Discussion $\boxed{6.4}$

$\boxed{18}$ Find the area of the surface generated by revolving the curve

$$x = \frac{1}{3} y^{\frac{3}{2}} - \sqrt{y} \quad , \; 1 \le y \le 3$$

about y-axis

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{where } x = g(y) \geq 0$$

Note that $x = \frac{1}{3} y^{\frac{3}{2}} - \sqrt{y} \le 0$ on $1 \le y \le 3$ so we take $x = -\left(\frac{1}{3} y^{\frac{3}{2}} - \sqrt{y}\right) = \sqrt{y} - \frac{1}{3} y^{\frac{3}{2}}$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}} - \frac{1}{2} y^{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)^2 = \frac{1}{4}\left(\frac{1}{y} - 2 + y\right)$$

$$S = \int_1^3 2\pi \left(\sqrt{y} - \frac{1}{3} y^{\frac{3}{2}}\right) \sqrt{1 + \frac{1}{4}\left(\frac{1}{y} - 2 + y\right)}\, dy$$

$1 + \frac{1}{4}\left(\frac{1}{y} - 2 + y\right) =$
$1 - \frac{1}{4}(2) + \frac{1}{4}\left(\frac{1}{y} + y\right) =$
$\frac{1}{2} + \frac{1}{4}\left(\frac{1}{y} + y\right) =$
$\frac{1}{4}\left(\frac{1}{y} + 2 + y\right)$

$$= 2\pi \int_1^3 \left(\sqrt{y} - \frac{1}{3} y^{\frac{3}{2}}\right) \sqrt{\frac{1}{4}\left(\frac{1}{y} + 2 + y\right)}\, dy$$

$$= \cancel{2}\pi \int_1^3 \left(\sqrt{y} - \frac{1}{3} y^{\frac{3}{2}}\right) \frac{1}{\cancel{2}} \sqrt{\left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)^2}\, dy$$

$$= \pi \int_1^3 \left(\sqrt{y} - \frac{1}{3} y^{\frac{3}{2}}\right)\left(\frac{1}{\sqrt{y}} + \sqrt{y}\right) dy$$

$$= \pi \int_1^3 \left(1 + y - \frac{1}{3} y - \frac{1}{3} y^2\right) dy = \pi \int_1^3 \left(1 + \frac{2}{3} y - \frac{1}{3} y^2\right) dy$$

$$= \pi \left(y + \frac{1}{3} y^2 - \frac{1}{9} y^3\right) \Big|_1^3$$

$$= \pi \left[3 + \frac{9}{3} - \frac{27}{9} - \left(1 + \frac{1}{3} - \frac{1}{9}\right)\right] = \frac{16\pi}{9}$$

20 Find the area of the surface generated by revolving the curve

$$x = \sqrt{2y-1}\ , \quad \frac{5}{8} \le y \le 1$$

about y-axis.

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{where } x = g(y) \ge 0$$

Note that $x = \sqrt{2y-1} \ge 0$ on $\frac{5}{8} \le y \le 1$

$$\frac{dx}{dy} = \frac{1}{2}(2y-1)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2y-1}}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1}$$

$$S = \int_{5/8}^{1} 2\pi \sqrt{2y-1}\sqrt{1 + \frac{1}{2y-1}}\, dy$$

$$= \int_{5/8}^{1} 2\pi \cancel{\sqrt{2y-1}}\, \frac{\sqrt{2y-1+1}}{\cancel{\sqrt{2y-1}}}$$

$$= 2\pi \int_{5/8}^{1} \sqrt{2y}\, dy \quad = 2\sqrt{2}\,\pi \int_{5/8}^{1} y^{\frac{1}{2}}\, dy$$

$$= 2\sqrt{2}\,\pi\, \frac{2}{3}\, y^{\frac{3}{2}} \Big|_{5/8}^{1} \quad = \frac{4\sqrt{2}\,\pi}{3} \sqrt{y^3} \Big|_{\frac{5}{8}}^{1}$$

$$= \frac{4\sqrt{2}\,\pi}{3}\left(\sqrt{1} - \sqrt{\left(\frac{5}{8}\right)^3}\right)$$

$$= \frac{\pi}{12}\left(16\sqrt{2} - 5\sqrt{5}\right)$$