

5.3 The Definite Integral

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Let $f(x)$ be a function defined on a closed interval $[a, b]$.

We say a number J is the definite integral of f over $[a, b]$

and we write $J = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k$ if Riemann sum

Given $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have $\left| \sum_{k=1}^n f(c_k) \Delta x_k - J \right| < \epsilon$

So, we say the Riemann sums of f on $[a, b]$ converge to the definite integral J . That is

$$\lim_{\|P\| \rightarrow 0} S_n = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = J = \int_a^b f(x) dx$$

b ← upper limit of integration
 $f(x)$ ← the integrand
 x ← the variable of integration
 a ← lower limit of integration
 dx ← "dummy variable"

• When the partition P has n equal subintervals each of width Δx ,
 $\Delta x = \frac{b-a}{n}$, then $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right) = J = \int_a^b f(x) dx$

$\frac{b-a}{n}$
 $a + k \left(\frac{b-a}{n} \right)$

• Note that $\int_a^b f(t) dt = \int_a^b f(x) dx = \int_a^b f(u) du$

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Example Express the following limits as definite integrals:

1) $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k^2 \Delta x_k$, where P is a partition of $[0, 2]$ = $\int_0^2 x^2 dx$

2) $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 - 3c_k) \Delta x_k$, where P is a partition of $[-7, 5]$ = $\int_{-7}^5 (x^2 - 3x) dx$

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Th (Integrability of Continuous Functions)

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If f is continuous on $[a, b]$ or
If f has finitely many jump discontinuities on $[a, b]$,
Then the definite integral $\int_a^b f(x) dx$ exists and f is integrable on $[a, b]$.

Th If f and g are integrable on $[a, b]$, then

$$[1] \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$[2] \int_a^a f(x) dx = 0$$

$$[3] \int_a^b k f(x) dx = k \int_a^b f(x) dx$$

$$[4] \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$[5] \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

[6] If f has max value M and min value m on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

[7] If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$

If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$

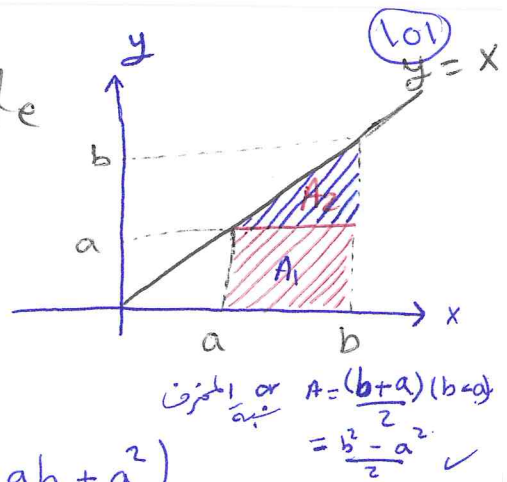
Example show that $\int_0^1 \sqrt{1+\sin x} dx \leq \sqrt{2}$

$$\int_0^1 \sqrt{1+\sin x} dx \leq \int_0^1 \sqrt{1+1} dx = \int_0^1 \sqrt{2} dx \leq \sqrt{2}(1-0) = \sqrt{2}$$

Def If $y = f(x)$ is nonnegative and integrable on $[a, b]$, then
the area under the curve $y = f(x)$ on $[a, b]$ is the integral of f
from a to b : $A = \int_a^b f(x) dx$.

Example: Use areas to evaluate

$$\begin{aligned} \textcircled{1} \int_a^b x \, dx &= A_1 + A_2 \\ &= (b-a)a + \frac{1}{2}(b-a)(b-a) \end{aligned}$$

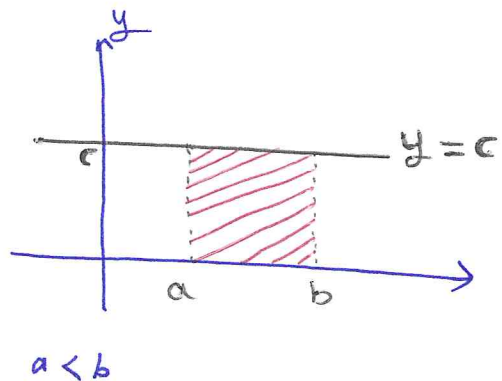


$$\int_a^b x \, dx = ab - a^2 + \frac{1}{2}(b^2 - 2ab + a^2)$$

$$= \cancel{ab} - a^2 + \frac{b^2}{2} - \cancel{ab} + \frac{a^2}{2}$$

$$\boxed{\int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}}$$

$$\textcircled{2} \int_a^b c \, dx = (b-a)c$$



$$\textcircled{3} \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}$$

Example: Use Riemann sum to calculate $\int_0^b x \, dx$

$$\int_0^b x \, dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(\frac{kb}{n}\right) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{kb}{n} \cdot \frac{b}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{b^2}{n^2} \sum_{k=1}^n k$$

$$= \lim_{n \rightarrow \infty} \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{b^2}{2} \left(1 + \frac{1}{n}\right)$$

$$= \frac{b^2}{2}$$

• Consider the partition P that divides $[0, b]$ into n subintervals of equal width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$

• $P = \left\{ 0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n} = b \right\}$
 $\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \dots, \left[\frac{(n-1)b}{n}, b\right]$

• choose c_k to be the right endpoint
 $c_k = \frac{kb}{n}$

Def If f is integrable on $[a, b]$, then the (102) average value of f on $[a, b]$ "Mean" is

$$av(f) = \frac{1}{b-a} \int_a^b f(x) dx$$

Example Find the average value of $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$.

$$av(f) = \frac{1}{\sqrt{3}-0} \int_0^{\sqrt{3}} (x^2 - 1) dx = \frac{1}{\sqrt{3}} \left[\int_0^{\sqrt{3}} x^2 dx - \int_0^{\sqrt{3}} dx \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{(\sqrt{3})^3}{3} - \frac{(0)^3}{3} - (\sqrt{3} - 0) \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{3\sqrt{3}}{3} - \sqrt{3} \right]$$

$$= \frac{1}{\sqrt{3}} [\sqrt{3} - \sqrt{3}]$$

$$= 0$$