition formula for the aut-correlati...

$$R_{xx}[k] = \sum_{n=-\infty}^{\infty} x[k-n]\, x[k-n-k]$$

$$= \sum_{n=-\infty}^{\infty} x[k-n]\, x[-n]$$

$$\triangleq x[k] * x[-k]$$

By assuming $m = n+k$ in the definition formula for the cross-correlation function

$$R_{xy}[k] = \sum_{m=-\infty}^{\infty} x[m]\, y[m-k]$$

$$= \sum_{n=-\infty}^{\infty} x[k-n]\, y[k-n-k]$$

$$= \sum_{n=-\infty}^{\infty} y[-n]\, x[k-n] \triangleq x[k] * y[-k]$$

- **Correlation Coefficient:** measures the similarity of two signals. It can be defined as

$$C_{xy} = R_{xy}[0] \Big/ \sqrt{R_{xx}[0]\, R_{yy}[0]}$$

- The correlation coefficient satisfies $-1 \le C_{xy} \le 1$

Example: show that the correlation coefficient satisfies

$$-1 \le C_{xy} \le 1$$

proof:

$$C_{xy} = \frac{R_{xy}[0]}{\sqrt{R_{xx}[0]\, R_{yy}[0]}}$$

Since $R_{xy}[0]$, $R_{xx}[0]$, and $R_{yy}[0]$ can be expressed in terms of inner product of two vectors where,

$$R_{xy}[0] = X \cdot Y \qquad R_{xx}[0] = X \cdot X = |X|^2$$

and $R_{yy}[0] = Y \cdot Y = |Y|^2$

where

- $|\cdot|$ defined Euclidean norm operation.

$\Rightarrow$

$$C_{xy} = \frac{X \cdot Y}{|X|\,|Y|} = \cos(\Theta_{xy})$$

Since $-1 \le \cos(\Theta_{xy}) \le 1$

$\Rightarrow$ $-1 \le C_{xy} \le 1$

It can be noted that:

- $C_{xy} = 1$: means that the signals are similar
- $C_{xy} = 0$: means that the signals are very different (orthogonal signals).

• $C_{XY} = -1$: means that the signals are similar but in opposite direction.

• The correlation coefficient can be also defined in terms of parameter k, that is

$$-1 \le C_{XY}[k] = \frac{R_{XY}[k]}{\sqrt{R_{XX}[0]\, R_{YY}[0]}} \le 1$$

in which case the same lower and upper bounds hold due to the fact that

$$|R_{XY}[k]| \le \sqrt{|R_{XX}[0]|\,|R_{YY}[0]|}$$

Example: Consider the following combined sequence

$$a\,x[n] + y[n-L]$$

which is finite and non-negative sequence since $x[n]$ and $y[n]$ are two finite-energy sequences.

• Linear Constant-Coefficient Difference Equations

An important subclass of Linear time-invariant systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an N^{th}-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \quad \text{------- } (0*)$$

where N^{th}-order denotes the no. of delays in output sequence.

- If we assume $N=0$, and $a_0 = 1$, we obtain

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] \quad \text{------- } (1*)$$

It can be noted that this equation is identical to convolution sum where,

$$h[n] = \begin{cases} b_n & n = 0, 1, \ldots, M \\ 0 & \text{o.w} \end{cases}$$

On the other hand, if we assume $N \neq 0$ and $a_0 = 1$, (0*) becomes

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] - \sum_{k=1}^{N} a_k y[n-k]$$

In this case we need initial condition to solve this equation

Example: Determine the impulse response of the first order System. Assume $y[n]=0$ for $n<0$

Ans: $y[n] = x[n] + a\,y[n-1]$

For impulse response $x[n] = \delta[n]$ and $y[n] = h[n]$

$\Rightarrow h[n] = \delta[n] + a\,h[n-1]$

By applying the initial condition where $y[n] = 0$ for $n < 0$

$\Rightarrow y[-1] = 0 = h[-1]$

$h[0] = y[0] = \delta[0] + a\,h[-1]$

$= 1$

$h[1] = y[1] = a$

$y[2] = a^2$

$\vdots$

$\Rightarrow$

$h[n] = a^n u[n]$

Now, let us assume $x[n] = \delta[n]$ and $y[n] = 0$ for $n > 0$

$\Rightarrow$ We have to use the following expression

$y[n-1] = a^{-1}\left[y[n] - \delta[n]\right]$

when $n = 2 \Rightarrow y[1] = 0$

$n = 1 \Rightarrow y[0] = 0$

$n = 0 \quad y[-1] = -a^{-1}$

$n = -1 \quad y[-2] = -a^{-2}$

$\vdots$

$y[n] = -a^n u[-n-1]$