chapter 12 : exercises

Q1: given an example of a finite noncommutative Ring. give an example of an infinite noncommutative Ring that does not have a unity.

For any $n>1$, the Ring $M_2(\mathbb{Z}_n)$ of 2×2 matrices with entries from $\mathbb{Z}_n$ is a finite noncommutative Ring Ring that does not have a unity.

Q3: give an example of a subset of a Ring that is a subgroup under addition but Not a subring.

In $\mathbb{R}$, consider $\{n\sqrt{2} : n \in \mathbb{Z}\}$.

Q4: Show, by example, that for fixed nonzero elements a and b in a Ring the equation $ax=b$ can have more than ~~not~~ one solution. How does this compare with groups.

In $(\mathbb{Z}_4, +, \cdot)$ the equation $2x=2$ has two sol. $x=1,3$.

But In group, the equation $ax=b$ has unique s.l. $x=a^{-1}b$.

Q5: prove Thm, If a Ring has a unity, it is unique. If a ring element has a multiplicative inverse, it is unique.

$\Rightarrow$ identity unique :

suppose that $e\in R$ and that $ea=a=ae$ $\forall a\in R$. suppose also that $f\in R$ and that $fa=a=af$ for all $a\in R$. then we have

$$f = ef = e.$$

There for $f=e$, Thuse, there can only be one element in R satisfying the requirements for the multiplicative identity of the ring R.

⟹ inverse unique :

suppose that $b, c \in R$ and that $ab = ba = 1$ and that $ac = ca = 1$. then $c = 1 \cdot c = (ba)c = b(ac) = b1 = b$.

Hence, we have $c = b$.

The multiplicative inverse of a is unique.

Q6: Find an integer n that shows that the Ring Z_n need not have the following properties that the Ring of integers has.

a. $a^2 = a$ implies $a=0$ or $a=1$.

$n=6$, $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$.

$\Rightarrow$ $3^2 = 3$ But 3 is neather 0 nor 1.

b. $ab=0$ implies $a=0$ or $b=0$.

$n=6$ $\Rightarrow$ $2.3 = 6$ but $2 \neq 0$ and $3 \neq 0$.

3. $ab = ac$ and $a \neq 0$ imply $b=c$.

$n=6$ $\Rightarrow$ $3.2 = 0 = 3.4$ but $3 \neq 0$ and $2 \neq 4$.

Q7: show that the three properties listed in Q6 ~~and~~ are valid for Z_p where p prime.

we know that in Z_p, every nonzero element ______ it multiplicative inverse.

a. given $a^2 = a$

For $a=0$, result is

if $a \neq 0 \rightarrow$ $a^{-1}(aa) = a^{-1}a$

$\Rightarrow$ $\underbrace{(a^{-1}a)}_{1} a = \underline{a^{-1}a}$,

$\underline{a = 1}$

b. Given $ab=0$

If $a \neq 0$, a^{-1} exists and $a^{-1}(ab) = a^{-1}\,0$

$\Rightarrow$ $(a^{-1}a)b = 0$

$\Rightarrow$ $b = 0$

If $b \neq 0$, b^{-1} exists and $(ab)b^{-1}$ exists and $(ab)b^{-1} = 0.b^{-1}$

$\Rightarrow$ $a(bb^{-1}) = 0$

$\Rightarrow$ $a.1 = 0$

c.

Q8: show that the Ring is commutative if it has the property that $ab = ca$ implies $b = c$ when $a \neq 0$.

consider $aba = ab \cdot a$

$a(ba) = (ab)a$

$a x = y \cdot a$

$\Rightarrow x = y$

$\Rightarrow ba = ab$

Q9: prove that the intersection of any collection of subrings of a ring R is a subring of R.

let G be the intersection of any collection of subrings.

→ $G \neq \emptyset$ since $0 \in G$.

→ let $a, b \in G$ then a and b are in each Rings

Thus, $a-b$ and ab are in each Ring.

But since these are in every Ring, $a-b$ and ab are also in the intersection. Hence, it is a subring.

Q10: Verify that exp 8 through 13 in this chapter are as stated.

Done ON Notes

Q11: Prove Rule 3 ?!!.

Q12: let a, b and c elements of a commutative Ring, and suppose that a is a unit. Prove that b divides c iff ab divides c.

($\Rightarrow$) suppose $b|c$ $\Rightarrow$ $c = d\,b$

$\Rightarrow$ $c = d\,(a^{-1}a)\,b$

$\Rightarrow$ $c = (d\,a^{-1})\,(ab)$

$\Rightarrow$ $ab\,|\,c$

($\Leftarrow$) suppose $ab|c$ $\Rightarrow$ $c = (ab)\,d$

$\Rightarrow$ $c = (ad)\,b$

$\Rightarrow$ $b\,|\,c$.

Q15: Show that if m and n are integers and a and b are elements from a Ring, then $(m.a)(n.b) = (mn).(ab)$.

case1 : if $m>0$, $n>0$ then

$$(ma)(nb) = (\underbrace{a+a+ --- +a}_{m\text{-times}})(\underbrace{b+b+ ---- + b}_{n\text{-times}})$$

$$= ab + ab + --- + ab \quad (nm\text{times}).$$

$$= mn(ab)$$

case 2 : If $m<0$, $n<0$ then

$$ma . nb = (\underbrace{-a + -a + --- + -a}_{-m\text{-times}})(\underbrace{-b + -b + ---- + -b}_{-n\text{-times}})$$

$$= (ab + --- + ab) \quad mn \text{ times}$$

$$= mn(ab).$$

case 3 : m or n is zero, then

either ma or $nb = 0$ and $ma . nb = 0 = mn . ab$.

case 4 : if $m>0$, $n<0$ or $m<0$, $n>0$, then

$$(ma)(nb) = (a + --- + a)(-b + --- + -b)$$

$$= -ab + --- + -ab \quad (mn\text{-time})$$

$$= mn . ab.$$

Q16: Show that if n is an integer and a is an element from a Ring then $n.(-a) = -(n.a)$.

$$n(-a) + na = 0$$
$$n(-a) = -na .$$

Q20: Describe the elements of $M_2(\mathbb{Z})$ that have multiplicative inverses:

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $M_2(\mathbb{Z})$. then A has a multiplicative inverse iff it has nonzero determinant so $ad - bc \neq 0$.

the inverse is only in $M_2(\mathbb{Z})$ if the $\frac{1}{\det(A)}$ is in $\mathbb{Z}$

Thus, the determinate of A must be ± 1.

Thus the elements with multiplicative inverse are $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1 \right\}$.

Exercises of chapter 12 :

Q2: The Ring $\{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 has a unity. Find it.

$\otimes_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

identity $= 6$.

Inverse : $2 \to 3$

$4 \to 4$

$6 \to 6$

$8 \to 2$

So the unity $= 6$. and units $= \{2, 4, 6, 8\}$.

Q17: show that a Ring that is cyclic under addition is commutative.

let $x, y \in R \Rightarrow x = na \quad , y = ma$.

$x \cdot y = (na) \cdot (ma)$

$= (nm) a^2$

$= (mn) a^2$

$= ma \cdot na$

$= y \cdot x \quad \to x \cdot y = y \cdot x$ so its comm. ⊠

Q18: let a belong to a Ring R. let $S = \{x \in R : ax = 0\}$. show that S is a subring of R.

let $a, b \in S \Rightarrow$ ① $(a-b)x = ax - bx = 0 - 0 = 0$

So $a-b \in S$.

② $(ab)x = a(bx) = a \cdot 0 = 0$ So $ab \in S$.

So S is subRing of R. ⊠

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Q19: let R be a Ring. the center of R is the set $\{x \in R : ax = xa, \forall a \in R\}$
Prove that the center of a ring is a subring.

let $a, b \in Z(R)$, $Z(R) \neq \emptyset$. $x \in Z \to ax = xa$ $(a-b)x = ax - bx$

① $(a-b)x = ax - bx$

$= xa - xb$

$= x(a-b)$.

So $(a-b) \in Z(R)$.

② $(ab)x = a(bx)$

$= a(xb)$

$= (ax)b$

$= (xa)b$

$= x(ab)$

$\to$ so $ab \in Z(R)$.

So By ① and ② $\to$ center of a ring is a subring

Q22: let R be a commutative Ring with unity and let $U(R)$ denote the set of units of R. Prove that $U(R)$ is a group under the multiplication of R.
(this group is called the group of units of R).

$U(R) = \{u \in R : \exists u^{-1} \in R\}$

let $x, y \in U(R) \Rightarrow \exists a, b \in R$ s.t $1 = x.a = ax$, $1 = y.b = by$

- Closure: $(xy)(ab) = x(yb)a = x(1)a = xa = 1 = (ba)(xy)$

So xy is unit $\to xy \in U(R)$.

- Associative: $(x.y).z = x.(y.z) \quad \forall x, y, z \in R$
- Identity: 1 is the unity of $R \Rightarrow 1.1 = 1$ so 1 is a unit $\Rightarrow 1 \in U(R)$
- Inverse: If $a \in U(R) \to \exists a^{-1}$ s.t $aa^{-1} = 1 \Rightarrow a^{-1} = a^{-1}(a) = 1$ So $a^{-1} \in U(R)$

Q23 : Determine $U(Z[i])$:

$U(Z[i]) = \{1, -1, i, -i\}$.

Q24 : If $R_1, R_2, \dots, R_n$ are commutative Rings with unity , show that
$U(R_1 \oplus R_2 \oplus \dots \oplus R_n) = U(R_1) \oplus U(R_2) \oplus \dots \oplus U(R_n)$.

($\supseteq$) suppose $u_1, u_2, \dots, u_n$ are units of $R_1, R_2, \dots R_n$ with inverses
$\sqrt{1}, \sqrt{2}, \dots \sqrt{n}$.

$(u_1, u_2, \dots, u_n)$ is units of $R_1 \oplus R_2 \oplus \dots \oplus R_n$

$U(R_1) \oplus U(R_2) \oplus \dots \oplus U(R_n) \subseteq U(R_1 \oplus R_2 \oplus \dots \oplus R_n)$.

($\subseteq$) let $X = (x_1, x_2, \dots, x_n) \in U(R_1 \oplus R_2 \oplus \dots \oplus R_n)$

So $\exists X^{-1} = (y_1, y_2, \dots y_n)$

$\Rightarrow x_1 \cdot y_1 = e = y_1 \cdot x_1$

$\Rightarrow x_i \in U(R_i)$.

$U(R_1 \oplus R_2 \dots \oplus R_n) \subseteq U(R_1) \oplus U(R_2) \dots \oplus U(R_n)$.

Q29 : suppose that a and b belong to a commutative Ring with unity . If a is a unit of R and $b^2 = 0$, show that $a+b$ is a unit of R .

$(a+b)(a-b) = a^2 - \cancel{ab} + \cancel{ba} - b^2$

$= a^2 - b^2$

$= a^2 - 0 = a^2$.

$(a+b)(a-b) \cdot a^{-2} = a^2 \cdot a^{-2}$

$= a^0$

$= 1$

So $a+b$ is a unit

Q30: suppose that there is an integer $n>1$ s.t $x^n = x$ for all elements x of some ring. If m is a positive integer and $a^m = 0$ for some a show that $a=0$.

case 1 : $n = m$

$a = a^n = a^m = 0$ and we are done.

case 2 : $n > m$

$$a = a^n = a^m a^{(n-m)}$$
$$= 0 \cdot a^{n-m}$$
$$= 0$$

case 3 : $n < m$

$$a = a^n = \underbrace{a \cdots a}_{n} = \underbrace{a^n \cdots a^n}_{n} = a^{n^n},$$

since we can do this arbitarily often, notice that $\exists K > 1$ in the form of repeated expo. of n s.t $K > m$ and $a^K = a^n = a$

Hence, this case reduces to case 2,

and we are done.

Q31: given an example of Ring elements a and b with the properties that $ab = 0$ But $ba \neq 0$.

$M_2(\mathbb{Z})$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ But $BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Q32: let n be an integer greater than 1. in a Ring in which $x^n = x$ for all x, show that $ab = 0$ implies $ba = 0$.

let $a, b \in R$ so $ab \in R$

$ba = (ba)^n$ since $ba \in R$

$= ba \cdot ba \cdot \ldots$ " n-times "

$= b \cdot (ab) \cdot (ab) \cdot \ldots$

$= b \cdot 0 \cdot 0 \cdot \ldots \cdot 0$

$= 0$

$ba = 0$ ☒

Q33: suppose that R is a ring s.t $x^3 = x$ for all x in R. Prove that $6x = 0$ for all x in R.

let $x \in R$ then $2x \in R$ since R closed under addition

$\Rightarrow \quad 2x = (2x)^3$

$2x = 8x^3$ But $x^3 = x$

$\underset{-2x}{2x} = \underset{-2x}{8x}$

$\Rightarrow 8x - 2x = 0 \Rightarrow 6x = 0$ ☒

Q50: suppose that R is a ring and that $a^2 = a$ for all a in R. show that R is commutative.

First Notice that,

$a + b = (a+b)(a+b)$

$= a^2 + ab + ba + b^2$

$\cancel{a} + \cancel{b} = \cancel{a} + ab + ba + \cancel{b}$

$0 = ab + ba$

$\Rightarrow ab = -ba$ --- ①

Next,

$-ba = (-ba)^2 = (ba)^2$

$-ba = ba$ --- ②

So By 1 and 2

$ab = ba$

☒