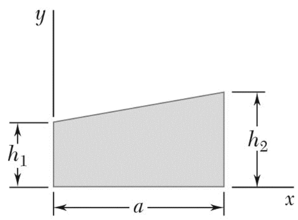
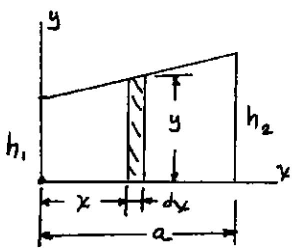
CHAPTER 9



PROBLEM 9.1

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y = h_1 + (h_2 - h_1)\frac{x}{a};\; dA = y\,dx$$

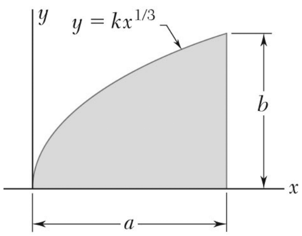
$$dI_y = x^2 dA = x^2\left[h_1 + (h_2 - h_1)\frac{x}{a}\right]dx$$

$$I_y = \int_0^a \left[h_1 x^2 + \frac{h_2 - h_1}{a}x^3\right]dx$$

$$= h_1\frac{a^3}{3} + \frac{h_2 - h_1}{a}\frac{a^4}{4}$$

$$= \frac{b_1 a^3}{12} + \frac{h_2 a^3}{4}$$

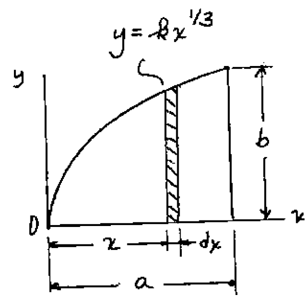
$$I_y = \frac{a^3}{12}(h_1 + 3h_2) \blacktriangleleft$$



PROBLEM 9.2

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y = kx^{1/3}$$

For $x = a$:

$$b = ka^{1/3}$$

$$k = b/a^{1/3}$$

Thus:

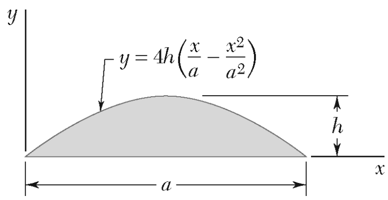
$$y = \frac{b}{a^{1/3}}x^{1/3}$$

$$dI_y = x^2 dA = x^2 y dx$$

$$dI_y = x^2 \frac{b}{a^{1/3}}x^{1/3}dx = \frac{b}{a^{1/3}}x^{7/3}dx$$

$$I_y = \int dI_y = \frac{b}{a^{1/3}}\int_0^a x^{7/3}dx = \frac{b}{a^{1/3}}\left(\frac{3}{10}a^{10/3}\right)$$

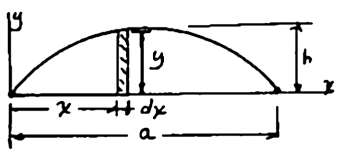
$$I_y = \frac{3}{10}a^3 b \blacktriangleleft$$



PROBLEM 9.3

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y = 4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)$$

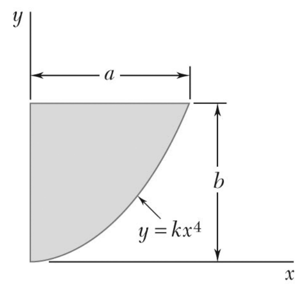
$$dA = y\,dx$$

$$dI_y = x^2 dA = 4hx^2\left(\frac{x}{a} - \frac{x^2}{a^2}\right)dx$$

$$I_y = 4h\int_0^a \left(\frac{x^3}{a} - \frac{x^4}{a^2}\right)dx$$

$$I_y = 4h\left[\frac{x^4}{4a} - \frac{x^5}{5a^2}\right]_0^a = 4h\left(\frac{a^3}{4} - \frac{a^3}{5}\right)$$

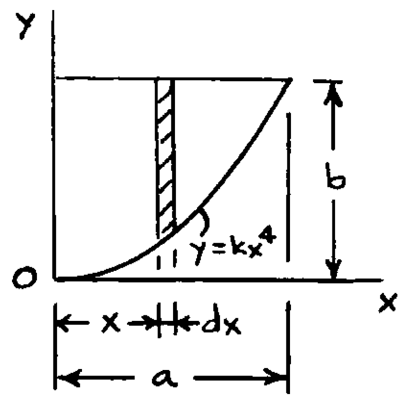
$$I_y = \frac{1}{5}ha^3 \blacktriangleleft$$



PROBLEM 9.4

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y = kx^4$$

For $x = a$: $$b = ka^4$$

$$k = \frac{b}{a^4}$$

Thus: $$y = \frac{b}{a^4}x^4$$

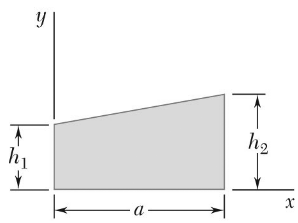
$$dA = (b - y)dx$$

$$dI_y = x^2 da = x^2(b - y)dx$$

$$= x^2\left(b - \frac{b}{a^4}x^4\right)dx$$

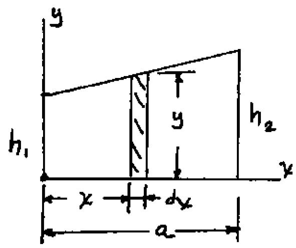
$$I_y = \int dI_y = \int_0^a \left(bx^2 - \frac{b}{a^4}x^6\right)dx = \frac{1}{3}a^3b - \frac{1}{7}a^3b \qquad I_y = 4a^3b/21 \blacktriangleleft$$

PROBLEM 9.5



Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



$$y = h_1 + (h_2 - h_1)\frac{x}{a} \qquad dI_x = \frac{1}{3}y^3 dx$$

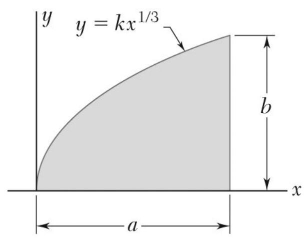
$$I_x = \int dI_y = \frac{1}{3}\int_0^a \left[h_1 + (h_2 - h_1)\frac{x}{a}\right]^3 dx$$

$$= \frac{1}{12}\left|\left[h_1 + (h_2 - h_1)\frac{x}{a}\right]^4 \left(\frac{a}{h_2 - h_1}\right)\right|_0^a$$

$$= \frac{a}{12(h_2 - h_1)}\left(h_2^4 - h_1^4\right) = \frac{a}{12}\cdot\frac{(h_2^2 + h_1^2)(h_2 + h_1)(h_2 - h_1)}{h_2 - h_1}$$

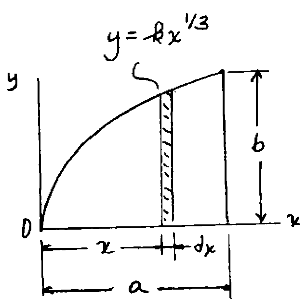
$$I_x = \frac{a}{12}(h_1^2 + h_2^2)(h_1 + h_2) \blacktriangleleft$$

PROBLEM 9.6



Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

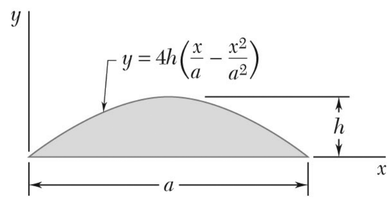


$$y = \frac{b}{a^{1/3}} x^{1/3}$$

$$dI_x = \frac{1}{3} y^3 dx = \frac{1}{3}\left(\frac{b}{a^{1/3}} x^{1/3}\right)^3 dx = \frac{1}{3}\frac{b^3}{a} x dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a} x dx = \frac{1}{3}\frac{b^3}{a}\frac{a^2}{2}$$

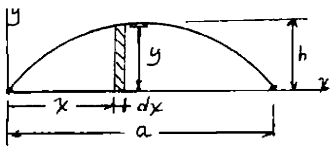
$$I_x = \frac{1}{6} ab^3 \blacktriangleleft$$



PROBLEM 9.7

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



$$y = 4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)$$

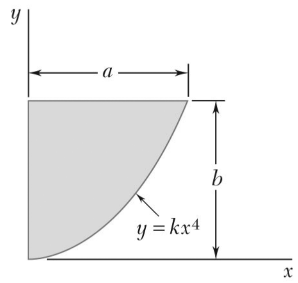
$$dI_x = \frac{1}{3}y^3 dx = \frac{1}{3}\left[4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)\right]^3 dx$$

$$I_x = \int dI_x = \frac{64h^3}{3}\int_0^a \left(\frac{x^3}{a^3} - 3\frac{x^4}{a^4} + 3\frac{x^5}{a^5} - \frac{x^6}{a^6}\right)dx$$

$$= \frac{64h^3}{3}\left|\frac{1}{4}\frac{x^4}{a^3} - \frac{3}{5}\frac{x^5}{a^4} + \frac{1}{2}\frac{x^6}{a^5} - \frac{1}{7}\frac{x^7}{a^6}\right|_0^a$$

$$= \frac{64h^3}{3}a\left(\frac{1}{4} - \frac{3}{5} + \frac{1}{2} - \frac{1}{7}\right) = \frac{64h^3}{3}a\left(\frac{3}{420}\right)$$

$$I_x = \frac{16}{105}ah^3 \blacktriangleleft$$

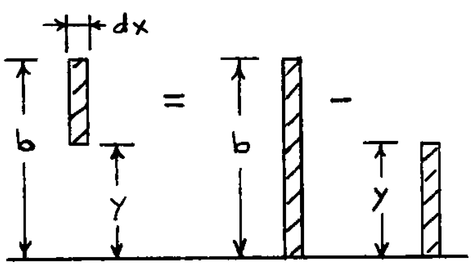


PROBLEM 9.8

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

See figure of solution of Problem 9.4.



$$y = \frac{b}{a^4}x^4$$

$$dI_x = \frac{1}{3}b^3dx - \frac{1}{3}y^3dx = \frac{1}{3}b^3dx - \frac{1}{3}\frac{b^3}{a^{12}}x^{12}dx$$

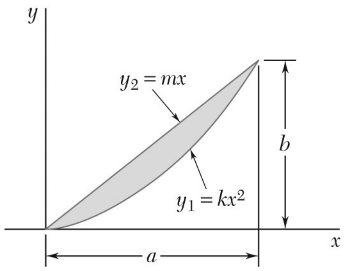
$$I_x = \int dI_x = \int_0^a \left(\frac{1}{3}b^3 - \frac{1}{3}\frac{b^3}{a^{12}}x^{12}\right)dx$$

$$= \frac{1}{3}b^3a - \frac{1}{3}\frac{b^3}{a^{12}}\frac{a^{13}}{13}$$

$$= \left(\frac{1}{3} - \frac{1}{39}\right)ab^3$$

$$= \left(\frac{13}{39} - \frac{1}{39}\right)ab^3 = \frac{12}{39}ab^3$$

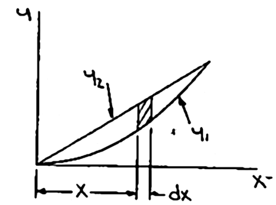
$I_x = 4ab^3/13$ ◀



PROBLEM 9.9

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



At
$$x_1 = a, \quad y_1 = y_2 = b$$

$$y_1: \quad b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

$$y_2: \quad b = ma \quad \text{or} \quad m = \frac{b}{a}$$

Then
$$y_1 = \frac{b}{a^2}x^2$$

$$y_2 = \frac{b}{a}x$$

Now
$$dI_x = \frac{1}{3}\left(y_2^3 - y_1^3\right)dx$$

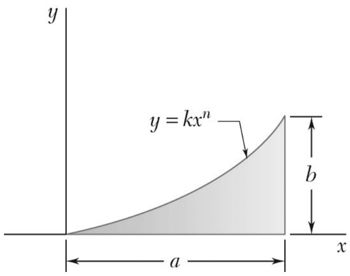
$$= \frac{1}{3}\left(\frac{b^3}{a^3}x^3 - \frac{b^3}{a^6}x^6\right)dx$$

Then
$$I_x = \int dI_x = \int_0^a \frac{b^3}{3}\left(\frac{1}{a^3}x^3 - \frac{1}{a^6}x^6\right)dx$$

$$= \frac{b^3}{3}\left[\frac{1}{4a^3}x^4 - \frac{1}{7a^6}x^7\right]_0^a$$

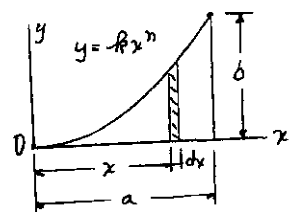
or $I_x = \dfrac{1}{28}ab^3$ ◀

PROBLEM 9.10



Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



$$y = kx^n$$

For $x = a$: $b = ka^n$

$$k = b/a^n$$

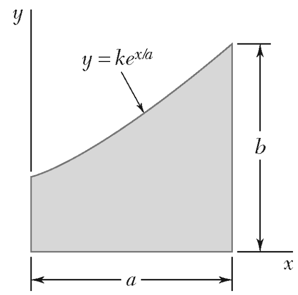
Thus: $y = \frac{b}{a^n} x^n$

$$y = \frac{b}{a^n} x^n$$

$$dI_x = \frac{1}{3} y^3 dx = \frac{1}{3} \frac{b^3}{a^{3n}} x^{3n} dx$$

$$I_x = \int dI_x = \frac{b^3}{3a^{3n}} \int_0^a x^{3n} dx = \frac{b^3}{3a^{3n}} \frac{a^{3n+1}}{(3n+1)}$$

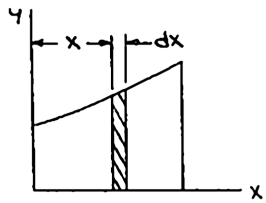
$$I_x = \frac{ab^3}{3(3n+1)} \blacktriangleleft$$



PROBLEM 9.11

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



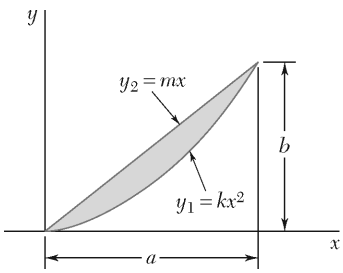
At $x = a,\ y = b$: $\quad b = ke^{a/a} \quad$ or $\quad k = \dfrac{b}{e}$

Then $\quad y = \dfrac{b}{e}e^{x/a} = be^{x/a-1}$

Now
$$dI_x = \frac{1}{3}y^3 dx = \frac{1}{3}(be^{x/a-1})^3 dx$$
$$= \frac{1}{3}b^3 e^{3(x/a-1)} dx$$

Then
$$I_x = \int dI_x = \int_0^a \frac{1}{3}b^3 e^{3(x/a-1)} dx = \frac{b^3}{3}\left[\frac{a}{3}e^{3(x/a-1)}\right]_0^a$$
$$= \frac{1}{9}ab^3(1-e^{-3})$$

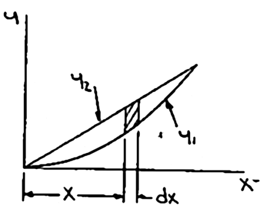
or $I_x = 0.1056ab^3$ ◀



PROBLEM 9.12

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $\quad x_1 = a, \quad y_1 = y_2 = b$

y_1: $\quad b = ka^2 \quad$ or $\quad k = \dfrac{b}{a^2}$

y_2: $\quad b = ma \quad$ or $\quad m = \dfrac{b}{a}$

Then $\quad y_1 = \dfrac{b}{a^2}x^2$

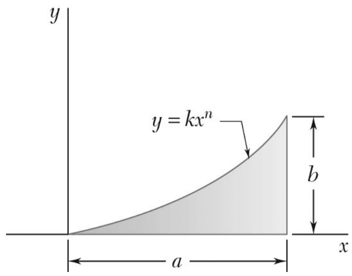
$$y_2 = \frac{b}{a}x$$

Now $\quad dI_y = x^2 dA = x^2[(y_2 - y_1)dx = x^2\left[\left(\frac{b}{a}x - \frac{b}{a^2}x^2\right)dx\right]$

Then $\quad I_y = \int dI_y = \int_0^a b\left(\frac{1}{a}x^3 - \frac{1}{a^2}x^4\right)dx$

$$= b\left[\frac{1}{4a}x^4 - \frac{1}{5a^2}x^5\right]_0^a$$

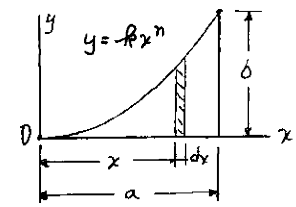
or $\quad I_y = \dfrac{1}{20}a^3 b$ ◀



PROBLEM 9.13

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



$$y = kx^n$$

For $x = a$:

$$b = ka^n$$

$$k = b/a^n$$

Thus:

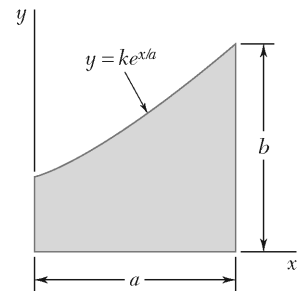
$$y = \frac{b}{a^n}x^n$$

$$dI_y = x^2 dA = x^2 y dx$$

$$dI_y = x^2 \frac{b}{a^n} x^n dx = \frac{b}{a^n} x^{n+2} dx$$

$$I_y = \int dI_y = \frac{b}{a^n}\int_0^a x^{n+2} dx = \frac{b}{a^n}\frac{a^{n+3}}{n+3}$$

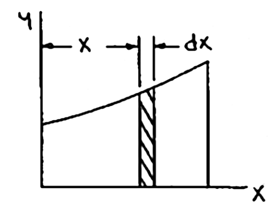
$$I_y = \frac{a^3 b}{n+3} \blacktriangleleft$$



PROBLEM 9.14

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $x = a, \quad y = b$: $\quad b = ke^{a/a}$

or $\quad k = \dfrac{b}{e}$

Then $\quad y = \dfrac{b}{e}e^{x/a} = be^{x/a-1}$

Now
$$dI_y = x^2 dA = x^2(ydx)$$
$$= x^2(be^{x/a-1}dx)$$

Then $\quad I_y = \int dI_y = \int_0^a bx^2 e^{x/a-1}dx$

Now use integration by parts with

$$u = x^2 \qquad dv = e^{x/a-1}dx$$
$$du = 2xdx \qquad v = ae^{x/a-1}$$

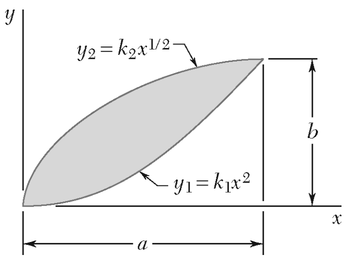
Then
$$\int_0^a x^2 e^{x/a-1}dx = \left[x^2 ae^{x/a-1}\right]_0^a - \int_0^a (ae^{x/a-1})2xdx$$
$$= a^3 - 2a\int_0^a xe^{x/a-1}dx$$

Using integration by parts with

$$u = x \qquad dv = e^{x/a-1}dx$$
$$du = dx \qquad v = ae^{x/a-1}$$

Then
$$I_y = b\left\{a^3 - 2a\left[(xae^{x/a-1})\big|_0^a - \int_0^a (ae^{x/a-1})dx\right]\right\}$$
$$= b\left\{a^3 - 2a\left[a^2 - (a^2e^{x/a-1})\big|_0^a\right]\right\}$$
$$= b\left\{a^3 - 2a\left[a^2 - (a^2 - a^2e^{-1})\right]\right\}$$

or $I_y = 0.264a^3b$ ◀



PROBLEM 9.15

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION

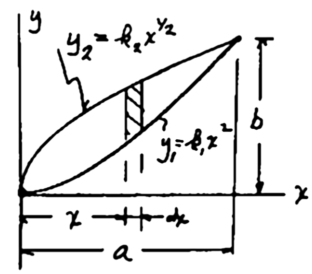
$$y_1 = k_1x^2 \qquad y_2 = k_2x^{1/2}$$

For

$$x = 0 \quad \text{and} \quad y_1 = y_2 = b$$

$$b = k_1a^2 \qquad b = k_2a^{1/2}$$

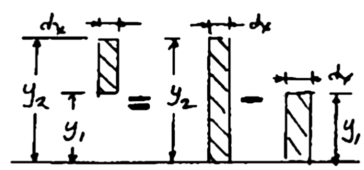
$$k_1 = \frac{b}{a^2} \qquad k_2 = \frac{b}{a^{1/2}}$$



Thus,

$$y_1 = \frac{b}{a^2}x^2 \qquad y_2 = \frac{b}{a^{1/2}}x^{1/2}$$

$$dA = (y_2 - y_1)dx$$



$$A = \int_0^a \left[\frac{b}{a^{1/2}}x^{1/2} - \frac{b}{a^2}x^2\right]dx$$

$$A = \frac{2}{3}\frac{ba^{3/2}}{a^{1/2}} - \frac{ba^3}{3a^2}$$

$$A = \frac{1}{3}ab$$

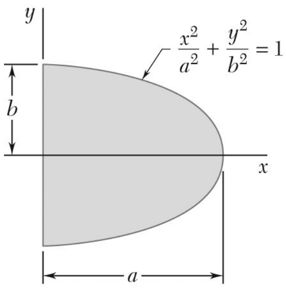
$$dI_x = \frac{1}{3}y_2^3dx - \frac{1}{3}y_1^3dx$$

$$= \frac{1}{3}\frac{b^3}{a^{3/2}}x^{3/2}dx - \frac{1}{3}\frac{b^3}{a^6}x^6dx$$

$$I_x = \int dI_x = \frac{b^3}{3a^{3/2}}\int_0^a x^{3/2}dx - \frac{b^3}{3a^6}\int_0^a x^6dx$$

$$= \frac{b^3}{3a^{3/2}}\frac{a^{5/2}}{\left(\frac{5}{2}\right)} - \frac{b^3}{3a^6}\frac{a^7}{7} = \left(\frac{2}{15} - \frac{1}{21}\right)ab^3 \qquad I_x = \frac{3}{35}ab^3 \blacktriangleleft$$

$$k_x^2 = \frac{I_x}{A} = \frac{\left(\frac{3}{35}ab^3\right)}{\frac{ab}{b}} \qquad k_x = b\sqrt{\frac{9}{35}} \blacktriangleleft$$

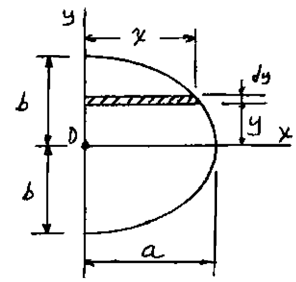


PROBLEM 9.16

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$x = a\sqrt{1-\frac{y^2}{b^2}}$$

$$dA = x\,dy$$

$$dI_x = y^2 dA = y^2 x\,dy$$

$$I_x = \int dI_x = \int_{-b}^{b} xy^2 dy = a\int_{-b}^{b} y^2\sqrt{1-\frac{y^2}{b^2}}\,dy$$

Set: $\qquad y = b\sin\theta \qquad dy = b\cos\theta\, d\theta$

$$I_x = a\int_{-\pi/2}^{\pi/2} b^2\sin^2\theta\sqrt{1-\sin^2\theta}\quad b\cos\theta\, d\theta$$

$$= ab^3\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta = ab^3\int_{-\pi/2}^{\pi/2}\frac{1}{4}\sin^2 2\theta\, d\theta$$

$$= \frac{1}{4}ab^3\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1-\cos 4\theta)d\theta = \frac{1}{8}ab^3\left[\theta - \frac{1}{4}\sin 4\theta\right]_{-\pi/2}^{\pi/2}$$

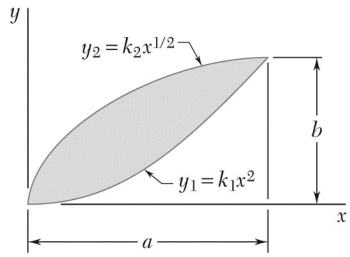
$$= \frac{1}{8}ab^3\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{8}ab^2 \qquad I_x = \frac{1}{8}\pi ab^3 \blacktriangleleft$$

$$A = \int dA = \int_{-b}^{+b} x\,dy = a\int_{-b}^{b}\sqrt{1-\frac{y^2}{b^2}}\,dy = a\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\theta}\; b\cos\theta\, d\theta$$

$$= ab\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta = ab\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= ab\left|\theta + \frac{1}{2}\sin 2\theta\right|_{-\pi/2}^{\pi/2} = \frac{ab}{2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \frac{1}{2}\pi ab$$

$$I_x = k_x^2 A \quad k_x^2 = \frac{I_x}{A} = \frac{\frac{1}{8}\pi ab^3}{\frac{1}{2}\pi ab} = \frac{1}{4}b^2 \qquad k_x = \frac{1}{2}b \blacktriangleleft$$



PROBLEM 9.17

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION

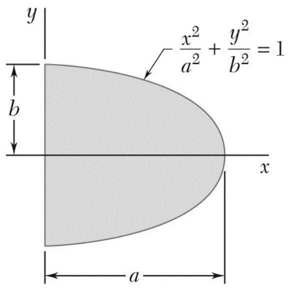
See figure of solution on Problem 9.15.

$$A = \frac{1}{3}ab \qquad dI_y = x^2 dA = x^2(y_2 - y_1)dx$$

$$I_y = \int_0^a x^2\left(\frac{b}{a^{1/2}}x^{1/2} - \frac{b}{a^2}x^2\right)dx = \frac{b}{a^{1/2}}\int_0^a x^{5/2}dx - \frac{b}{a^2}\int_0^a x^4 dx$$

$$I_y = \frac{b}{a^{1/2}} \cdot \frac{b^{7/2}}{\left(\frac{7}{2}\right)} - \frac{b}{a^2} \cdot \frac{a^5}{5} = \left(\frac{2}{7} - \frac{1}{5}\right)a^3 b \qquad I_y = \frac{3}{35}a^3 b \blacktriangleleft$$

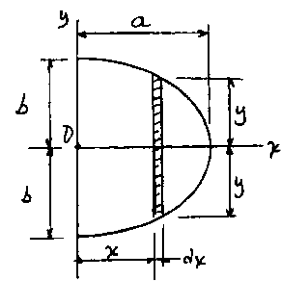
$$k_y^2 = \frac{I_y}{A} = \frac{\left(\frac{3}{35}a^3 b\right)}{\frac{ab}{3}} \qquad k_y = a\sqrt{\frac{9}{35}} \blacktriangleleft$$



PROBLEM 9.18

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$dA=2y\,dx$$

$$dI_y=x^2dA=2x^2y\,dx$$

$$I_y=\int dI_y=\int_0^a 2x^2y\,dx=2b\int_0^a x^2\sqrt{1-\frac{x^2}{a^2}}\,dx$$

Set: $\quad x=a\sin\theta \qquad dx=a\cos\theta\,d\theta$

$$I_y=2b\int_0^{\pi/2}a^2\sin^2\theta\sqrt{1-\sin^2\theta}\quad a\cos\theta\,d\theta$$

$$=2a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta=2a^3b\int_0^{\pi/2}\frac{1}{4}\sin^2 2\theta\,d\theta$$

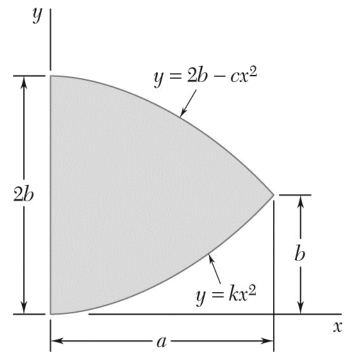
$$=\frac{1}{2}a^3b\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)d\theta=\frac{1}{4}a^3b\left|\theta-\frac{1}{4}\sin 4\theta\right|_0^{\pi/2}$$

$$=\frac{1}{4}a^3b\left[\frac{\pi}{2}-0\right]=\frac{\pi}{8}a^3b \qquad I_y=\frac{1}{8}\pi a^3b \blacktriangleleft$$

From solution of Problem 9.16: $\quad A=\frac{1}{2}\pi ab$

Thus:

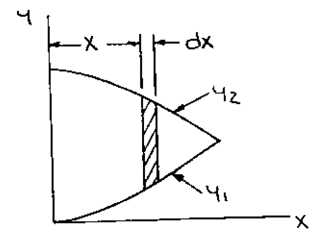
$$I_y=k_y^2A \quad k_y^2=\frac{I_y}{A}=\frac{\frac{1}{8}\pi a^3b}{\frac{1}{2}\pi ab}=\frac{1}{4}a^2 \qquad k_y=\frac{1}{2}a \blacktriangleleft$$



PROBLEM 9.19

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION



At $x = a_1,\ y_1 = y_2 = b$: $\quad y_1:\ b = ka^2 \quad$ or $\quad k = \dfrac{b}{a^2}$

$y_2:\ b = 2b - ca^2 \quad$ or $\quad c = \dfrac{b}{a^2}$

Then $$y_1 = \frac{b}{a^2}x^2 \qquad y_2 = b\left(2 - \frac{x^2}{a^2}\right)$$

Now $$dA = (y_2 - y_1)dx = \left[b\left(2 - \frac{x^2}{a^2}\right) - \frac{b}{a^2}x^2\right]dx = \frac{2b}{a^2}(a^2 - x^2)dx$$

Then $$A = \int dA = \int_0^a \frac{2b}{a^2}(a^2 - x^2)dx = \frac{2b}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}ab$$

Now $$dI_x = \left(\frac{1}{3}y_2^3 - \frac{1}{3}y_1^3\right)dx = \frac{1}{3}\left\{\left[b\left(2 - \frac{x^2}{a^2}\right)\right]^3 - \left[\frac{b}{a^2}x^2\right]^3\right\}dx$$

$$= \frac{1}{3}\frac{b^3}{a^6}(8a^6 - 12a^4x^2 + 6a^2x^4 - x^6 - x^6)dx$$

$$= \frac{2}{3}\frac{b^3}{a^6}(4a^6 - 6a^4x^2 + 3a^2x^4 - x^6)dx$$

Then $$I_x = \int dI_x = \int_0^a \frac{2}{3}\frac{b^3}{a^6}(4a^6 - 6a^4x^2 + 3a^2x^4 - x^6)dx$$

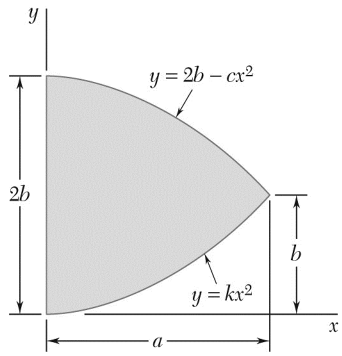
$$= \frac{2}{3}\frac{b^3}{a^6}\left[4a^6x - 2a^4x^3 + \frac{3}{5}a^6x^5 - \frac{1}{7}x^7\right]_0^a$$

$$= \frac{172}{105}ab^3$$

or $I_x = 1.638ab^3$ ◀

and $$k_x^2 = \frac{I_x}{A} = \frac{\frac{172}{105}ab^3}{\frac{4}{3}ab} = \frac{43}{35}b^2$$

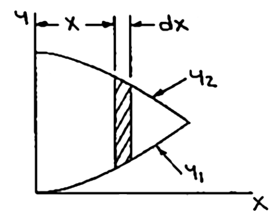
or $k_x = 1.108b$ ◀



PROBLEM 9.20

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION



At $x = a, \quad y_1 = y_2 = b$: $\qquad y_1: \quad b = ka^2 \quad \text{or} \quad k = \dfrac{b}{a^2}$

$$y_2: \quad b = 2b - ca^2 \quad \text{or} \quad c = \frac{b}{a^2}$$

Then

$$y_1 = \frac{b}{a^2}x^2$$

$$y_2 = b\left(2 - \frac{x^2}{a^2}\right)$$

Now

$$dA = (y_2 - y_1)dx$$

$$= \left[b\left(2 - \frac{x^2}{a^2}\right) - \frac{b}{a^2}x^2\right]dx$$

$$= \frac{2b}{a^2}(a^2 - x^2)\,dx$$

Then

$$A = \int dA = \int_0^a \frac{2b}{a^2}(a^2 - x^2)dx$$

$$= \frac{2b}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{4}{3}ab$$

Now

$$dI_y = x^2 dA = x^2\left[\frac{2b}{a^2}(a^2 - x^2)dx\right]$$

PROBLEM 9.20 (Continued)

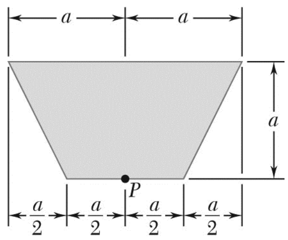
Then $$I_y = \int dI_y = \int_0^a \frac{2b}{a^2} x^2 (a^2 - x^2) dx$$

$$= \frac{2b}{a^2} \left[\frac{1}{3} a^2 x^3 - \frac{1}{5} x^5 \right]_0^a$$

or $$I_y = \frac{4}{15} a^3 b \blacktriangleleft$$

and $$k_y^2 = \frac{I_y}{A} = \frac{\frac{4}{15} a^3 b}{\frac{4}{3} ab} = \frac{1}{5} a^2$$

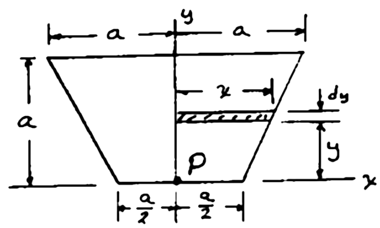
or $$k_y = \frac{a}{\sqrt{5}} \blacktriangleleft$$



PROBLEM 9.21

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



$$x = \frac{a}{2} + \frac{a}{2}\frac{y}{a} = \frac{1}{2}(a+y)$$

$$dA = x\,dy = \frac{1}{2}(a+y)dy$$

$$\frac{1}{2}A = \int_0^a dA = \int_0^a \frac{1}{2}(a+y)dy = \frac{1}{2}\left| ay + \frac{y^2}{2}\right|_0^a = \frac{1}{2}\left(a^2 + \frac{a^2}{2}\right) = \frac{3}{4}a^2 \qquad A = \frac{3}{2}a^2 \ \triangleleft$$

$$\frac{1}{2}I_x = \int y^2 dA = \int_0^a y^2 \frac{1}{2}(a+y)dy = \frac{1}{2}\int_0^a (ay^2 + y^3)dy$$

$$= \frac{1}{2}\left| a\frac{y^3}{3} + \frac{y^4}{4}\right|_0^a = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right)a^4 = \frac{1}{2}\frac{7}{12}a^4 \qquad I_x = \frac{7}{12}a^4 \ \triangleleft$$

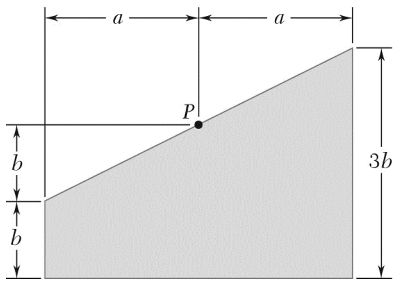
$$\frac{1}{2}I_y = \int_0^a \frac{1}{3}x^3 dy = \frac{1}{3}\int_0^a \left(\frac{1}{2}(a+y)\right)^3 dy$$

$$\frac{1}{2}I_y = \frac{1}{24}\int_0^a (a+y)^3 dy = \frac{1}{24}\left|\frac{1}{4}(a+y)^4\right|_0^a = \frac{1}{96}\left[(2a)^4 - a^4\right] = \frac{15}{96}a^4$$

$$\frac{1}{2}I_y = \frac{5}{32}a^4 \qquad I_y = \frac{5}{16}a^4 \ \triangleleft$$

From Eq. (9.4):
$$J_O = I_x + I_y = \frac{7}{12}a^4 + \frac{5}{16}a^4 = \left(\frac{28+15}{48}\right)a^4 \qquad J_O = \frac{43}{48}a^4 \ \blacktriangleleft$$

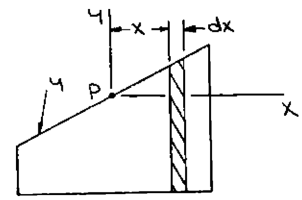
$$J_O = k_O^2 A \qquad k_O^2 = \frac{J_O}{A} = \frac{\frac{43}{48}a^4}{\frac{3}{2}a^2} = \frac{43}{72}a^2 \qquad k_O = a\sqrt{\frac{43}{72}} \qquad k_O = 0.773a \ \blacktriangleleft$$



PROBLEM 9.22

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



By observation $$y = \frac{b}{a}x$$

First note $$dA = (y + 2b)dx$$
$$= \frac{b}{a}(x + 2a)dx$$

Now $$dI_x = \left(\frac{1}{3}y_{\text{top}}^3 - \frac{1}{3}y_{\text{bottom}}^3\right)dx$$
$$= \frac{1}{3}\left[\left(\frac{b}{a}x\right)^3 - (-2b)^3\right]dx$$
$$= \frac{1}{3}\frac{b^3}{a^3}(x^3 + 8a^3)dx$$

Then $$I_x = \int dI_x = \int_{-a}^{a}\frac{1}{3}\frac{b^3}{a^3}(x^2 + 8a^3)dx$$
$$= \frac{1}{3}\frac{b^3}{a^3}\left[\frac{1}{4}x^4 + 8a^3x\right]_{-a}^{a}$$
$$= \frac{1}{3}\frac{b^3}{a^3}\left\{\left[\frac{1}{4}(a)^4 + 8a^3(a)\right] - \left[\frac{1}{4}(-a)^4 + 8a^3(-a)\right]\right\} = \frac{16}{3}ab^3$$

Also $$dI_y = x^2dA = x^2\left[\frac{b}{a}(x + 2a)dx\right]$$

Then $$I_y = \int dI_y = \int_{-a}^{a}\frac{b}{a}x^2(x + 2a)dx$$
$$= \frac{b}{a}\left[\frac{1}{4}x^4 + \frac{2}{3}ax^3\right]_{-a}^{a}$$
$$= \frac{b}{a}\left\{\left[\frac{1}{4}(a)^4 + \frac{2}{3}a(a)^3\right] - \left[\frac{1}{4}(-a)^4 + \frac{2}{3}a(-a)^3\right]\right\} = \frac{4}{3}a^3b$$

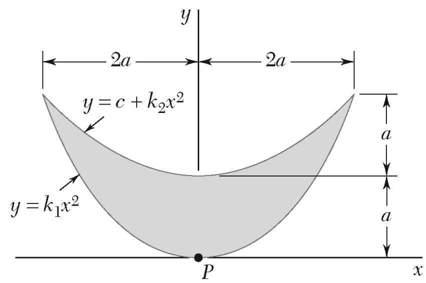
PROBLEM 9.22 (Continued)

Now $$J_P = I_x + I_y = \frac{16}{3}ab^3 + \frac{4}{3}a^3b$$

$$J_P = \frac{4}{3}ab(a^2 + 4b^2) \blacktriangleleft$$

and $$k_P^2 = \frac{J_P}{A} = \frac{\frac{4}{3}ab(a^2 + 4b^2)}{(2a)(3b) - \frac{1}{2}(2a)(2b)}$$

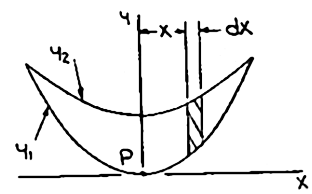
$$= \frac{1}{3}(a^2 + 4b^2)$$ or $$k_P = \sqrt{\frac{a^2 + 4b^2}{3}} \blacktriangleleft$$



PROBLEM 9.23

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



y_1: At $x = 2a$, $y = 2a$:

$$2a = k_1(2a)^2 \quad \text{or} \quad k_1 = \frac{1}{2a}$$

y_2: At $x = 0$, $y = a$: $\quad a = c$

At $x = 2a$, $y = 2a$:

$$2a = a + k_2(2a)^2 \quad \text{or} \quad k_2 = \frac{1}{4a}$$

Then
$$y_1 = \frac{1}{2a}x^2 \qquad y_2 = a + \frac{1}{4a}x^2$$
$$= \frac{1}{4a}(4a^2 + x^2)$$

Now
$$dA = (y_2 - y_1)dx = \left[\frac{1}{4a}(4a^2 + x^2) - \frac{1}{2a}x^2\right]dx$$
$$= \frac{1}{4a}(4a^2 - x^2)dx$$

Then
$$A = \int dA = 2\int_0^{2a} \frac{1}{4a}(4a^2 - x^2)dx = \frac{1}{2a}\left[4a^2x - \frac{1}{3}x^3\right]_0^{2a} = \frac{8}{3}a^2$$

Now
$$dI_x = \left(\frac{1}{3}y_2^3 - \frac{1}{3}y_1^3\right)dx = \frac{1}{3}\left\{\left[\frac{1}{4a}(4a^2 + x^2)\right]^3 - \left[\frac{1}{2a}x^2\right]^3\right\}dx$$
$$= \frac{1}{3}\left[\frac{1}{64a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 + x^6) - \frac{1}{8a^3}x^6\right]dx$$
$$= \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx$$

PROBLEM 9.23 (Continued)

Then $$I_x = \int dI_x = 2\int_0^{2a} \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx$$

$$= \frac{1}{96a^3}\left[64a^6x + 16a^4x^3 + \frac{12}{5}a^2x^5 - x^7\right]_0^{2a}$$

$$= \frac{1}{96a^3}\left[64a^6(2a) + 16a^4(2a)^3 + \frac{12}{5}a^2(2a)^5 - (2a)^7\right]$$

$$= \frac{1}{96}a^4\left(128 + 128 + \frac{12}{5}\times 32 - 128\right) = \frac{32}{15}a^4$$

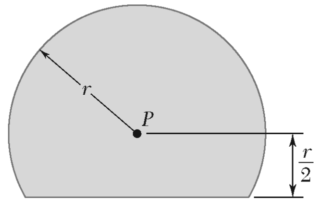
Also $$dI_y = x^2 dA = x^2\left[\frac{1}{4a}(4a^2 - x^2)dx\right]$$

Then $$I_y = \int dI_y = 2\int_0^{2a} \frac{1}{4a}x^2(4a^2 - x^2)dx = \frac{1}{2a}\left[\frac{4}{3}a^2x^3 - \frac{1}{5}x^5\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{4}{3}a^2(2a)^3 - \frac{1}{5}(2a)^5\right] = \frac{32}{2}a^4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{32}{15}a^4$$

Now $$J_P = I_x + I_y = \frac{32}{15}a^4 + \frac{32}{15}a^4$$ or $J_P = \frac{64}{15}a^4$ ◀

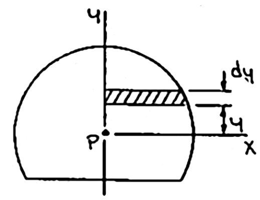
and $$k_P^2 = \frac{J_P}{A} = \frac{\frac{64}{15}a^4}{\frac{8}{3}a^2} = \frac{8}{5}a^2$$ or $k_P = 1.265a$ ◀



PROBLEM 9.24

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



The equation of the circle is

$$x^2 + y^2 = r^2$$

So that $$x = \sqrt{r^2 - y^2}$$

Now $$dA = xdy = \sqrt{r^2 - y^2}\,dy$$

Then $$A = \int dA = 2\int_{-r/2}^{r} \sqrt{r^2 - y^2}\,dy$$

Let $$y = r\sin\theta; \quad dy = r\cos\theta\, d\theta$$

Then $$A = 2\int_{-\pi/6}^{\pi/2} \sqrt{r^2 - (r\sin\theta)^2}\, r\cos\, d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} r^2\cos^2\theta\, d\theta = 2r^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\pi/6}^{\pi/2}$$

$$= 2r^2\left[\frac{\frac{\pi}{2}}{2} - \left(\frac{-\frac{\pi}{6}}{2} + \frac{\sin -\frac{\pi}{3}}{4}\right)\right] = 2r^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)$$

$$= 2.5274r^2$$

Now $$dI_x = y^2 dA = y^2\left(\sqrt{r^2 - y^2}\,dy\right)$$

Then $$I_x = \int dI_x = 2\int_{-r/2}^{r} y^2\sqrt{r^2 - y^2}\,dy$$

Let $$y = r\sin\theta; \quad dy = r\cos\theta\, d\theta$$

Then $$I_x = 2\int_{-\pi/6}^{\pi/2} (r\sin\theta)^2\sqrt{r^2 - (r\sin\theta)^2}\, r\cos\theta\, d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} r^2\sin^2\theta(r\cos\theta) r\cos\theta\, d\theta$$

Now $$\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin^2\theta\cos^2\theta = \frac{1}{4}\sin 2\theta$$

Then $$I_x = 2\int_{-\pi/6}^{\pi/2} r^4\left(\frac{1}{4}\sin^2 2\theta\right)d\theta = \frac{r^4}{2}\left[\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right]_{-\pi/6}^{\pi/2}$$
$$= \frac{r^4}{2}\left[\frac{\frac{\pi}{2}}{2}-\left(\frac{\frac{\pi}{6}}{2}-\frac{\sin -\frac{2\pi}{3}}{8}\right)\right]$$
$$= \frac{r^4}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{16}\right)$$

Also $$dI_y = \frac{1}{3}x^3 dy = \frac{1}{3}\left(\sqrt{r^2-y^2}\right)^3 dy$$

Then $$I_y = \int dI_y = 2\int_{-r/2}^{r}\frac{1}{3}(r^2-y^2)^{3/2}dy$$

Let $$y = r\sin\theta; \quad dy = r\cos\theta\, d\theta$$

Then $$I_y = \frac{2}{3}\int_{-\pi/6}^{\pi/2}[r^2-(r\sin\theta)^2]^{3/2} r\cos\theta\, d\theta$$

$$I_y = \frac{2}{3}\int_{-\pi/6}^{\pi/2}(r^3\cos^3\theta) r\cos\theta\, d\theta$$

Now $$\cos^4\theta = \cos^2\theta(1-\sin^2\theta) = \cos^2\theta - \frac{1}{4}\sin^2 2\theta$$

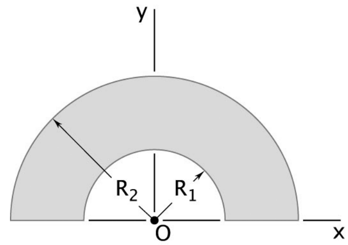
Then $$I_y = \frac{2}{3}\int_{-\pi/6}^{\pi/2} r^4\left(\cos^2\theta - \frac{1}{4}\sin^2 2\theta\right)d\theta$$
$$= \frac{2}{3}r^4\left[\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)-\frac{1}{4}\left(\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right)\right]_{-\pi/6}^{\pi/2}$$
$$= \frac{2}{3}r^4\left\{\left[\frac{\frac{\pi}{2}}{2}-\frac{1}{4}\left(\frac{\frac{\pi}{2}}{2}\right)\right]-\left[\frac{-\frac{\pi}{6}}{2}+\frac{\sin -\frac{\pi}{3}}{4}-\frac{1}{4}\left(\frac{-\frac{\pi}{6}}{2}-\frac{\sin -\frac{2\pi}{3}}{8}\right)\right]\right\}$$
$$= \frac{2}{3}r^4\left[\frac{\pi}{4}-\frac{\pi}{16}+\frac{\pi}{12}+\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-\frac{\pi}{48}+\frac{1}{32}\left(\frac{\sqrt{3}}{2}\right)\right]$$
$$= \frac{2}{3}r^4\left(\frac{\pi}{4}+\frac{9\sqrt{3}}{64}\right)$$

PROBLEM 9.24 (Continued)

Now $$J_P = I_x + I_y = \frac{r^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{16}\right) + \frac{2}{3}r^4\left(\frac{\pi}{4} + \frac{9\sqrt{3}}{64}\right)$$

$$= r^4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{16}\right) = 1.15545r^4$$ or $J_P = 1.155r^4$ ◀

and $$k_P^2 = \frac{J_P}{A} = \frac{1.15545r^4}{2.5274r^2}$$ or $k_P = 0.676r$ ◀



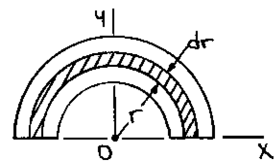
PROBLEM 9.25

(*a*) Determine by direct integration the polar moment of inertia of the semiannular area shown with respect to Point *O*. (*b*) Using the result of part *a*, determine the moments of inertia of the given area with respect to the *x* and *y* axes.

SOLUTION

(*a*) By definition

$$dJ_O = r^2 dA = r^2(\pi r dr)$$
$$= \pi r^3 dr$$



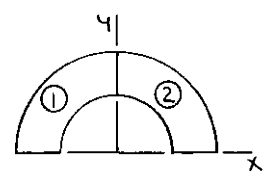
Then

$$J_O = \int dJ_O = \int_{R_1}^{R_2} \pi r^3 dr$$
$$= \left[\frac{1}{4} r^4\right]_{R_1}^{R_2}$$

or $J_O = \dfrac{\pi}{4}\left(R_2^4 - R_1^4\right)$ ◀

(*b*) First note that symmetry implies

$$(I_x)_1 = (I_y)_1 \qquad (I_x)_2 = (I_y)_2$$



Also have

$$I_x = (I_x)_1 + (I_x)_2$$

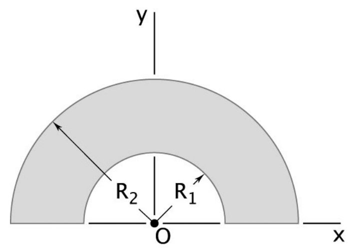
and $I_y = (I_y)_1 + (I_y)_2 = (I_x)_1 + (I_x)_2 = I_x$

Now $J_O = I_x + I_y \qquad I_x = I_y$

$$= 2I_x$$

$\therefore \quad I_x = I_y = \dfrac{1}{2} J_O$ or $I_x = I_y = \dfrac{\pi}{8}\left(R_2^4 - R_1^4\right)$ ◀

PROBLEM 9.26



(*a*) Show that the polar radius of gyration k_O of the semiannular area shown is approximately equal to the mean radius $R_m = (R_1 + R_2)/2$ for small values of the thickness $t = R_2 - R_1$. (*b*) Determine the percentage error introduced by using R_m in place of k_O for the following values of t/R_m: $1, \frac{1}{2}$, and $\frac{1}{10}$.

SOLUTION

(*a*) From the solution to Problem 9.25 have

$$J_O = \frac{\pi}{4}\left(R_2^4 - R_1^4\right)$$

Now
$$k_O^2 = \frac{J_O}{A} = \frac{\frac{\pi}{4}\left(R_2^4 - R_1^4\right)}{\frac{\pi}{2}\left(R_2^2 - R_1^2\right)} = \frac{1}{2}\frac{\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)}{R_2^2 - R_1^1}$$

$$= \frac{1}{2}\left(R_2^2 + R_1^2\right) \qquad (1)$$

Now
$$R_m = \frac{1}{2}(R_1 + R_2) \qquad t = R_2 - R_1$$

Then
$$R_1 = R_m - \frac{1}{2}t \qquad R_2 = R_m + \frac{1}{2}t$$

Substituting into Eq. (1),

$$k_O^2 = \frac{1}{2}\left[\left(R_m + \frac{1}{2}t\right)^2 + \left(R_m - \frac{1}{2}t\right)^2\right]$$

$$= \frac{1}{2}\left(R_m^2 + R_m t + \frac{1}{4}t^2 + R_m^2 - R_m t + \frac{1}{4}t^2\right)$$

$$= R_m^2 + \frac{1}{4}t^2 \qquad \left(k_O = \sqrt{R_m^2 + \frac{1}{4}t^2}\right)$$

If $t << R_1, R_2$

Then $t << R_m$

So that
$$k_O^2 \approx R_m^2 \qquad \text{or} \quad k_O \approx R_m \quad \triangleleft$$

(*b*) The percentage error, % error, is given by

$$\%\ \text{error} = \frac{R_m - k_O}{k_O} \times 100\%$$

$$= \frac{R_m - \sqrt{R_m^2 + \frac{1}{4}t^2}}{\sqrt{R_m^2 + \frac{1}{4}t^2}} \times 100\%$$

$$= \left(\frac{1}{\sqrt{1 + \left(\frac{1}{2}\frac{t}{R_m}\right)^2}} - 1 \right) \times 100\%$$

Then, for

$$\frac{t}{R_m} = 1: \quad \%\ \text{error} = \left(\frac{1}{\sqrt{1 + \left(\frac{1}{2} + 1\right)^2}} - 1 \right) \times 100\%$$

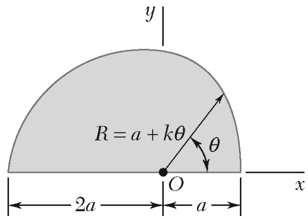
or $\quad \%\ \text{error} = -10.56\%$ ◀

$$\frac{t}{R_m} = \frac{1}{2}: \quad \%\ \text{error} = \left(\frac{1}{\sqrt{1 + \left(\frac{1}{2} \times \frac{1}{2}\right)^2}} - 1 \right) \times 100\%$$

or $\quad \%\ \text{error} = -2.99\%$ ◀

$$\frac{t}{R_m} = \frac{1}{10}: \quad \%\ \text{error} = \left(\frac{1}{\sqrt{1 + \left(\frac{1}{2} \times \frac{1}{10}\right)^2}} - 1 \right) \times 100\%$$

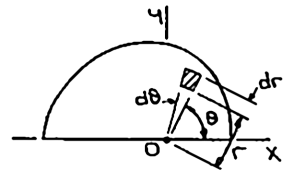
or $\quad \%\ \text{error} = -0.1248\%$ ◀



PROBLEM 9.27

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point O.

SOLUTION



At $\theta = \pi, \quad R = 2a$: $\quad 2a = a + k(\pi)$

or $\quad k = \dfrac{a}{\pi}$

Then $\quad R = a + \dfrac{a}{\pi}\theta = a\left(1 + \dfrac{\theta}{\pi}\right)$

Now
$$dA = (dr)(r\,d\theta)$$
$$= r dr\,d\theta$$

Then
$$A = \int dA = \int_0^{\pi}\int_0^{a(1+\theta/\pi)} r dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{a(1+\theta/\pi)} d\theta$$

$$A = \int_0^{\pi}\frac{1}{2}a^2\left(1+\frac{\theta}{\pi}\right)^2 d\theta = \frac{1}{2}a^2\left[\frac{\pi}{3}\left(1+\frac{\theta}{\pi}\right)^3\right]_0^{\pi}$$

$$= \frac{1}{6}\pi a^2\left[\left(1+\frac{\pi}{\pi}\right)^3 - (1)^3\right] = \frac{7}{6}\pi a^2$$

Now $\quad dJ_O = r^2 dA = r^2(r dr\,d\theta)$

Then
$$J_O = \int dJ_O = \int_0^{\pi}\int_0^{a(1+\theta/\pi)} r^3 dr\,d\theta$$

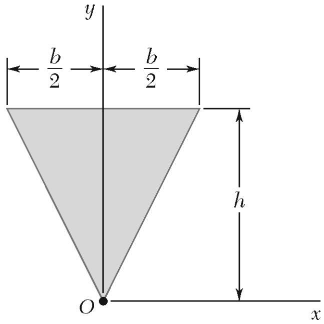
$$= \int_0^{\pi}\left[\frac{1}{4}r^4\right]_0^{a(1+\theta/\pi)} d\theta = \int_0^{\pi}\frac{1}{4}a^4\left(1+\frac{\theta}{\pi}\right)^4 d\theta$$

$$= \frac{1}{4}a^4\left[\frac{\pi}{5}\left(1+\frac{\theta}{\pi}\right)^5\right]_0^{\pi} = \frac{1}{20}\pi a^4\left[\left(1+\frac{\pi}{\pi}\right)^5 - (1)^5\right]$$

or $\quad J_O = \dfrac{31}{20}\pi a^4$ ◀

and
$$k_O^2 = \frac{J_O}{A} = \frac{\frac{31}{20}\pi a^4}{\frac{7}{6}\pi a^2} = \frac{93}{70}a^2$$

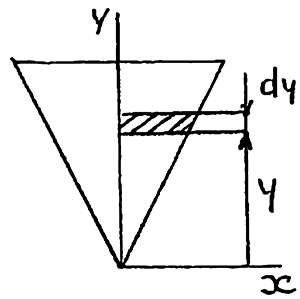
or $\quad k_O = 1.153a$ ◀



PROBLEM 9.28

Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O.

SOLUTION



By observation: $$y = \frac{h}{\frac{b}{2}}x$$

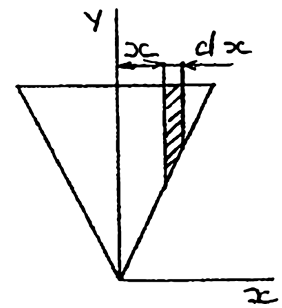
or $$x = \frac{b}{2h}y$$

Now $$dA = xdy = \left(\frac{b}{2h}y\right)dy$$

and $$dI_x = y^2 dA = \frac{b}{2h}y^3 dy$$

Then $$I_x = \int dI_x = 2\int_0^h \frac{b}{2h}y^3 dy$$

$$= \frac{b}{h}\frac{y^4}{4}\bigg|_0^h = \frac{1}{4}bh^3$$



From above: $$y = \frac{2h}{b}x$$

Now $$dA = (h-y)dx = \left(h - \frac{2h}{b}x\right)dx$$

$$= \frac{h}{b}(b-2x)dx$$

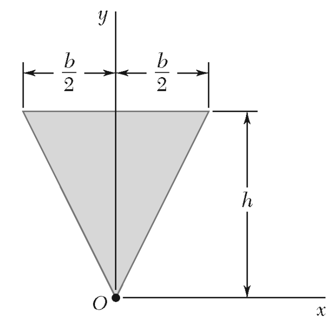
and $$dI_y = x^2 dA = x^2\frac{h}{b}(b-2x)dx$$

PROBLEM 9.28 (Continued)

Then
$$I_y = \int dI_y = 2\int_0^{b/2} \frac{h}{b} x^2 (b - 2x)dx$$
$$= 2\frac{h}{b}\left[\frac{1}{3}bx^3 - \frac{1}{2}x^4\right]_0^{b/2}$$
$$= 2\frac{h}{b}\left[\frac{b}{3}\left(\frac{b}{2}\right)^3 - \frac{1}{2}\left(\frac{b}{2}\right)^4\right] = \frac{1}{48}b^3 h$$

Now
$$J_O = I_x + I_y = \frac{1}{4}bh^3 + \frac{1}{48}b^3 h \qquad \text{or} \quad J_O = \frac{bh}{48}(12h^2 + b^2) \blacktriangleleft$$

and
$$k_O^2 = \frac{J_O}{A} = \frac{\frac{bh}{48}(12h^2 + b^2)}{\frac{1}{2}bh} = \frac{1}{24}(12h^2 + b^2) \qquad \text{or} \quad k_O = \frac{\sqrt{12h^2 + b^2}}{24} \blacktriangleleft$$



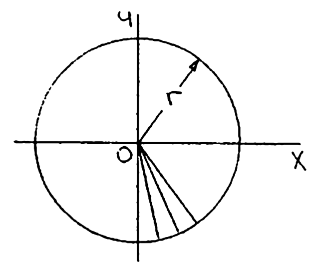
PROBLEM 9.29*

Using the polar moment of inertia of the isosceles triangle of Problem 9.28, show that the centroidal polar moment of inertia of a circular area of radius r is $\pi r^4/2$. (*Hint:* As a circular area is divided into an increasing number of equal circular sectors, what is the approximate shape of each circular sector?)

PROBLEM 9.28 Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O.

SOLUTION

First the circular area is divided into an increasing number of identical circular sectors. The sectors can be approximated by isosceles triangles. For a large number of sectors the approximate dimensions of one of the isosceles triangles are as shown.



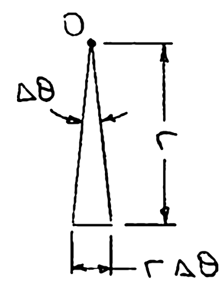
For an isosceles triangle (see Problem 9.28):

$$J_O = \frac{bh}{48}(12h^2 + b^2)$$

Then with $\quad b = r\Delta\theta \quad$ and $\quad h = r$

$$(\Delta J_O)_{\text{sector}} \simeq \frac{1}{48}(r\Delta\theta)(r)[12r^2 + (r\Delta\theta)^2]$$

$$= \frac{1}{48}r^4\Delta\theta\left[(12 + \Delta\theta^2)\right]$$



Now
$$\frac{dJ_{O\,\text{sector}}}{d\theta} = \lim_{\Delta\theta\to 0}\left(\frac{\Delta J_{O\,\text{sector}}}{\Delta\theta}\right) = \lim_{\Delta\theta\to 0}\left\{\frac{1}{48}r^4[12 + (\Delta\theta)^2]\right\}$$

$$= \frac{1}{4}r^4$$

Then
$$(J_O)_{\text{circle}} = \int dJ_{O\,\text{sector}} = \int_0^{2\pi}\frac{1}{4}r^4 d\theta = \frac{1}{4}r^4\,[\theta]_0^{2\pi}$$

or $\quad (J_O)_{\text{circle}} = \dfrac{\pi}{2}r^4$ ◀

PROBLEM 9.30*

Prove that the centroidal polar moment of inertia of a given area A cannot be smaller than $A^2/2\pi$. (*Hint:* Compare the moment of inertia of the given area with the moment of inertia of a circle that has the same area and the same centroid.)

SOLUTION

From the solution to sample Problem 9.2, the centroidal polar moment of inertia of a circular area is

$$(J_C)_{\text{cir}} = \frac{\pi}{2} r^4$$

The area of the circle is

$$A_{\text{cir}} = \pi r^2$$

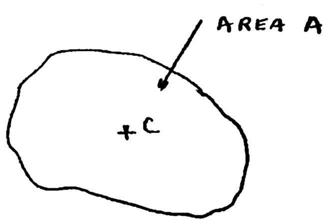
So that

$$[J_C(A)]_{\text{cir}} = \frac{A^2}{2\pi}$$

Two methods of solution will be presented. However, both methods depend upon the observation that as a given element of area dA is moved closer to some Point C. The value of J_C will be decreased ($J_C = \int r^2 dA$; as r decreases, so must J_C).

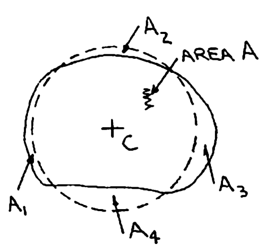
Solution 1

Imagine taking the area A and drawing it into a thin strip of negligible width and of sufficient length so that its area is equal to A. To minimize the value of $(J_C)_A$, the area would have to be distributed as closely as possible about C. This is accomplished by winding the strip into a tightly wound roll with C as its center; any voids in the roll would place the corresponding area farther from C than is necessary, thus increasing the value of $(J_C)_A$. (The process is analogous to rewinding a length of tape back into a roll.) Since the shape of the roll is circular, with the centroid of its area at C, it follows that



$$(J_C)_A \geq \frac{A^2}{2\pi} \quad \text{Q.E.D.} \blacktriangleleft$$

where the equality applies when the original area is circular.



PROBLEM 9.30* (Continued)

Solution 2

Consider an area A with its centroid at Point C and a circular area of area A with its center (and centroid) at Point C. Without loss of generality, assume that

$$A_1 = A_2 \qquad A_3 = A_4$$

It then follows that

$$(J_C)_A = (J_C)_{\text{cir}} + [J_C(A_1) - J_C(A_2) + J_C(A_3) - J_C(A_4)]$$

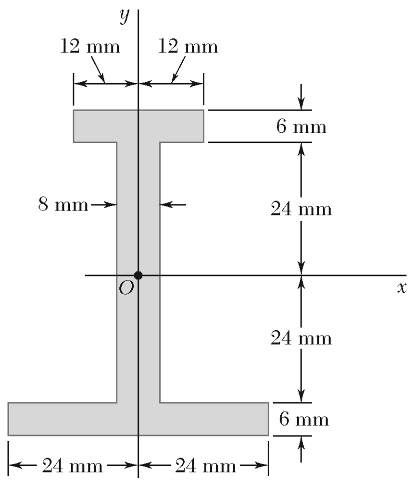
Now observe that

$$J_C(A_1) - J_C(A_2) \geq 0$$

$$J_C(A_3) - J_C(A_4) \geq 0$$

since as a given area is moved farther away from C its polar moment of inertia with respect to C must increase.

$$(J_C)_A \geq (J_C)_{\text{cir}} \qquad \text{or} \quad (J_C)_A \geq \frac{A^2}{2\pi} \quad \text{Q.E.D.} \blacktriangleleft$$



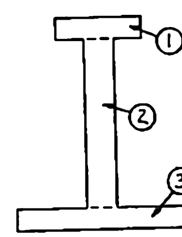
PROBLEM 9.31

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis.

SOLUTION

First note that

$$A = A_1 + A_2 + A_3$$
$$= [(24)(6) + (8)(48) + (48)(6)] \text{ mm}^2$$
$$= (144 + 384 + 288) \text{ mm}^2$$
$$= 816 \text{ mm}^2$$



Now $\qquad I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$

where

$$(I_x)_1 = \frac{1}{12}(24 \text{ mm})(6 \text{ mm})^3 + (144 \text{ mm}^2)(27 \text{ mm})^2$$
$$= (432 + 104{,}976) \text{ mm}^4$$
$$= 105{,}408 \text{ mm}^4$$

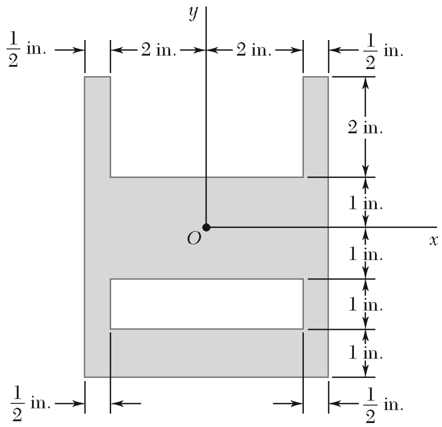
$$(I_x)_2 = \frac{1}{12}(8 \text{ mm})(48 \text{ mm})^3 = 73{,}728 \text{ mm}^4$$

$$(I_x)_3 = \frac{1}{12}(48 \text{ mm})(6 \text{ mm})^3 + (288 \text{ mm}^2)(27 \text{ mm})^2$$
$$= (864 + 209{,}952) \text{ mm}^4 = 210{,}816 \text{ mm}^4$$

Then $\qquad I_x = (105{,}408 + 73{,}728 + 210{,}816) \text{ mm}^4$

$$= 389{,}952 \text{ mm}^4 \qquad \text{or} \quad I_x = 390 \times 10^3 \text{ mm}^4 \blacktriangleleft$$

and $\qquad k_x^2 = \frac{I_x}{A} = \frac{389{,}952 \text{ mm}^4}{816 \text{ mm}^2} \qquad \text{or} \quad k_x = 21.9 \text{ mm} \blacktriangleleft$



PROBLEM 9.32

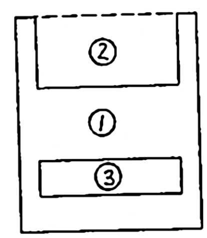
Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis.

SOLUTION

First note that

$$A = A_1 - A_2 - A_3$$
$$= [(5)(6) - (4)(2) - (4)(1)]\ \text{in}^2$$
$$= (30 - 8 - 4)\ \text{in}^2$$
$$= 18\ \text{in}^2$$

Now $\qquad I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$



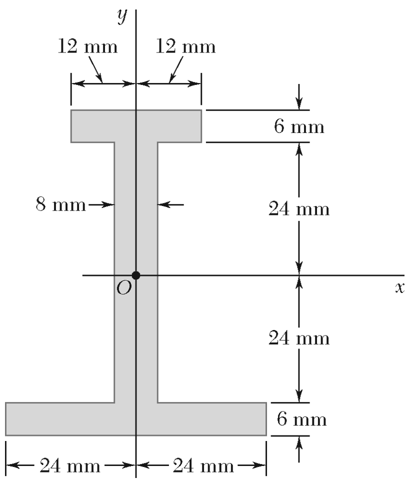
where $\qquad (I_x)_1 = \dfrac{1}{12}(5\ \text{in.})(6\ \text{in.})^3 = 90\ \text{in}^4$

$$(I_x)_2 = \frac{1}{12}(4\ \text{in.})(2\ \text{in.})^3 + (8\ \text{in}^2)(2\ \text{in.})^2$$
$$= 34\frac{2}{3}\ \text{in}^4$$

$$(I_x)_3 = \frac{1}{12}(4\ \text{in.})(1\ \text{in.})^3 + (4\ \text{in}^2)\left(\frac{3}{2}\ \text{in.}\right)^2$$
$$= 9\frac{1}{3}\ \text{in}^4$$

Then $\qquad I_x = \left(90 - 34\dfrac{2}{3} - 9\dfrac{1}{3}\right)\text{in}^4 \qquad$ or $\quad I_x = 46.0\ \text{in}^4$ ◀

and $\qquad k_x^2 = \dfrac{I_x}{A} = \dfrac{46.0\ \text{in}^4}{18\ \text{in}^4} \qquad$ or $\quad k_x = 1.599\ \text{in.}$ ◀



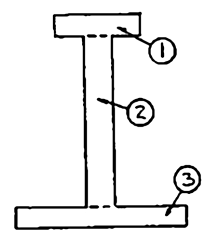
PROBLEM 9.33

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the y axis.

SOLUTION

First note that

$$A = A_1 + A_2 + A_3$$
$$= [(24\times6)+(8)(48)+(48)(6)]\ \text{mm}^2$$
$$= (144+384+288)\ \text{mm}^2$$
$$= 816\ \text{mm}^2$$



Now $$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

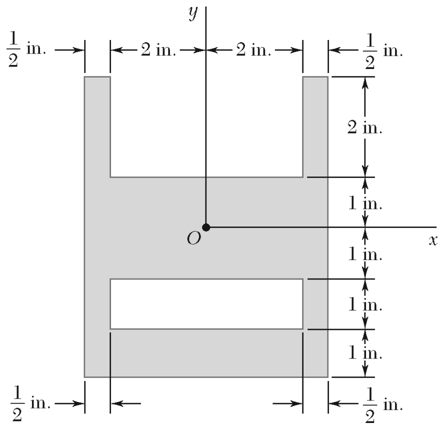
where

$$(I_y)_1 = \frac{1}{12}(6\ \text{mm})(24\ \text{mm})^3 = 6912\ \text{mm}^4$$
$$(I_y)_2 = \frac{1}{12}(48\ \text{mm})(8\ \text{mm})^3 = 2048\ \text{mm}^4$$
$$(I_y)_3 = \frac{1}{12}(6\ \text{mm})(48\ \text{mm})^3 = 55{,}296\ \text{mm}^4$$

Then $$I_y = (6912+2048+55{,}296)\ \text{mm}^4 = 64{,}256\ \text{mm}^4$$

or $I_y = 64.3\times10^3\ \text{mm}^4$ ◀

and $$k_y^2 = \frac{I_y}{A} = \frac{64{,}256\ \text{mm}^4}{816\ \text{mm}^2}$$ or $k_y = 8.87$ mm ◀



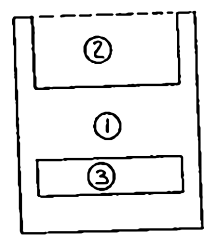
PROBLEM 9.34

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the *y* axis.

SOLUTION

First note that

$$A = A_1 - A_2 - A_3$$
$$= [(5)(6) - (4)(2) - (4)(1)]\ \text{in}^2$$
$$= (30 - 8 - 4)\ \text{in}^2$$
$$= 18\ \text{in}^2$$



Now $\qquad I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$

where

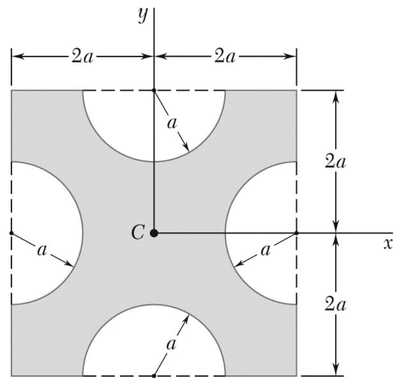
$$(I_y)_1 = \frac{1}{12}(6\ \text{in.})(5\ \text{in.})^3 = 62.5\ \text{in}^4$$

$$(I_y)_2 = \frac{1}{12}(2\ \text{in.})(4\ \text{in.})^3 = 10\frac{2}{3}\ \text{in}^4$$

$$(I_y)_3 = \frac{1}{12}(1\ \text{in.})(4\ \text{in.})^3 = 5\frac{1}{3}\text{in}^4$$

Then $\qquad I_y = \left(62.5 - 10\frac{2}{3} - 5\frac{1}{3}\right)\text{in}^4 \qquad$ or $\quad I_y = 46.5\ \text{in}^4$ ◀

and $\qquad k_y^2 = \frac{I_y}{A} = \frac{46.5\ \text{in}^4}{18\ \text{in}^2} \qquad$ or $\quad k_y = 1.607$ in. ◀



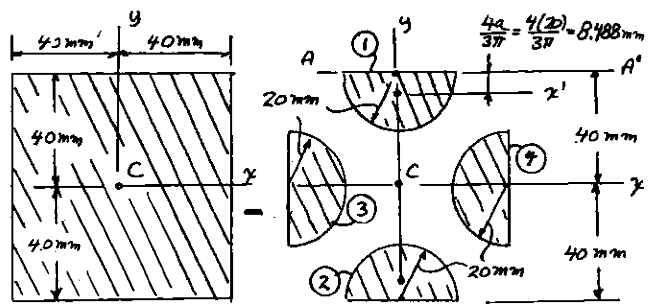
PROBLEM 9.35

Determine the moments of inertia of the shaded area shown with respect to the x and y axes when $a = 20$ mm.

SOLUTION

By symmetry: $I_x = I_y$

Given area = square − 4(semicircles)



Square $$I_x = \frac{1}{12}(80\text{ mm})^4 = 3.413\times 10^6\text{ mm}^4$$

Semicircle 1: $$I_{AA'} = \frac{\pi}{8}(20\text{ mm})^4 = 62.83\times 10^3\text{ mm}^4$$

$$I_{AA'} = \bar{I}_{x'} + Ad^2$$

$$62.83\times 10^3\text{ mm}^4 = \bar{I}_{x'} + \frac{\pi}{2}(20\text{ mm})^2(8.488\text{ mm})^2$$

$$\bar{I}_{x'} = 17.56\times 10^3\text{ mm}^4$$

$$I_x = \bar{I}_{x'} + A(40\text{ mm} - 8.488\text{ mm})^2$$

$$= 17.56\times 10^3\text{ mm}^4 + \frac{\pi}{2}(20\text{ mm})^2(31.512\text{ mm})^2$$

$$I_x = 641.5\times 10^3\text{ mm}^4$$

PROBLEM 9.35 (Continued)

Semicircle 2: Same as 1: $I_x = 641.5 \times 10^3 \text{ mm}^4$

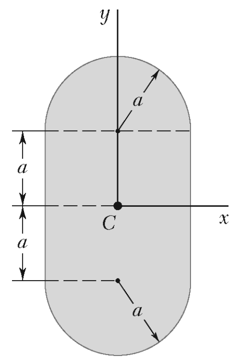
Semicircles 3 and 4 are equivalent to one 20-mm circle when computing I_x

$$I_x = \frac{\pi}{4}(20 \text{ mm})^4 = 125.66 \times 10^3 \text{ mm}^4$$

Entire area = square − 4 (semicircles)

$$\bar{I}_x = 3.413 \times 10^6 \text{ mm}^4 - 2(641.5 \times 10^3 \text{ mm}^4) - 125.66 \times 10^{-3} \text{ mm}^4$$
$$= 2.0047 \times 10^6 \text{ mm}^4$$

$$\bar{I}_x = \bar{I}_y = 2.00 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



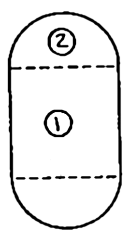
PROBLEM 9.36

Determine the moments of inertia of the shaded area shown with respect to the x and y axes when $a = 20$ mm.

SOLUTION

We have $$I_x = (I_x)_1 + 2(I_x)_2$$

where $$(I_x)_1 = \frac{1}{12}(40\text{ mm})(40\text{ mm})^3$$
$$= 213.33 \times 10^3\text{ mm}^4$$



$$(I_x)_2 = \left[\frac{\pi}{8}(20\text{ mm})^4 - \frac{\pi}{2}(20\text{ mm})^2\left(\frac{4\times 20}{3\pi}\text{ mm}\right)^2\right]$$
$$+\frac{\pi}{2}(20\text{ mm})^2\left[\left(\frac{4\times 20}{3\pi}+20\right)\text{mm}\right]^2$$
$$= 527.49\times 10^3\text{ mm}^4$$

Then $$I_x = [213.33 + 2(527.49)]\times 10^3\text{ mm}^4$$

or $I_x = 1.268\times 10^6\text{ mm}^4$ ◀

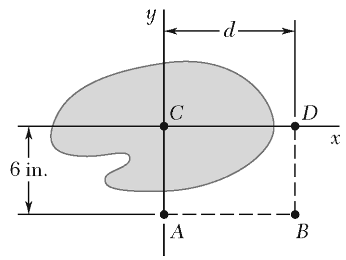
Also $$I_y = (I_y)_1 + 2(I_y)_2$$

where $$(I_y)_1 = \frac{1}{12}(40\text{ mm})(40\text{ mm})^3 = 213.33\times 10^3\text{ mm}^4$$

$$(I_y)_2 = \frac{\pi}{8}(20\text{ mm})^4 = 62.83\times 10^3\text{ mm}^4$$

Then $$I_y = [213.33 + 2(62.83)]\times 10^3\text{ mm}^4$$

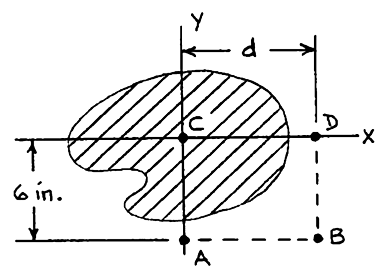
or $I_y = 339\times 10^3\text{ mm}^4$ ◀



PROBLEM 9.37

The shaded area is equal to 50 in^2. Determine its centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$, knowing that $\bar{I}_y = 2\bar{I}_x$ and that the polar moment of inertia of the area about Point A is $J_A = 2250 \text{ in}^4$.

SOLUTION



Given: $A = 50 \text{ in}^2 \quad \bar{I}_y = 2\bar{I}_x, \quad J_A = 2250 \text{ in}^4$

$$J_A = \bar{J}_C + A(6 \text{ in.})^2$$

$$2250 \text{ in}^4 = \bar{J}_C + (50 \text{ in}^2)(6 \text{ in.})^2$$

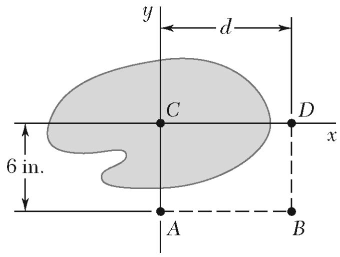
$$\bar{J}_C = 450 \text{ in}^4$$

$$\bar{J}_C = \bar{I}_x + \bar{I}_y \quad \text{with} \quad \bar{I}_y = 2\bar{I}_x$$

$$450 \text{ in}^4 = \bar{I}_x + 2\bar{I}_x$$

$\bar{I}_x = 150.0 \text{ in}^4$ ◀

$\bar{I}_y = 2\bar{I}_x = 300 \text{ in}^4$ ◀



PROBLEM 9.38

The polar moments of inertia of the shaded area with respect to Points A, B, and D are, respectively, $J_A = 2880\text{ in}^4$, $J_B = 6720\text{ in}^4$, and $J_D = 4560\text{ in}^4$. Determine the shaded area, its centroidal moment of inertia $\bar{J}_C$, and the distance d from C to D.

SOLUTION

See figure at solution of Problem 9.39.

Given: $J_A = 2880\text{ in}^4, \quad J_B = 6720\text{ in}^4, \quad J_D = 4560\text{ in}^4$

$$J_B = \bar{J}_C + A(CB)^2; \quad 6720\text{ in}^4 = \bar{J}_C + A(6^2 + d^2) \qquad (1)$$

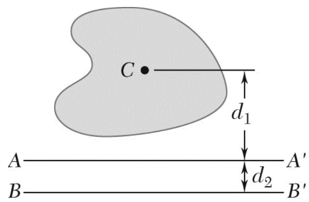
$$J_D = \bar{J}_C + A(CD)^2; \quad 4560\text{ in}^4 = \bar{J}_C + Ad^2 \qquad (2)$$

Eq. (1) subtracted by Eq. (2): $J_B - J_D = 2160\text{ in}^4 = A(6)^2$ $\qquad A = 60.0\text{ in}^2$ ◀

$J_A = \bar{J}_C + A(AC)^2; \quad 2880\text{ in}^4 = \bar{J}_C + (60\text{ in}^2)(6\text{ in.})^2$ $\qquad \bar{J}_C = 720\text{ in}^4$ ◀

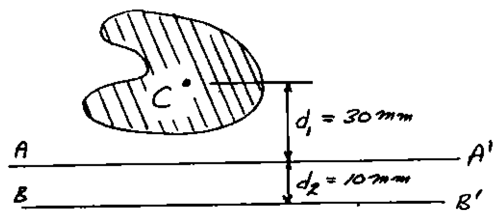
Eq. (2): $4560\text{ in}^4 = 720\text{ in}^4 + (60\text{ in}^2)d^2$ $\qquad d = 8.00\text{ in.}$ ◀

PROBLEM 9.39



Determine the shaded area and its moment of inertia with respect to the centroidal axis parallel to AA', knowing that $d_1 = 30$ mm and $d_2 = 10$ mm, and that the moments of inertia with respect to AA' and BB' are 4.1×10^6 mm^4 and 6.9×10^6 mm^4, respectively.

SOLUTION



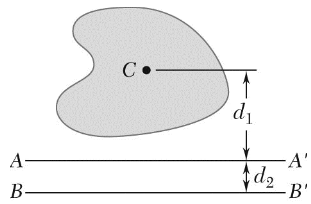
$$I_{AA'} = 4.1\times10^6 \text{ mm}^4 = \overline{I} + A(30 \text{ mm})^2 \tag{1}$$

$$I_{BB'} = 6.9\times10^6 \text{ mm}^4 = \overline{I} + A(40 \text{ mm})^2$$

$$I_{BB'} - I_{AA'} = (6.9-4.1)\times10^6 = A(40^2 - 30^2)$$

$$2.8\times10^6 = A(700) \qquad A = 4000 \text{ mm}^2 \blacktriangleleft$$

Eq. (1): $$4.1\times10^6 = \overline{I} + (4000)(30)^2 \qquad \overline{I} = 500\times10^3 \text{ mm}^4 \blacktriangleleft$$

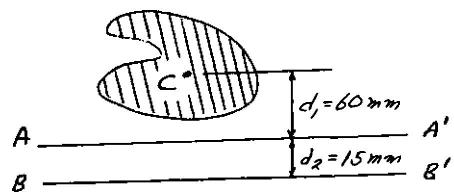


PROBLEM 9.40

Knowing that the shaded area is equal to 7500 mm^2 and that its moment of inertia with respect to AA' is 31×10^6 mm^4, determine its moment of inertia with respect to BB', for $d_1 = 60$ mm and $d_2 = 15$ mm.

SOLUTION

Given: $A = 7500 \text{ mm}^2$

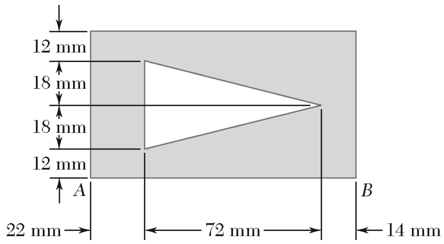


$$I_{AA'} = \bar{I} + Ad_1^2; \quad 31\times10^6 \text{ mm}^2 = \bar{I} + (7500 \text{ mm}^2)(60 \text{ mm})^2$$

$$\bar{I} = 4\times10^6 \text{ mm}^4$$

$$I_{BB'} = \bar{I} + Ad^2 = 4\times10^6 \text{ mm}^4 + (7500 \text{ mm}^2)(60 \text{ mm} + 15 \text{ mm})^2$$
$$= 4\times10^6 + 7500(75)^2 = 46.188\times10^6 \text{ mm}^4$$

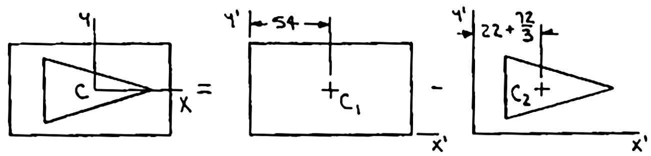
$$I_{BB'} = 46.2\times10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.41

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side *AB*.

SOLUTION



Dimensions in mm

First locate centroid *C* of the area.

Symmetry implies $\bar{Y} = 30$ mm.

	A, mm^2	$\bar{x}$, mm	$\bar{x}A$, mm^3
1	$108 \times 60 = 6480$	54	349,920
2	$-\frac{1}{2} \times 72 \times 36 = -1296$	46	–59,616
Σ	5184		290,304

Then $\bar{X}\,\Sigma A = \Sigma \bar{x} A: \quad \bar{X}(5184 \text{ mm}^2) = 290{,}304 \text{ mm}^3$

or $\bar{X} = 56.0$ mm

Now $\bar{I}_x = (I_x)_1 - (I_x)_2$

where $(I_x)_1 = \frac{1}{12}(108 \text{ mm})(60 \text{ mm})^3 = 1.944 \times 10^6 \text{ mm}^4$

$$(I_x)_2 = 2\left[\frac{1}{36}(72 \text{ mm})(18 \text{ mm})^3 + \left(\frac{1}{2} \times 72 \text{ mm} \times 18 \text{ mm}\right)(6 \text{ mm})^2\right]$$

$$= 2(11{,}664 + 23{,}328) \text{ mm}^4 = 69.984 \times 10^3 \text{ mm}^4$$

[$(I_x)_2$ is obtained by dividing A_2 into ⊳]

Then $\bar{I}_x = (1.944 - 0.069984) \times 10^6 \text{ mm}^4$

or $\bar{I}_x = 1.874 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.41 (Continued)

Also $\bar{I}_y = (I_y)_1 - (I_y)_2$

where $(I_y)_1 = \frac{1}{12}(60\text{ mm})(108\text{ mm})^3 + (6480\text{ mm}^2)[(56.54)\text{ mm}]^2$

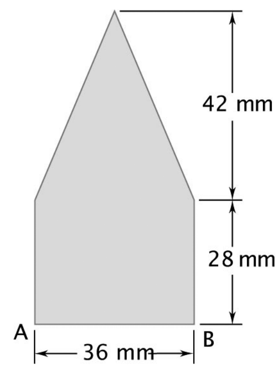
$= (6,298,560 + 25,920)\text{ mm}^4 = 6.324 \times 10^6\text{ mm}^4$

$(I_y)_2 = \frac{1}{36}(36\text{ mm})(72\text{ mm})^3 + (1296\text{ mm}^2)[(56-46)\text{ mm}]^2$

$= (373,248 + 129,600)\text{ mm}^4 = 0.502 \times 10^6\text{ mm}^4$

Then $\bar{I}_y = (6.324 - 0.502)10^6\text{ mm}^4$

or $\bar{I}_y = 5.82 \times 10^6\text{ mm}^4$ ◀



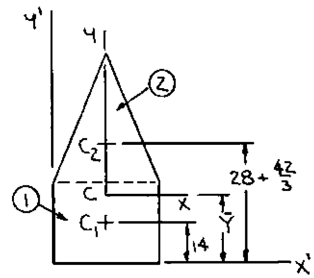
PROBLEM 9.42

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION

First locate C of the area.

Symmetry implies $\bar{X} = 18$ mm.



Dimensions in mm

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$36 \times 28 = 1008$	14	14,112
2	$\frac{1}{2} \times 36 \times 42 = 756$	42	31,752
Σ	1764		45,864

Then $\qquad \bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(1764 \text{ mm}^2) = 45,864 \text{ mm}^3$

or $\qquad \bar{Y} = 26.0$ mm

Now $\qquad \bar{I}_x = (I_x)_1 + (I_x)_2$

where
$$(I_x)_1 = \frac{1}{12}(36 \text{ mm})(28 \text{ mm})^3 + (1008 \text{ mm}^2)[(26-14)\text{ mm}]^2$$
$$= (65,856 + 145,152)\text{ mm}^4 = 211.008 \times 10^3 \text{ mm}^4$$
$$(I_x)_2 = \frac{1}{36}(36 \text{ mm})(42 \text{ mm})^3 + (756 \text{ mm}^2)[(42-26)\text{ mm}]^2$$
$$= (74,088 + 193,536)\text{ mm}^4 = 267.624 \times 10^3 \text{ mm}^4$$

Then $\qquad \bar{I}_x = (211.008 + 267.624) \times 10^3 \text{ mm}^4$

or $\qquad \bar{I}_x = 479 \times 10^3 \text{ mm}^4$ ◀

Also $$\bar{I}_y = (I_y)_1 + (I_y)_2$$

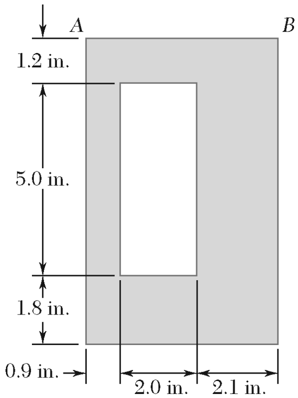
where $$(I_y)_1 = \frac{1}{12}(28\text{ mm})(36\text{ mm})^3 = 108.864\times 10^3\text{ mm}^4$$

$$(I_y)_2 = 2\left[\frac{1}{36}(42\text{ mm})(18\text{ mm})^3 + \left(\frac{1}{2}\times 18\text{ mm}\times 42\text{ mm}\right)(6\text{ mm})^2\right]$$

$$= 2(6804 + 13{,}608)\text{ mm}^4 = 40.824\times 10^3\text{ mm}^4$$

[$(I_y)_2$ is obtained by dividing A_2 into ]

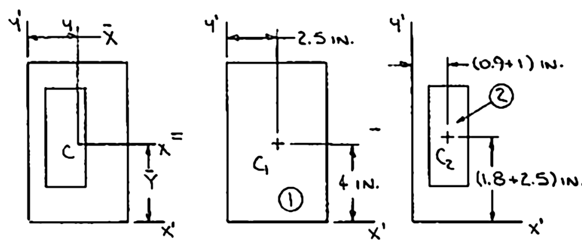
Then $$\bar{I}_y = (108.864 + 40.824\times 10^3\text{ mm}^4$$ or $$\bar{I}_y = 149.7\times 10^3\text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.43

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate centroid C of the area.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$5\times 8 = 40$	2.5	4	100	160
2	$-2\times 5 = -10$	1.9	4.3	–19	–43
Σ	30			81	117

Then $\qquad \bar{X}\Sigma A = \Sigma\bar{x}A: \quad \bar{X}(30\text{ in}^2) = 81\text{ in}^3$

or $\qquad \bar{X} = 2.70\text{ in.}$

and $\qquad \bar{Y}\Sigma A = \Sigma\bar{y}A: \quad \bar{Y}(30\text{ in}^2) = 117\text{ in}^3$

or $\qquad \bar{Y} = 3.90\text{ in.}$

Now $\qquad \bar{I}_x = (I_x)_1 - (I_x)_2$

where

$$(I_x)_1 = \frac{1}{12}(5\text{ in.})(8\text{ in.})^3 + (40\text{ in}^2)[(4-3.9)\text{ in.}]^2$$
$$= (213.33 + 0.4)\text{ in}^4 = 213.73\text{ in}^4$$
$$(I_x)_2 = \frac{1}{12}(2\text{ in.})(5\text{ in.})^3 + (10\text{ in}^2)[(4.3-3.9)\text{ in.}]^2$$
$$= (20.83 + 1.60) = 22.43\text{ in}^4$$

PROBLEM 9.43 (Continued)

Then $\bar{I}_x = (213.73 - 22.43)\text{ in}^4$ or $\bar{I}_x = 191.3\text{ in}^4$ ◀

Also $\bar{I}_y = (I_y)_1 - (I_y)_2$

where $(I_y)_1 = \frac{1}{12}(8\text{ in.})(5\text{ in.})^3 + (40\text{ in}^2)[(2.7 - 2.5)\text{ in.}]^2$

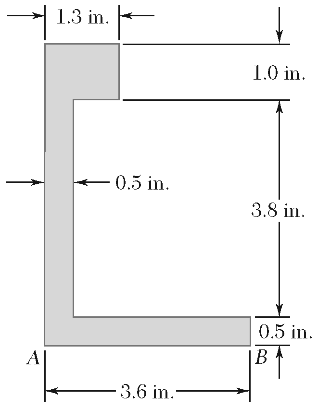
$= (83.333 + 1.6)\text{ in}^4 = 84.933\text{ in}^4$

$(I_y)_2 = \frac{1}{12}(5\text{ in.})(2\text{ in.})^3 + (10\text{ in}^2)[(2.7 - 1.9)\text{ in.}]^2$

$= (3.333 + 6.4)\text{ in}^4 = 9.733\text{ in}^4$

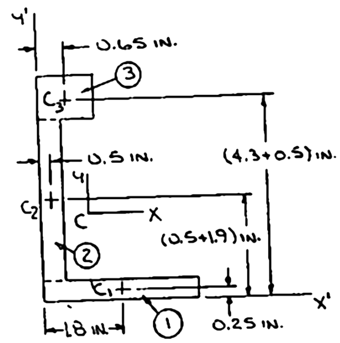
Then $\bar{I}_y = (84.933 - 9.733)\text{in}^4$ or $\bar{I}_y = 75.2\text{ in}^4$ ◀

PROBLEM 9.44



Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate centroid C of the area.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$3.6 \times 0.5 = 1.8$	1.8	0.25	3.24	0.45
2	$0.5 \times 3.8 = 1.9$	0.25	2.4	0.475	4.56
3	$1.3 \times 1 = 1.3$	0.65	4.8	0.845	6.24
Σ	5.0			4.560	11.25

Then $\qquad \bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(5\text{ in}^2) = 4.560\text{ in}^3$

or $\qquad \bar{X} = 0.912\text{ in.}$

and $\qquad \bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(5\text{ in}^2) = 11.25\text{ in}^3$

or $\qquad \bar{Y} = 2.25\text{ in.}$

PROBLEM 9.44 (Continued)

Now $\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3$

where
$$(I_x)_1 = \frac{1}{12}(3.6\text{ in.})(0.5\text{ in.})^3 + (1.8\text{ in}^2)[(2.25 - 0.25)\text{ in.}]^2$$
$$= (0.0375 + 7.20)\text{ in}^4 = 7.2375\text{ in}^4$$
$$(I_x)_2 = \frac{1}{12}(0.5\text{ in.})(3.8\text{ in.})^3 + (1.9\text{ in}^2)[(2.4 - 2.25)\text{ in.}]^2$$
$$= (2.2863 + 0.0428)\text{ in}^4 = 2.3291\text{ in}^4$$
$$(I_x)_3 = \frac{1}{12}(1.3\text{ in.})(1\text{ in.})^3 + (1.3\text{ in}^2)[(4.8 - 2.25\text{ in.})]^2$$
$$= (0.1083 + 8.4533)\text{ in}^4 = 8.5616\text{ in}^4$$

Then $\bar{I}_x = (7.2375 + 2.3291 + 8.5616)\text{ in}^4 = 18.1282\text{ in}^4$

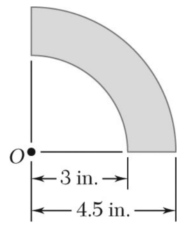
or $\bar{I}_x = 18.13\text{ in}^4$ ◀

Also $\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3$

where
$$(I_y)_1 = \frac{1}{12}(0.5\text{ in.})(3.6\text{ in.})^3 + (1.8\text{ in}^2)[(1.8 - 0.912)\text{ in.}]^2$$
$$= (1.9440 + 1.4194)\text{ in}^4 = 3.3634\text{ in}^4$$
$$(I_y)_2 = \frac{1}{12}(3.8\text{ in.})(0.5\text{ in.})^3 + (1.9\text{ in}^2)[(0.912 - 0.25)\text{ in.}]^2$$
$$= (0.0396 + 0.8327)\text{ in}^4 = 0.8723\text{ in}^4$$
$$(I_y)_3 = \frac{1}{12}(1\text{ in.})(1.3\text{ in.})^3 + (1.3\text{ in}^2)[(0.912 - 0.65)\text{ in.}]^2$$
$$= (0.1831 + 0.0892)\text{ in}^4 = 0.2723\text{ in}^4$$

Then $\bar{I}_y = (3.3634 + 0.8723 + 0.2723)\text{ in}^4 = 4.5080\text{ in}^4$

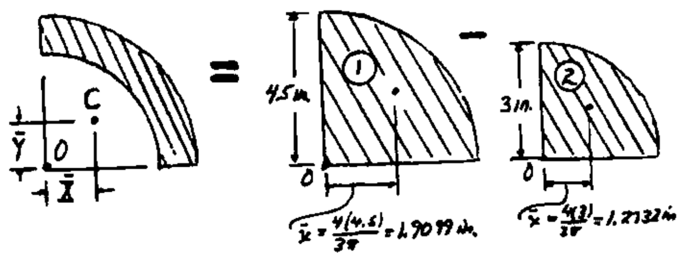
or $\bar{I}_y = 4.51\text{ in}^4$ ◀



PROBLEM 9.45

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION



Section	Area, in^2	$\bar{x}$, in.	$\bar{x}A$, in^3
1	$\frac{\pi}{4}(4.5)^2 = 15.904$	1.9099	30.375
2	$-\frac{\pi}{4}(3)^2 = -7.069$	1.2732	–9.00
Σ	8.835		21.375

Then $\bar{X}A = \Sigma\bar{x}A$: $\quad \bar{X}(8.835 \text{ in}^2) = 21.375 \text{ in}^3$

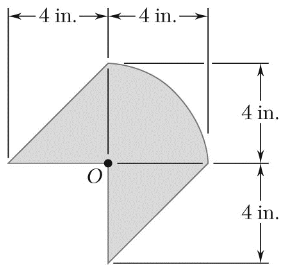
or $\bar{X} = 2.419$ in.

Then $J_O = \frac{\pi}{8}(4.5 \text{ in.})^4 - \frac{\pi}{8}(3 \text{ in.})^4 = 129.22 \text{ in}^4$ $\qquad J_O = 129.2 \text{ in}^4$ ◀

$$\overline{OC} = \sqrt{2}\bar{X} = \sqrt{2}(2.419 \text{ in.}) = 3.421 \text{ in.}$$

$$J_O = \bar{J}_C + A(\overline{OC})^2:$$

$129.22 \text{ in}^4 = \bar{J}_C + (8.835 \text{ in.})(3.421 \text{ in.})^2$ $\qquad \bar{J}_C = 25.8 \text{ in}^4$ ◀



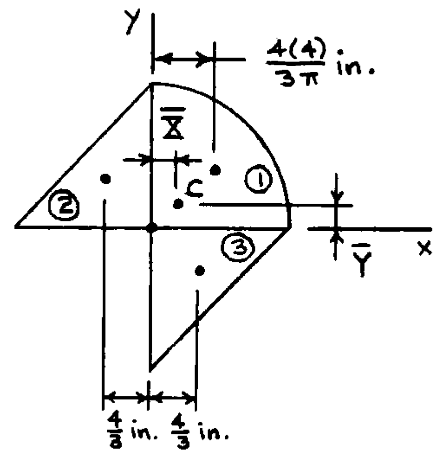
PROBLEM 9.46

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

Determination of centroid *C* of entire section:

Section	Area, in^2	$\bar{x}$, in.	$\bar{x}A$, in^3
1	$\frac{\pi}{4}(4)^2 = 4\pi$	$\frac{16}{3\pi}$	21.333
2	$\frac{1}{2}(4)(4) = 8$	$-\frac{4}{3}$	−10.6667
3	$\frac{1}{2}(4)(4) = 8$	$\frac{4}{3}$	10.6667
Σ	28.566		21.333



$$\bar{X}\Sigma A = \Sigma\bar{x}A: \quad \bar{X}(28.566\text{ in}^2) = 21.333\text{ in}^3$$

$$\bar{X} = 0.74680\text{ in.}; \quad \text{by symmetry,} \quad \bar{Y} = \bar{X} = 0.74680\text{ in.}$$

(*a*) For section ①: $I_x = \frac{1}{4}\left(\frac{1}{4}\pi r^4\right) = \frac{1}{16}\pi(4)^4 = 50.265\text{ in}^4$

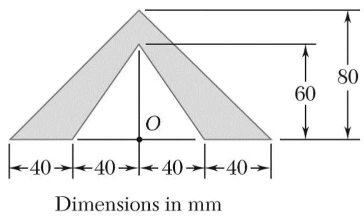
For sections ②,③: $I_x = \frac{1}{12}bh^3 = \frac{1}{12}(4)(4)^3 = 21.333\text{ in}^4$

For total area, $I_x = 50.265 + 2(21.333) = 92.931\text{ in}^4$

By symmetry, $I_y = I_x; \quad J_O = I_x + I_y = 185.862\text{ in}^4$ $\quad J_O = 185.9\text{ in}^4$ ◀

(*b*) Parallel axis theorem: $J_C = J_O - A(OC)^2 = J_O - A(\bar{X}^2 + \bar{Y}^2)$

$J_C = 185.862 - (28.566)(0.74680^2 + 0.74680^2)$ $\quad J_C = 154.0\text{ in}^4$ ◀

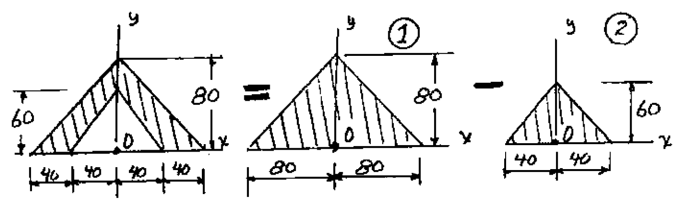


PROBLEM 9.47

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

Dimensions in mm   Symmetry: $\bar{X} = 0$



Determination of centroid *C* of entire section

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(4000 \text{ mm}^2) = 122.67 \times 10^3 \text{ mm}^3$$

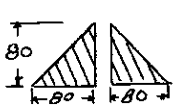
$$\bar{Y} = 30.667 \text{ mm}$$

	Area mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{1}{2}(160)(80) = 6400$	$\frac{80}{3}$	170.67×10^3
2	$-\frac{1}{2}(80)(60) = -2400$	20	-48×10^3
Σ	4000		122.67×10^3

(*a*) Polar moment of inertia J_O:

Section ①: $$I_x = \frac{1}{12}(160)(80)^3 = 6.8267 \times 10^6 \text{ mm}^4$$

For I_y consider the following two triangles



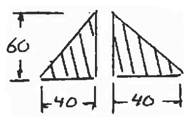
$$I_y = 2\left[\frac{1}{12}(80)(80)^3\right] = 6.8267 \times 10^6 \text{ mm}^4$$

$$J_O = I_x + I_y = (6.8267 + 6.8267)10^6 = 13.653 \times 10^6 \text{ mm}^4$$

(*b*) Section ②:

$$I_x = \frac{1}{12}(80)(60)^3 = 1.44\times10^6 \text{ mm}^4$$

For I_y consider the following two triangles



$$I_y = 2\left[\frac{1}{12}(60)(40)^3\right] = 0.640\times10^6 \text{ mm}^4$$

$$J_O = I_x + I_y = (1.44+0.640)10^6 = 2.08\times10^6 \text{ mm}^4$$

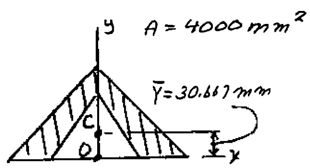
Entire section

$$J_O = (J_O)_1 - (J_O)_2 = 13.653\times10^6 - 2.08\times10^6$$

$$J_O = 11.573\times10^6 \text{ mm}^4$$

$J_O = 11.57\times10^6 \text{ mm}^4$ ◀

(*c*) Polar moment of inertia J_O of intire area

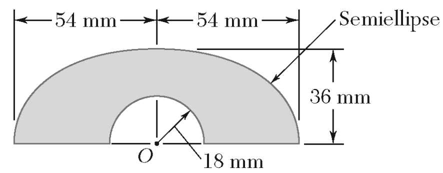


$$J_O = J_C + A\overline{Y}^2$$

$$11.573\times10^6 = J_C + (4000)(30.667)^2$$

$$J_C = 7.811\times10^6 \text{ mm}^4$$

$J_C = 7.81\times10^6 \text{ mm}^4$ ◀

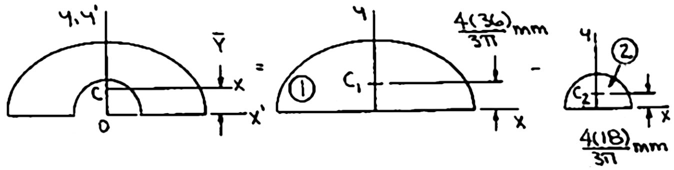


PROBLEM 9.48

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

First locate centroid *C* of the area.



	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{\pi}{2}(54)(36) = 3053.6$	$\frac{48}{\pi} = 15.2789$	$46,656$
2	$-\frac{\pi}{2}(18)^2 = -508.9$	$\frac{24}{\pi} = 7.6394$	-3888
Σ	2544.7		42,768

Then $\qquad \bar{Y}\Sigma A = \Sigma \bar{y} A: \quad \bar{Y}(2544.7 \text{ mm}^2) = 42,768 \text{ mm}^3$

or $\qquad \bar{Y} = 16.8067 \text{ mm} \triangleleft$

(*a*)

$$J_O = (J_O)_1 - (J_O)_2$$

$$= \frac{\pi}{8}(54 \text{ mm})(36 \text{ mm})[(54 \text{ mm})^2 + (36 \text{ mm})^2] - \frac{\pi}{4}(18 \text{ mm})^4$$

$$= (3.2155 \times 10^6 - 0.0824 \times 10^6) \text{ mm}^4$$

$$= 3.1331 \times 10^6 \text{ mm}^4$$

or $\qquad J_O = 3.13 \times 10^6 \text{ mm}^4 \blacktriangleleft$

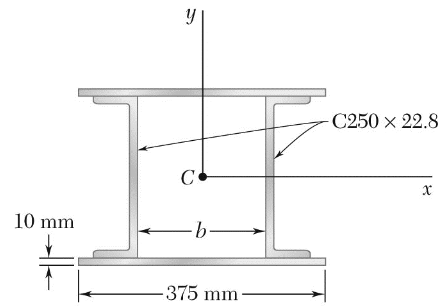
(*b*)

$$J_O = \bar{J}_C + A(\bar{Y})^2$$

or

$$\bar{J}_C = 3.1331 \times 10^6 \text{ mm}^4 - (2544.7 \text{ mm}^2)(16.8067 \text{ mm})^2$$

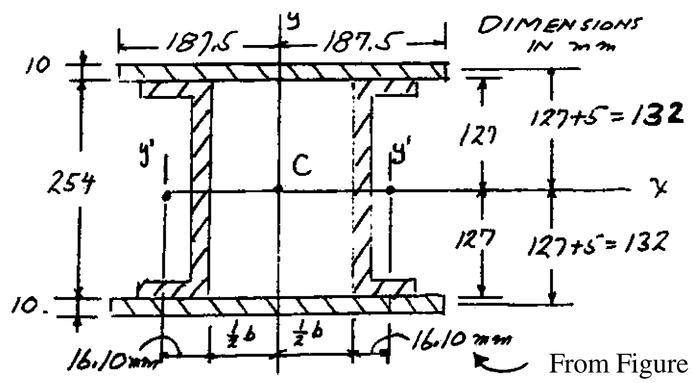
or $\qquad \bar{J}_C = 2.41 \times 10^6 \text{ mm}^4 \blacktriangleleft$



PROBLEM 9.49

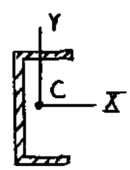
Two channels and two plates are used to form the column section shown. For b = 200 mm, determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION



From Figure 9.13B

For C250 × 22.8



$$A = 2890 \text{ mm}^2$$
$$I_x = 28.0 \times 10^6 \text{ mm}^4$$
$$I_y = 0.945 \times 10^6 \text{ mm}^4$$

Total area $\quad A = 2[2890 \text{ mm}^4 + (10 \text{ mm})(375 \text{ mm})] = 13.28 \times 10^3 \text{ mm}^2$

Given $b = 200$ mm: $\quad \bar{I}_x = 2[28.0 \times 10^6 \text{ mm}^4] + 2\left[\frac{1}{12}(375 \text{ mm})(10 \text{ mm})^3 + (375 \text{ mm})(10 \text{ mm})(132 \text{ mm})^2\right]$

$$= 186.743 \times 10^6$$

$$\bar{I}_x = 186.7 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{k}_x^2 = \frac{\bar{I}_x}{A} = \frac{186.743 \times 10^6}{13.28 \times 10^3} = 14.0620 \times 10^3 \text{ mm}^2$$

$$\bar{k}_x = 118.6 \text{ mm} \blacktriangleleft$$

PROBLEM 9.49 (Continued)

$$\bar{I}_y = \Sigma(\bar{I}_y + Ad^2) = \overset{\text{Channel}}{2[\bar{I}_y + Ad^2]} + \overset{\text{Plate}}{2\left[\frac{1}{12}(10\text{ mm})(375\text{ mm})^3\right]}$$

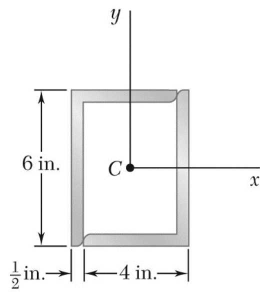
$$= 2\left[0.945\times 10^6\text{ mm}^4 + (2890\text{ mm}^2)\left(\frac{200\text{ mm}}{2} + 16.10\text{ mm}\right)^2\right] + 87.891\times 10^6\text{ mm}^4$$

$$= 2[0.945\times 10^6 + 38.955\times 10^6] + 87.891\times 10^6$$

$$= 167.691\times 10^6\text{ mm}^4 \qquad \bar{I}_y = 167.7\times 10^6\text{ mm}^4 \blacktriangleleft$$

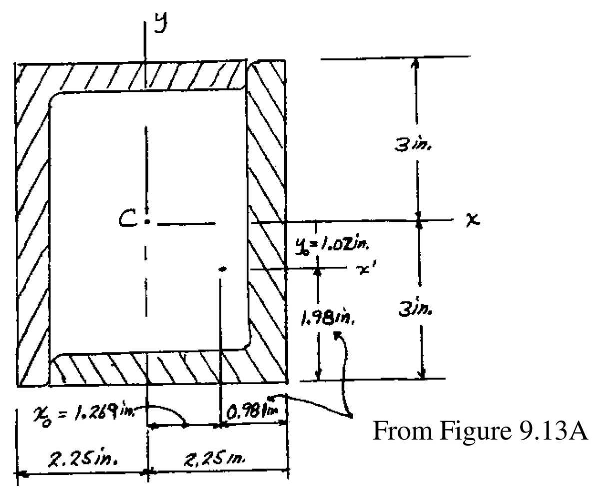
$$k_y^2 = \frac{I_y}{A} = \frac{167.691\times 10^6}{13.28\times 10^3} = 12.6273\times 10^3 \qquad \bar{k}_y = 112.4\text{ mm} \blacktriangleleft$$

PROBLEM 9.50

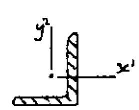


Two L6 × 4 × $\frac{1}{2}$-in. angles are welded together to form the section shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION



From Figure 9.13A: Area $= A = 4.75\text{ in}^2$



$$I_{x'} = 17.3\text{ in}^4$$

$$I_{y'} = 6.22\text{ in}^4$$

$$\bar{I}_x = 2[\bar{I}_{x'} + Ay_O^2] = 2[17.3\text{ in}^4 + (4.75\text{ in}^2)(1.02\text{ in.})^2] = 44.484\text{ in}^4$$

$\bar{I}_x = 44.5\text{ in}^4$ ◀

Total area = 2(4.75) = 9.50 in^2

$$\bar{k}_x^2 = \frac{\bar{I}_x}{\text{area}} = \frac{44.484\text{ in}^4}{9.50\text{ in}^2} = 4.6825\text{ in}^2$$

$\bar{k}_x = 2.16\text{ in.}$ ◀

$$\bar{I}_y = 2[\bar{I}_{y'} + Ax_O^2] = 2[6.22\text{ in}^4 + (4.75\text{ in}^2)(1.269\text{ in.})^2]$$

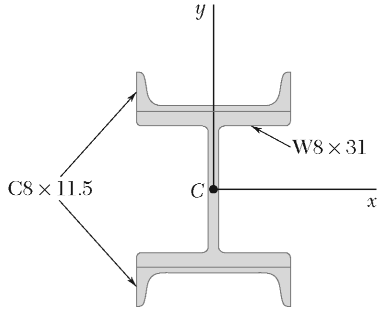
$$= 27.738\text{ in}^4$$

$\bar{I}_y = 27.7\text{ in}^4$ ◀

Total area = 2(4.75) = 9.50 in^2

$$\bar{k}_y^2 = \frac{\bar{I}_y}{\text{area}} = \frac{27.738\text{ in}^4}{9.50\text{ in}^2} = 2.9198$$

$\bar{k}_y = 1.709\text{ in}$ ◀



PROBLEM 9.51

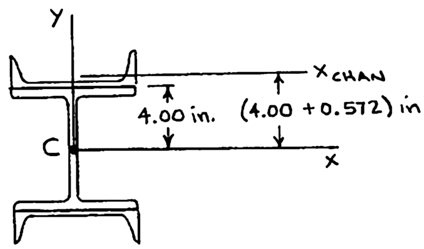
Two channels are welded to a rolled W section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION

W section: $A = 9.12\text{ in}^2$

$\bar{I}_x = 110\text{ in}^4$

$\bar{I}_y = 37.1\text{ in}^4$



Channel: $A = 3.37\text{ in}^2$

$\bar{I}_x = 1.31\text{ in}^4$

$\bar{I}_y = 32.5\text{ in}^4$

$$A_{\text{total}} = A_{\text{W}} + 2A_{\text{chan}}$$
$$= 9.12 + 2(3.37) = 15.86\text{ in}^2$$

Now $\bar{I}_x = (\bar{I}_x)_{\text{W}} + 2(I_x)_{\text{chan}}$

where $(I_x)_{\text{chan}} = \bar{I}_{x_{\text{chan}}} + Ad^2$

$$= 1.31\text{ in}^4 + (3.37\text{ in}^2)(4.572\text{ in.})^2 = 71.754\text{ in}^4$$

Then $\bar{I}_x = (110 + 2 \times 71.754)\text{ in}^4 = 253.51\text{ in}^4$ $\qquad \bar{I}_x = 254\text{ in}^4$ ◀

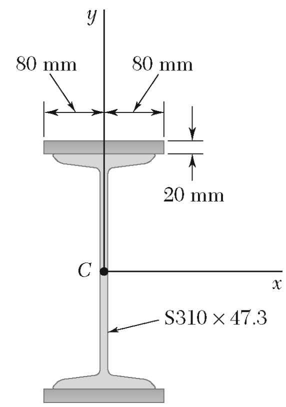
and $\bar{k}_x^2 = \dfrac{\bar{I}_x}{A_{\text{total}}} = \dfrac{253.51\text{ in}^4}{15.86\text{ in}^2}$ $\qquad \bar{k}_x = 4.00\text{ in.}$ ◀

Also $\bar{I}_y = (\bar{I}_y)_{\text{W}} + 2(\bar{I}_y)_{\text{chan}}$

$$= (37.1 + 2 \times 32.5)\text{ in}^4 = 102.1\text{ in}^4$$ $\qquad \bar{I}_y = 102.1\text{ in}^4$ ◀

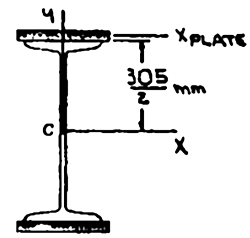
and $\bar{k}_y^2 = \dfrac{\bar{I}_y}{A_{\text{total}}} = \dfrac{102.1\text{ in}^4}{15.86\text{ in}^2}$ $\qquad \bar{k}_y = 2.54\text{ in.}$ ◀

PROBLEM 9.52



Two 20-mm steel plates are welded to a rolled S section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION



S section: $A = 6010 \text{ mm}^2$

$$\bar{I}_x = 90.3 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 3.88 \times 10^6 \text{ mm}^4$$

Note: $A_{\text{total}} = A_S + 2A_{\text{plate}}$

$$= 6010 \text{ mm}^2 + 2(160 \text{ mm})(20 \text{ mm})$$

$$= 12,410 \text{ mm}^2$$

Now $\bar{I}_x = (\bar{I}_x)_S + 2(I_x)_{\text{plate}}$

where $(I_x)_{\text{plate}} = \bar{I}_{x_{\text{plate}}} + Ad^2$

$$= \frac{1}{12}(160 \text{ mm})(20 \text{ mm})^3 + (3200 \text{ mm}^2)[(152.5 + 10) \text{ mm}]^2$$

$$= 84.6067 \times 10^6 \text{ mm}^4$$

Then $\bar{I}_x = (90.3 + 2 \times 84.6067) \times 10^6 \text{ mm}^4$

$$= 259.5134 \times 10^6 \text{ mm}^4 \qquad \text{or} \quad \bar{I}_x = 260 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

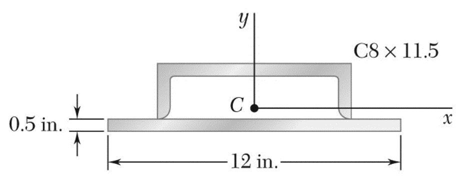
and $\bar{k}_x^2 = \dfrac{\bar{I}_x}{A_{\text{total}}} = \dfrac{259.5134 \times 10^6 \text{ mm}^4}{12410 \text{ mm}^2} \qquad \text{or} \quad \bar{k}_x = 144.6 \text{ mm} \blacktriangleleft$

Also $\bar{I}_y = (\bar{I}_y)_S + 2(\bar{I}_y)_{\text{plate}}$

$$= 3.88 \times 10^6 \text{ mm}^4 + 2\left[\frac{1}{12}(20 \text{ mm})(160 \text{ mm})^3\right]$$

$$= 17.5333 \times 10^6 \text{ mm}^4 \qquad \text{or} \quad \bar{I}_y = 17.53 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

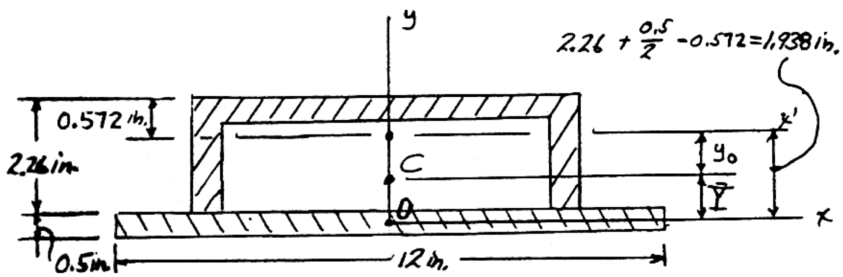
and $\bar{k}_y^2 = \dfrac{\bar{I}_y}{A_{\text{total}}} = \dfrac{17.5333 \times 10^6 \text{ mm}^4}{12,410 \text{ mm}^2} \qquad \text{or} \quad \bar{k}_y = 37.6 \text{ mm} \blacktriangleleft$



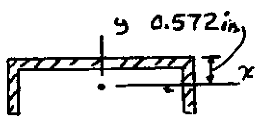
PROBLEM 9.53

A channel and a plate are welded together as shown to form a section that is symmetrical with respect to the y axis. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

SOLUTION



From Figure 9.13B



$B = 3.37 \text{ in}^2$

For C8 × 11.5: $\bar{I}_y = 1.31 \text{ in}^4$

(Note change of axes) $\bar{I}_y = 32.5 \text{ in}^4$

Location of centroid

$$\bar{Y}\Sigma A = \Sigma \bar{y} A: \qquad \bar{Y}[3.7 \text{ in}^2 + (12 \text{ in.})(0.5 \text{ in.})] = (3.37 \text{ in}^2)(1.938 \text{ in.})$$

$$\bar{Y} = 0.69702 \text{ in.}$$

Moment of inertia with respect to x axis.

Plate:
$$I_x = \bar{I}_x + A\bar{Y}^2 = \frac{1}{12}(12 \text{ in.})(0.5 \text{ in.})^3 + (12 \text{ in.})(0.5 \text{ in.})(0.69702 \text{ in.})^2$$
$$= 0.125 + 2.9150 = 3.0400 \text{ in}^4$$

Channel:
$$I_x = \bar{I}_{x'} + A\bar{Y}_0^2 = 1.31 \text{ in}^4 + (3.37 \text{ in.})(1.938 \text{ in.} - 0.69702 \text{ in.})^2$$
$$= 1.31 + 5.1899 = 6.4999 \text{ in}^4$$

Entire section:
$$I_x = 3.0400 \text{ in}^4 + 6.4999 \text{ in}^4 = 9.5399 \text{ in}^4$$

$\bar{I}_x = 9.54 \text{ in}^4$ ◀

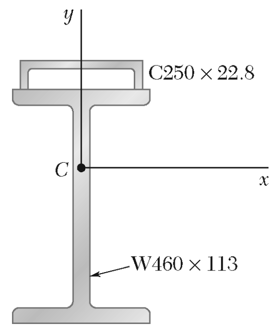
Moment of inertia with respect to y axis.

Plate:
$$\bar{I}_y = \frac{1}{12}(0.5 \text{ in.})(12 \text{ in.})^3 = 72 \text{ in}^4$$

Channel:
$$\bar{I}_y = 32.5 \text{ in}^4$$

Entire section:
$$\bar{I}_y = 72 \text{ in}^4 + 32.5 \text{ in}^4 = 104.5 \text{ in}^4$$

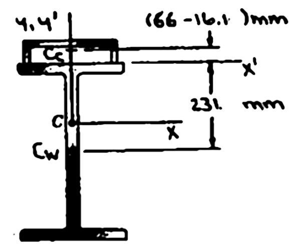
$\bar{I}_y = 104.5 \text{ in}^4$ ◀



PROBLEM 9.54

The strength of the rolled W section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

SOLUTION



W section: $A = 14,400 \text{ mm}^2$

$$\bar{I}_x = 554 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 63.3 \times 10^6 \text{ mm}^4$$

Channel: $A = 2890 \text{ mm}^2$

$$\bar{I}_x = 0.945 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 28.0 \times 10^6 \text{ mm}^4$$

First locate centroid C of the section.

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
W Section	14,400	−231	−33,26,400
Channel	2,890	49.9	1,44,211
Σ	17,290		−31,82,189

Then $\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(17,290 \text{ mm}^2) = -3,182,189 \text{ mm}^3$

or $\bar{Y} = -184.047 \text{ mm}$

Now $\bar{I}_x = (I_x)_W + (I_x)_C$

where

$$(I_x)_W = \bar{I}_x + Ad^2$$
$$= 554 \times 10^6 \text{ mm}^4 + (14,400 \text{ mm}^2)(231 - 184.047)^2 \text{ mm}^2$$
$$= 585.75 \times 10^6 \text{ mm}^4$$

$$(I_x)_C = \bar{I}_x - Ad^2$$
$$= 0.945 \times 10^6 \text{ mm}^4 + (2,890 \text{ mm}^2)(49.9 + 184.047)^2 \text{ mm}^2$$
$$= 159.12 \times 10^6 \text{ mm}^4$$

PROBLEM 9.54 (Continued)

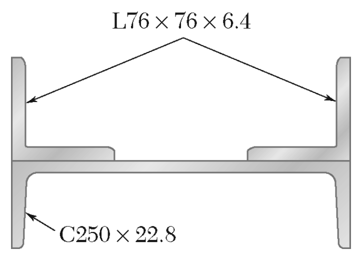
Then $\bar{I}_x = (585.75 + 159.12) \times 10^6 \text{ mm}^4$

or $\bar{I}_x = 745 \times 10^6 \text{ mm}^4$ ◀

also $\bar{I}_y = (I_y)_\text{W} + (I_y)_C$

$= (63.3 + 28.0) \times 10^6 \text{ mm}^4$

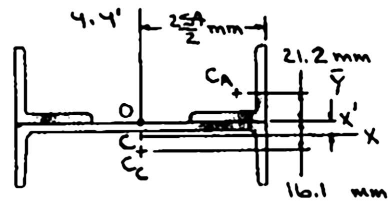
or $\bar{I}_y = 91.3 \times 10^6 \text{ mm}^4$ ◀



PROBLEM 9.55

Two L76 × 76 × 6.4-mm angles are welded to a C250 × 22.8 channel. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the web of the channel.

SOLUTION



Angle: $A = 929 \text{ mm}^2$

$\bar{I}_x = \bar{I}_y = 0.512 \times 10^6 \text{ mm}^4$

Channel: $A = 2890 \text{ mm}^2$

$\bar{I}_x = 0.945 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 28.0 \times 10^6 \text{ mm}^4$

First locate centroid C of the section

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
Angle	2(929) = 1858	21.2	39,389.6
Channel	2890	−16.1	−46,529
Σ	4748		−7139.4

Then $\bar{Y}\Sigma A = \Sigma \bar{y} A: \quad \bar{Y}(4748 \text{ mm}^2) = -7139.4 \text{ mm}^3$

or $\bar{Y} = -1.50366 \text{ mm}$

Now $\bar{I}_x = 2(I_x)_L + (I_x)_C$

where $(I_x)_L = \bar{I}_x + Ad^2 = 0.512 \times 10^6 \text{ mm}^4 + (929 \text{ mm}^2)[(21.2 + 1.50366) \text{ mm}]^2$

$= 0.990859 \times 10^6 \text{ mm}^4$

$(I_x)_C = \bar{I}_x + Ad^2 = 0.949 \times 10^6 \text{ mm}^4 + (2890 \text{ mm}^2)[(16.1 - 1.50366) \text{ mm}]^2$

$= 1.56472 \times 10^6 \text{ mm}^4$

Then $\bar{I}_x = [2(0.990859) + 1.56472 \times 10^6] \text{ mm}^4$

or $\bar{I}_x = 3.55 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.55 (Continued)

Also $$\bar{I}_y = 2(I_y)_L + (I_y)_C$$

where $$(I_y)_L = \bar{I}_y + Ad^2 = 0.512\times10^6\,\text{mm}^4 + (929\,\text{mm}^2)[(127-21.2)\,\text{mm}]^2$$

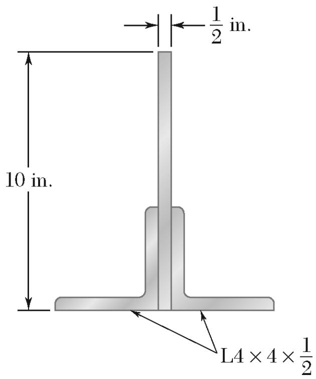
$$= 10.9109\times10^6\,\text{mm}^4$$

$$(I_y)_C = \bar{I}_y$$

Then $$\bar{I}_y = [2(10.9109)+28.0]\times10^6\,\text{mm}^4$$

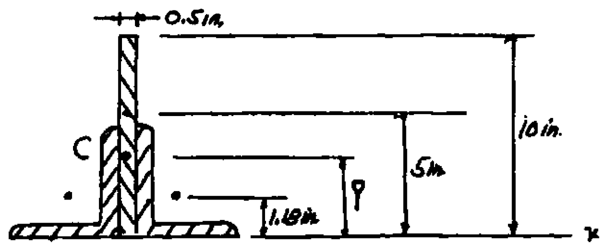
or $\bar{I}_y = 49.8\times10^6\,\text{mm}^4$ ◀

PROBLEM 9.56



Two L4 × 4 × $\frac{1}{2}$-in. angles are welded to a steel plate as shown. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the plate.

SOLUTION



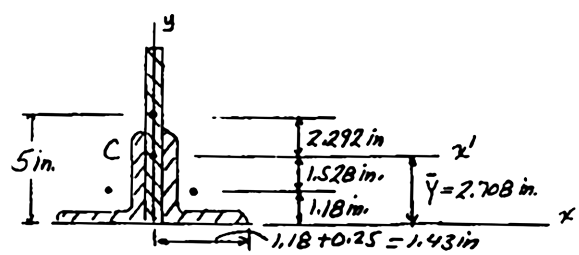
For $4\times4\times\frac{1}{2}$-in. angle: $A = 3.75\text{ in}^2,\quad \bar{I}_x = \bar{I}_y = 5.52\text{ in}^4$

$$\bar{Y}A = \Sigma \bar{y}A$$

$$\bar{Y}(12.5\text{ in}^2) = 33.85\text{ in}^3$$

$$\bar{Y} = 2.708\text{ in.}$$

Section	Area, in^2	$\bar{y}$ in.	$\bar{y}A$, in^3
Plate	(0.5)(10) = 5	5	25
Two angles	2(3.75) = 7.5	1.18	8.85
Σ	12.5		33.85



PROBLEM 9.56 (Continued)

Entire section:

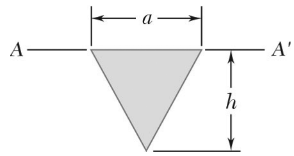
$$\bar{I}_x = \Sigma(\bar{I}_{x'} + Ad^2) = \left[\frac{1}{12}(0.5)(10)^3 + (0.5)(10)(2.292)^2\right] + 2[5.52 + (3.75)(1.528)^2]$$

$$= 41.667 + 26.266 + 1604 + 17.511 = 96.48 \text{ in}^4 \qquad \bar{I}_x = 96.5 \text{ in}^4 \blacktriangleleft$$

$$\bar{I}_y = \frac{1}{12}(10)(0.5)^3 + 2[5.52 + (3.75)(1.43)^2]$$

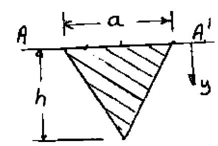
$$= 0.104 + 11.04 + 15.367 = 26.51 \text{ in}^4 \qquad \bar{I}_y = 26.5 \text{ in}^4 \blacktriangleleft$$

PROBLEM 9.57



The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION



From section 9.2: $R = \gamma \int y\,dA, \quad M_{AA'} = \gamma \int y^2\,dA$

Let y_P = distance of center of pressure from AA'. We must have:

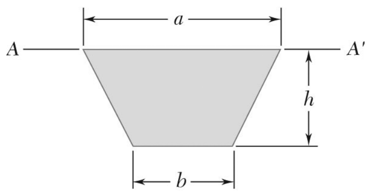
$$Ry_P = M_{AA'}: \quad y_P = \frac{M_{AA'}}{R} = \frac{\gamma \int y^2\,dA}{\gamma \int y\,dA} = \frac{I_{AA'}}{\bar{y}A} \qquad (1)$$

For triangular panel:

$$I_{AA'} = \frac{1}{12}ah^3 \qquad \bar{y} = \frac{1}{3}h \qquad A = \frac{1}{2}ah$$

$$y_P = \frac{I_{AA'}}{\bar{y}A} = \frac{\frac{1}{12}ah^3}{\left(\frac{1}{3}h\right)\left(\frac{1}{2}ah\right)}$$

$$y_P = \frac{1}{2}h \blacktriangleleft$$

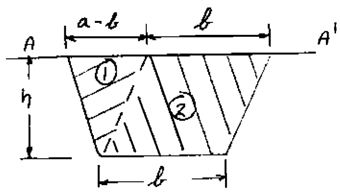


PROBLEM 9.58

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

See solution of Problem 9.57 for derivation of Eq. (1):



$$y_P = \frac{I_{AA'}}{\bar{y}A} \qquad (1)$$

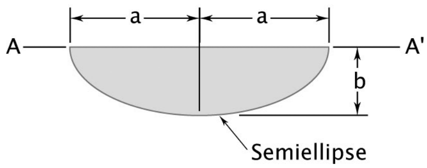
Divide trapezoid as shown:

$$I_{AA'} = \frac{1}{12}(a-b)h^3 + \frac{1}{3}bh^3$$

$$= \frac{1}{2}ah^3 + \frac{1}{4}bh^3$$

$$\bar{y}A = y_1A_1 + y_2A_2 = \frac{1}{3}h\left[\frac{1}{2}(a-b)h\right] + \frac{1}{2}h(bh) = \frac{1}{6}ah^2 + \frac{1}{3}bh^2$$

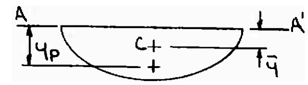
$$y_P = \frac{I_{AA'}}{\bar{y}A} = \frac{\frac{1}{12}ah^3 + \frac{1}{4}bh^3}{\frac{1}{6}ah^2 + \frac{1}{3}bh^2} \qquad y_P = \frac{a+3b}{2a+4b}h \blacktriangleleft$$

	**PROBLEM 9.59** The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 475 of the text have:

$$y_P = \frac{I_{AA'}}{\bar{y}A}$$



For a semi ellipse:

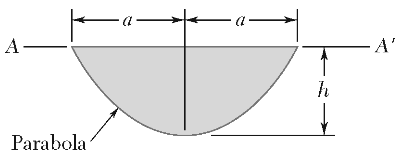
$$I_{AA'} = \frac{\pi}{8}ab^3$$

$$\bar{y} = \frac{4b}{3\pi} \qquad A = \frac{\pi}{2}ab$$

Then

$$y_P = \frac{\frac{\pi}{8}ab^3}{\frac{4b}{3\pi} \times \frac{\pi}{2}ab} \text{ or}$$

or $\quad y_P = \dfrac{3\pi}{16}b$ ◀



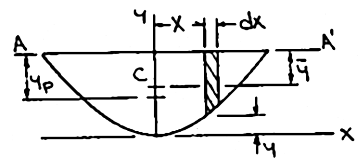
PROBLEM 9.60*

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 491 of the text:

$$y_P = \frac{I_{AA'}}{\bar{y}A}$$



For a parabola: $$\bar{y} = \frac{2}{5}h \qquad A = \frac{4}{3}ah$$

Now $$dI_{AA'} = \frac{1}{3}(h-y)^3\,dx$$

By observation $$y = \frac{h}{a^2}x^2$$

So that $$dI_{AA'} = \frac{1}{3}\left(h - \frac{h}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{h^3}{a^6}(a^2 - x^2)^3\,dx$$

$$= \frac{1}{3}\frac{h}{a^6}(a^6 - 3a^4x^2 + 3a^2x^4 - x^6)dx$$

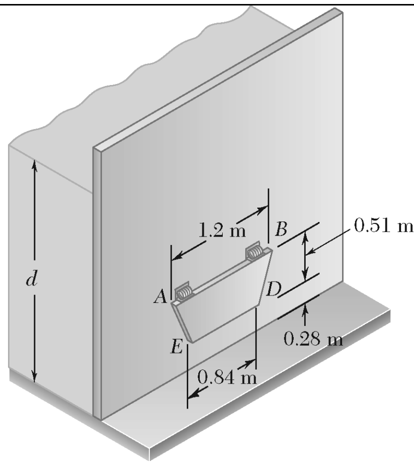
Then $$I_{AA'} = 2\int_0^a \frac{1}{3}\frac{h^3}{a^6}(a^6 - 3a^4x^2 + 3a^2x^4 - x^6)dx$$

$$= \frac{2}{3}\frac{h^3}{a^6}\left[a^6x - a^4x^3 + \frac{3}{5}a^2x^5 - \frac{1}{7}x^7\right]_0^a$$

$$= \frac{32}{105}ah^3$$

Finally, $$y_p = \frac{\frac{32}{105}ah^3}{\frac{2}{5}h \times \frac{4}{3}ah}$$

or $$y_P = \frac{4}{7}h \blacktriangleleft$$



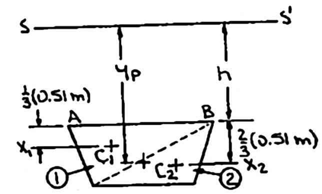
PROBLEM 9.61

A vertical trapezoidal gate that is used as an automatic valve is held shut by two springs attached to hinges located along edge *AB*. Knowing that each spring exerts a couple of magnitude 1470 N · m, determine the depth *d* of water for which the gate will open.

SOLUTION

From section 9.2:

$$R = \gamma \bar{y} A \qquad y_P = \frac{I_{SS'}}{\bar{y}A}$$



where R is the resultant of the hydrostatic forces acting on the gate and y_P is the depth to the point of application of R. Now

$$\bar{y}A = \Sigma \bar{y}A = [(h+0.17)\text{ m}]\left(\frac{1}{2}\times 1.2\text{ m}\times 0.51\text{ m}\right) + [(h+0.34)\text{ m}]\left(\frac{1}{2}\times 0.84\times 0.51\text{ m}\right)$$

$$= (0.5202h + 0.124848)\text{ m}^3$$

Recalling that $\gamma = py$, we have

$$R = (10^3\text{ kg/m}^3 \times 9.81\text{ m/s}^2)(0.5202h + 0.124848)\text{ m}^3$$

$$= 5103.162(h + 0.24)\text{ N}$$

Also $$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$$

where $$(I_{SS'})_1 = I_x + Ad^2$$

$$= \frac{1}{36}(1.2\text{ m})(0.51\text{ m})^3 + \left(\frac{1}{2}\times 1.2\text{ m}\times 0.51\text{ m}\right)[(h+0.17)\text{ m}]^2$$

$$= [0.0044217 + 0.306(h+0.17)^2]\text{ m}^4$$

$$= (0.306h^2 + 0.10404h + 0.0132651)\text{ m}^4$$

$$(I_{SS'})_2 = \bar{I}_{X_2} + Ad^2$$

$$= \frac{1}{36}(0.84\text{ m})(0.51\text{ m})^3 + \left(\frac{1}{2}\times 0.84\text{ m}\times 0.51\text{ m}\right)[(h+0.34)\text{ m}]^2$$

$$= [0.0030952 + 0.2142(h+0.34)^2]\text{ m}^4$$

$$= (0.2142h^2 + 0.145656h + 0.0278567)\text{ m}^4$$

PROBLEM 9.61 (Continued)

Then
$$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$$
$$= (0.5202h^2 + 0.249696h + 0.0411218)\ \text{m}^4$$

and
$$y_P = \frac{(0.5202h^2 + 0.244696h + 0.0411218)\ \text{m}^4}{(0.5202h + 0.124848)\ \text{m}^3}$$
$$= \frac{h^2 + 0.48h + 0.07905}{h + 0.24}\ \text{m}$$

For the gate to open, require that

$$\Sigma M_{AB}: \quad M_{\text{open}} = (y_P - h)R$$

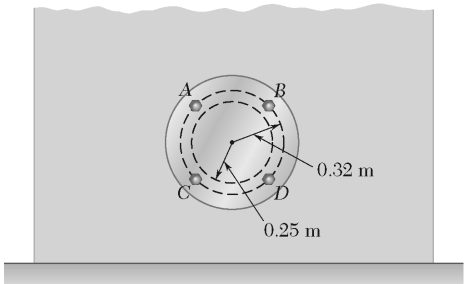
Substituting
$$2940\ \text{N} \cdot \text{m} = \left(\frac{h^2 + 0.48h + 0.07905}{h + 0.24} - h\right) \text{m} \times 5103.162(h + 0.24)\ \text{N}$$

or
$$5103.162(0.24h + 0.07905) = 2940$$

or
$$h = 2.0711\ \text{m}$$

Then
$$d = (2.0711 + 0.79)\ \text{m}$$

or $d = 2.86$ m ◀

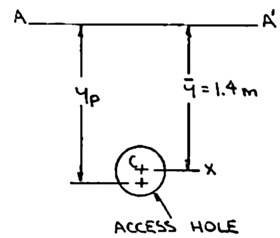


PROBLEM 9.62

The cover for a 0.5-m-diameter access hole in a water storage tank is attached to the tank with four equally spaced bolts as shown. Determine the additional force on each bolt due to the water pressure when the center of the cover is located 1.4 m below the water surface.

SOLUTION

From section 9.2:



$$R = \gamma \bar{y} A \qquad y_P = \frac{I_{AA'}}{\bar{y}A}$$

where R is the resultant of the hydrostatic forces acting on the cover and y_P is the depth to the point of application of R.

Recalling that $\gamma = p \cdot y$, we have

$$R = (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2)(1.4 \text{ m})[\pi(0.25 \text{ m})^2]$$
$$= 2696.67 \text{ N}$$

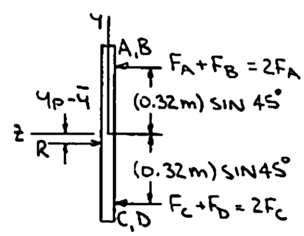
Also $$I_{AA'} = \bar{I}_x + A\bar{y}^2 = \frac{\pi}{4}(0.25 \text{ m})^4 + [\pi(0.25 \text{ m})^2](1.4 \text{ m})^2$$
$$= 0.387913 \text{ m}^4$$

Then $$y_P = \frac{0.387913 \text{ m}^4}{(1.4 \text{ m})[\pi(0.25 \text{ m})^2]} = 1.41116 \text{ m}$$

Now note that symmetry implies

$$F_A = F_B \qquad F_C = F_D$$

Next consider the free-body of the cover.



We have $+\circlearrowleft \Sigma M_{CD} = 0$: $[2(0.32 \text{ m}) \sin 45°](2F_A)$
$- [0.32 \sin 45° - (1.41116 - 1.4)] \text{ m}$
$\times (2696.67 \text{ N}) = 0$

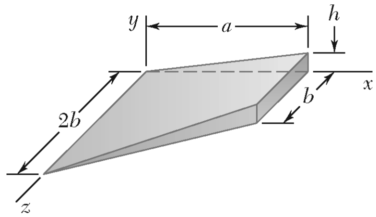
or $$F_A = 640.92 \text{ N}$$

Then $\overset{+}{\leftarrow} \Sigma F_z = 0$: $2(640.92 \text{ N}) + 2F_C - 2696.67 \text{ N} = 0$

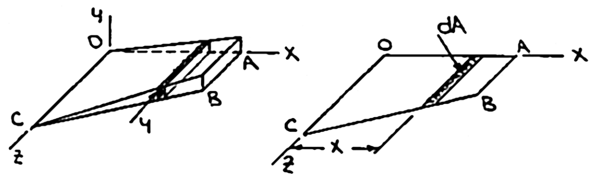
or $$F_C = 707.42 \text{ N}$$

$F_A = F_B = 641 \text{ N}$ ◀

and $F_C = F_D = 707 \text{ N}$ ◀

	**PROBLEM 9.63*** Determine the x coordinate of the centroid of the volume shown. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION



First note that $$y = \frac{h}{a}x$$

Now $$\bar{x}\int dV = \int \bar{x}_{EL}\,dV$$

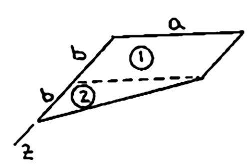
where $$\bar{x}_{EL} = x \quad dV = y\,dA = \left(\frac{h}{a}x\right)dA$$

Then $$\bar{x} = \frac{\int x\left(\frac{h}{a}x\,dA\right)}{\int \frac{h}{a}x\,dA} = \frac{\int x^2 dA}{\int x\,dA} = \frac{(I_z)_A}{(\bar{x}A)_A}$$

where $(I_z)_A$ and $(\bar{x}_A)_A$ pertain to area.

OABC: $(I_z)_A$ is the moment of inertia of the area with respect to the z axis, $\bar{x}_A$ is the x coordinate of the centroid of the area, and A is the area of *OABC*. Then

$$(I_z)_A = (I_z)_{A_1} + (I_z)_{A_2}$$
$$= \frac{1}{3}(b)(a)^3 + \frac{1}{12}(b)(a)^3$$
$$= \frac{5}{12}a^3 b$$



and $$(\bar{x}A)_A = \Sigma\bar{x}A$$
$$= \left[\left(\frac{a}{2}\right)(a \times b)\right] + \left[\left(\frac{a}{3}\right)\left(\frac{1}{2}\times a \times b\right)\right]$$
$$= \frac{2}{3}a^2 b$$

PROBLEM 9.63* (Continued)

Finally, $$\bar{x} = \frac{\frac{5}{12}a^3b}{\frac{2}{3}a^2b}$$

or $\bar{x} = \frac{5}{8}a$ ◀

Analogy with hydrostatic pressure on a submerged plate:

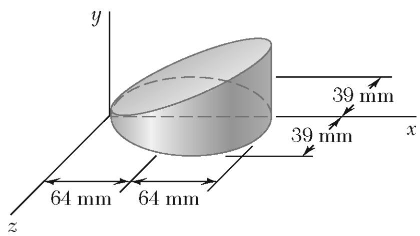
Recalling that $P = \gamma y$, it follows that the following analogies can be established.

Height $y \sim P$

$$dV = ydA \sim pdA = dF$$
$$xdV = x(ydA) \sim ydF = dM$$

Recalling that $$y_P = \frac{M_x}{R}\left(= \frac{\int dM}{\int dF}\right)$$

It can then be concluded that $x \sim y_P$



PROBLEM 9.64*

Determine the x coordinate of the centroid of the volume shown; this volume was obtained by intersecting an elliptic cylinder with an oblique plane. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION

Following the "Hint," it can be shown that (see solution to Problem 9.63)

$$\bar{x} = \frac{(I_z)_A}{(\bar{x}A)_A} x$$

where $(I_z)_A$ and $(\bar{x}A)_A$ are the moment of inertia and the first moment of the area, respectively, of the elliptical area of the base of the volume with respect to the z axis. Then

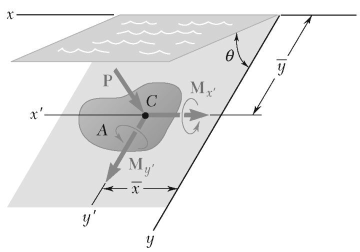
$$\begin{aligned}(I_z)_A &= \bar{I}_z + Ad^2 \\ &= \frac{\pi}{4}(39 \text{ mm})(64 \text{ mm})^3 + [\pi(64 \text{ mm})(39 \text{ mm})](64 \text{ mm})^2 \\ &= 12.779520\pi \times 10^6 \text{ mm}^4\end{aligned}$$

$$\begin{aligned}(\bar{x}A)_A &= (64 \text{ mm})[\pi(64 \text{ mm})(39 \text{ mm})] \\ &= 0.159744\pi \times 10^6 \text{ mm}^3\end{aligned}$$

Finally $$\bar{x} = \frac{12.779520\pi \times 10^6 \text{ mm}^4}{0.159744\pi \times 10^6 \text{ mm}^4}$$

or $\bar{x} = 80.0 \text{ mm}$ ◀

PROBLEM 9.65*



Show that the system of hydrostatic forces acting on a submerged plane area A can be reduced to a force $\mathbf{P}$ at the centroid C of the area and two couples. The force $\mathbf{P}$ is perpendicular to the area and is of magnitude $P = \gamma A\bar{y}\sin\theta,$ where γ is the specific weight of the liquid, and the couples are $\mathbf{M}_{x'} = (\gamma\bar{I}_{x'}\sin\theta)\mathbf{i}$ and $\mathbf{M}_{y'} = (\gamma\bar{I}_{x'y'}\sin\theta)\mathbf{j}$, where $\bar{I}_{x'y'} = \int x'y'\,dA$ (see section 9.8). Note that the couples are independent of the depth at which the area is submerged.

SOLUTION

The pressure p at an arbitrary depth $(y\sin\theta)$ is

$$p = \gamma(y\sin\theta)$$

so that the hydrostatic force dF exerted on an infinitesimal area dA is

$$dF = (\gamma y\sin\theta)dA$$

Equivalence of the force $\mathbf{P}$ and the system of infinitesimal forces dF requires

$$\Sigma F: \quad P = \int dF = \int \gamma y\sin\theta\,dA = \gamma\sin\theta\int y\,dA$$

or $\qquad P = \gamma A\bar{y}\sin\theta$ ◀

Equivalence of the force and couple $(\mathbf{P},\ \mathbf{M}_{x'} + \mathbf{M}_{y'})$ and the system of infinitesimal hydrostatic forces requires

$$\Sigma M_x: \quad -\bar{y}P - M_{x'} = \int(-y\,dF)$$

Now

$$-\int y\,dF = -\int y(\gamma y\sin\theta)dA = -\gamma\sin\theta\int y^2\,dA$$
$$= -(\gamma\sin\theta)I_x$$

Then

$$-\bar{y}P - M_{x'} = -(\gamma\sin\theta)I_x$$

or

$$M_{x'} = (\gamma\sin\theta)I_x - \bar{y}(\gamma A\bar{y}\sin\theta)$$
$$= \gamma\sin\theta(I_x - A\bar{y}^2)$$

or $\qquad M_{x'} = \gamma\bar{I}_{x'}\sin\theta$ ◀

$$\Sigma M_y: \quad \bar{x}P + M_{y'} = \int x\,dF$$

Now

$$\int x\,dF = \int x(\gamma y\sin\theta)dA = \gamma\sin\theta\int xy\,dA$$
$$= (\gamma\sin\theta)I_{xy} \qquad \text{(Equation 9.12)}$$

PROBLEM 9.65* (Continued)

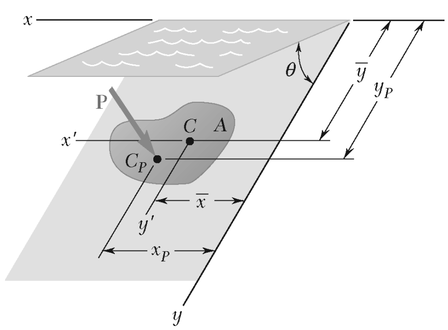
Then $$\bar{x}P + M_{y'} = (\gamma \sin\theta) I_{xy}$$

or $$M_{y'} = (\gamma \sin\theta) I_{xy} - \bar{x}(\gamma A \bar{y} \sin\theta)$$

$$= \gamma \sin\theta (I_{xy} - A\bar{x}\,\bar{y})$$

or $$M_{y'} = \gamma \bar{I}_{x'y'} \sin\theta \blacktriangleleft$$

PROBLEM 9.66*



Show that the resultant of the hydrostatic forces acting on a submerged plane area A is a force **P** perpendicular to the area and of magnitude $P = \gamma A\overline{y} \sin\theta = \overline{p}A$, where γ is the specific weight of the liquid and $\overline{p}$ is the pressure at the centroid C of the area. Show that **P** is applied at a Point C_P, called the center of pressure, whose coordinates are $x_P = I_{xy}/A\overline{y}$ and $y_P = I_x/A\overline{y}$, where $\overline{I}_{xy} = \int xydA$ (see section 9.8). Show also that the difference of ordinates $y_P - \overline{y}$ is equal to $\overline{k}_{x'}^2/\overline{y}$ and thus depends upon the depth at which the area is submerged.

SOLUTION

The pressure P at an arbitrary depth $(y\sin\theta)$ is

$$P = \gamma(y\sin\theta)$$

so that the hydrostatic force dP exerted on an infinitesimal area dA is

$$dP = (\gamma y\sin\theta)dA$$

The magnitude **P** of the resultant force acting on the plane area is then

$$P = \int dP = \int \gamma y \sin\theta\, dA = \gamma \sin\theta \int ydA$$
$$= \gamma\sin\theta(\overline{y}A)$$

Now $\qquad \overline{p} = \gamma\overline{y}\sin\theta \qquad\qquad P = \overline{p}A$ ◀

Next observe that the resultant **P** is equivalent to the system of infinitesimal forces d**P**. Equivalence then requires

$$\Sigma M_x: \quad -y_P P = -\int ydP$$

Now
$$\int ydP = \int y(\gamma y\sin\theta)dA = \gamma\sin\theta\int y^2 dA$$
$$= (\gamma\sin\theta)I_x$$

Then
$$y_P P = (\gamma\sin\theta)I_x$$

or
$$y_P = \frac{(\gamma\sin\theta)I_x}{\gamma\sin\theta(\overline{y}A)}$$

or $\qquad y_P = \dfrac{I_x}{A\overline{y}}$ ◀

$$\Sigma M_y: \quad x_P P = \int x\,dP$$

Now $$\int x\,dP = \int x(\gamma y \sin\theta)\,dA = \gamma \sin\theta \int xy\,dA$$

$$= (\gamma \sin\theta) I_{xy} \qquad \text{(Equation 9.12)}$$

Then $$x_P P = (\gamma \sin\theta) I_{xy}$$

or $$x_P = \frac{(\gamma \sin\theta) I_{xy}}{\gamma \sin\theta(\bar{y}A)}$$

or $$x_P = \frac{I_{xy}}{A\bar{y}} \blacktriangleleft$$

Now $$I_x = \bar{I}_{x'} + A\bar{y}^2$$

From above $$I_x = (A\bar{y})y_P$$

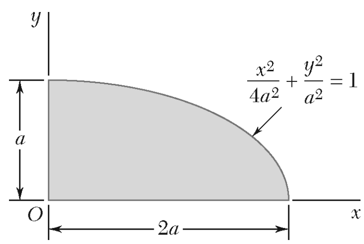
By definition $$\bar{I}_{x'} = \bar{k}_{x'}^2 A$$

Substituting $$(A\bar{y})y_P = \bar{k}_{x'}^2 A + A\bar{y}^2$$

Rearranging yields $$y_P - \bar{y} = \frac{\bar{k}_{x'}^2}{\bar{y}} \blacktriangleleft$$

Although $\bar{k}_{x'}$ is not a function of the depth of the area (it depends only on the shape of A), $\bar{y}$ is dependent on the depth.

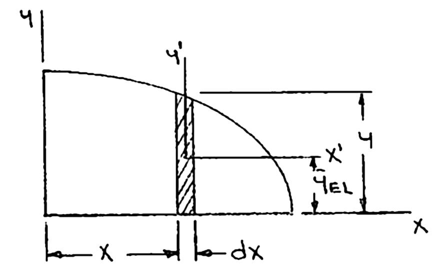
$$(y_P - \bar{y}) = f(\text{depth})$$



PROBLEM 9.67

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION



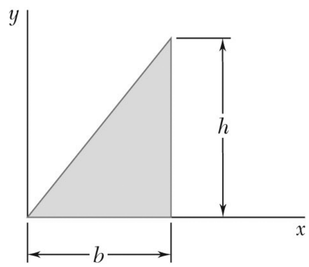
First note
$$y = a\sqrt{1 - \frac{x^2}{4a^2}}$$
$$= \frac{1}{2}\sqrt{4a^2 - x^2}$$

We have
$$dI_{xy} = d\overline{I}_{x'y'} + \overline{x}_{EL}\overline{y}_{EL}dA$$

where
$$d\overline{I}_{x'y'} = 0 \quad \text{(symmetry)} \quad \overline{x}_{EL} = x$$
$$\overline{y}_{EL} = \frac{1}{2}y = \frac{1}{4}\sqrt{4a^2 - x^2}$$
$$dA = ydx = \frac{1}{2}\sqrt{4a^2 - x^2 dx}$$

Then
$$I_{xy} = \int dI_{xy} = \int_0^{2a} x\left(\frac{1}{4}\sqrt{4a^2 - x^2}\right)\left(\frac{1}{2}\sqrt{4a^2 - x^2}\right)dx$$
$$= \frac{1}{8}\int_0^{2a}(4a^2x - x^3)dx = \frac{1}{8}\left[2a^2x^2 - \frac{1}{4}x^4\right]_0^{2a}$$
$$= \frac{a^4}{8}\left[2(2)^2 - \frac{1}{4}(2)^4\right]$$

or $\quad I_{xy} = \frac{1}{2}a^4$ ◀

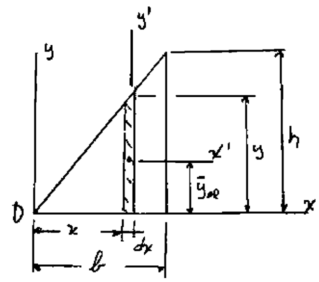


PROBLEM 9.68

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$$



But $\qquad d\bar{I}_{x'y'} = 0 \quad \text{(by symmetry)} \quad \bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2}y$

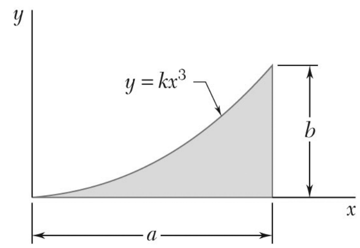
$$dA = yd_y$$

$$\frac{y}{x} = \frac{h}{b} \qquad y = \frac{h}{b}x$$

$$I_{xy} = \int dI_{xy} = \int_0^b x\left(\frac{1}{2}y\right)ydx = \frac{1}{2}\int_0^b x\left(\frac{h}{b}x\right)^2 dx$$

$$= \frac{1}{2}\frac{h^2}{b^2}\int_0^b x^3 dx = \frac{1}{2}\frac{h^2}{b^2}\frac{b^4}{4}$$

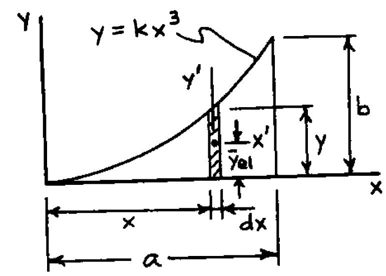
$$I_{xy} = \frac{1}{8}b^2h^2 \blacktriangleleft$$



PROBLEM 9.69

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION



For $\quad x = a, \quad b = ka^3 \quad \text{or} \quad k = \dfrac{b}{a^3}$

Thus, $\quad y = \dfrac{b}{a^3}x^3$

$$d\bar{I}_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA, \quad \text{But} \quad dI_{x'y'} = 0 \quad \text{(by symmetry)}$$

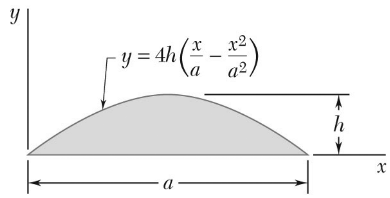
with $\quad \bar{x}_{EL} = x \quad \bar{y}_{EL} = \dfrac{y}{2} \quad dA = ydx$

$$I_{xy} = \int dI_{xy} = \int_0^a x\left(\frac{y}{2}\right)ydx = \frac{1}{2}\int_0^a x\left(\frac{b}{a^3}x^3\right)^2 dx$$

$$= \frac{1}{2}\frac{b^2}{a^6}\left[\frac{x^8}{8}\right]_0^a$$

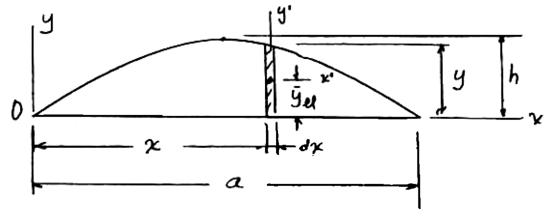
$$I_{xy} = a^2b^2/16 \blacktriangleleft$$

PROBLEM 9.70



Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION



$$y = 4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)$$

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$$

But $\quad d\bar{I}_{x'y'} = 0 \quad$ (by symmetry)

$$\bar{x}_{EL} = x \qquad \bar{y}_{EL} = \frac{1}{2}y \qquad dA = ydx$$

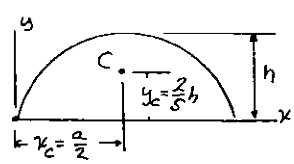
$$I_{xy} = \int dI_{xy} = \int_0^a x\left(\frac{1}{2}y\right)ydx = \frac{1}{2}\int_0^a 16h^2 x\left(\frac{x}{a} - \frac{x^2}{a^2}\right)dx$$

$$= 8h^2 \int_0^a \left[\frac{x^3}{a^2} - 2\frac{x^4}{a^3} + \frac{x^5}{a^4}\right]dx = 8h^2 \left|\frac{x^4}{4a^2} - \frac{2x^5}{5a^3} + \frac{x^6}{6a^4}\right|_0^a$$

$$= 8h^2a^2\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = 8h^2a^2\left(\frac{15-24+10}{60}\right)$$

$$I_{xy} = \frac{2}{15}a^2h^2 \blacktriangleleft$$

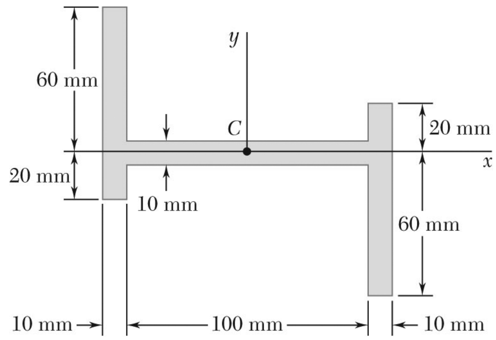
Check: Apply parallel axis theorem to area see Figure 5.8:



$$\text{Area} = \frac{2}{3}a\,h$$

$$\text{by symmetry } I_{x'y'} = 0$$

$$I_{xy} = I_{x'y'} + x_c y_c A = 0 + \left(\frac{a}{2}\right)\left(\frac{2}{5}h\right)\left(\frac{2}{3}ah\right) = \frac{2}{15}a^2h^2$$

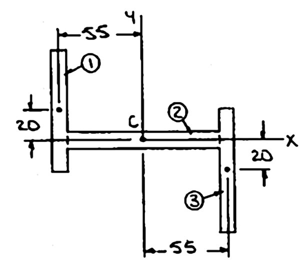


PROBLEM 9.71

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

Dimensions in mm



We have $\quad \bar{I}_{xy} = (I_{xy})_1 + (\bar{I}_{xy})_2 + (I_{xy})_3$

Now symmetry implies

$$(\bar{I}_{xy})_2 = 0$$

and for the other rectangles

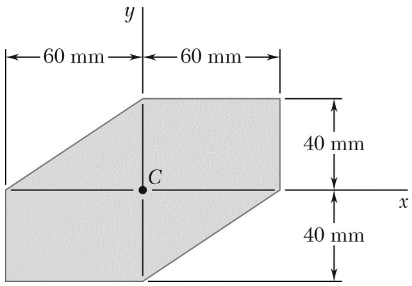
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

where $\quad I_{x'y'} = 0 \quad$ (symmetry)

Thus $\quad \bar{I}_{xy} = (\bar{x}\,\bar{y}A)_1 + (\bar{x}\,\bar{y}A)_3$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$10 \times 80 = 800$	-55	20	$-880,000$
3	$10 \times 80 = 800$	55	-20	$-880,000$
Σ				$-1,760,000$

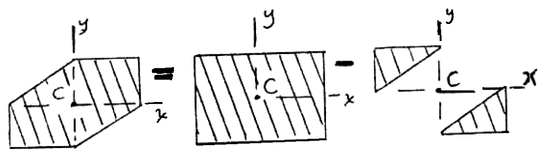
$$\bar{I}_{xy} = -1.760 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.72

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

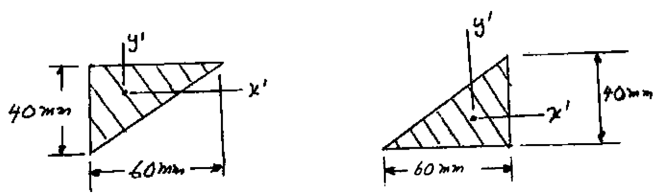
SOLUTION



Given area = Rectangle − (Two triangles)

For rectangle $I_{xy} = 0$, by symmetry

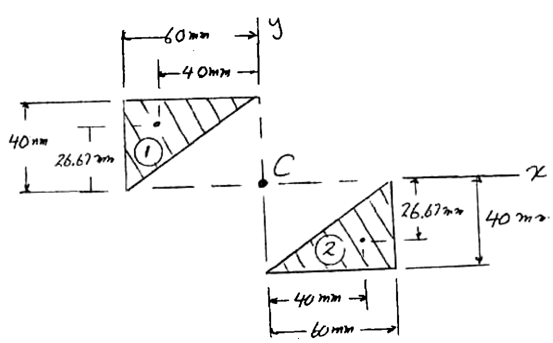
For each triangle:



Compare these triangles with the triangle of sample Problem 9.6, where $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$.

For orientation of axes of this problem,

$$\bar{I}_{x'y'} = +\frac{1}{72}b^2h^2 = \frac{1}{72}(60\text{ mm})^2(40\text{ mm})^2 = +80\times10^3\text{ mm}^4$$

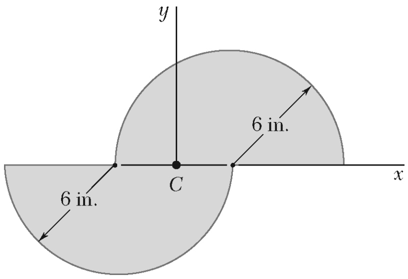


	Area, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$\frac{1}{2}(60)(40) = 1200$	-40	$+26.67$	-1.280×10^6
2	$\frac{1}{2}(60)(40) = 1200$	$+40$	-26.67	-1.280×10^6

$$\Sigma\,\bar{x}\,\bar{y}\,A = -2.56\times 10^6\,\text{mm}^4$$

For two triangles:
$$I_{xy} = \Sigma(\bar{I}_{x'y'} + \bar{x}\,\bar{y}\,A) = +2(80\times 10^3) - 2.56\times 10^6$$
$$= -2.40\times 10^6\,\text{mm}^4$$

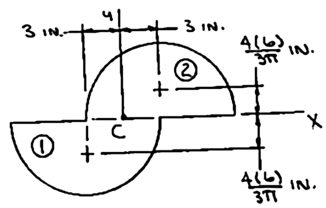
Since we must subtract triangles,
$$I_{xy} = +2.40\times 10^6\,\text{mm}^4 \blacktriangleleft$$



PROBLEM 9.73

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

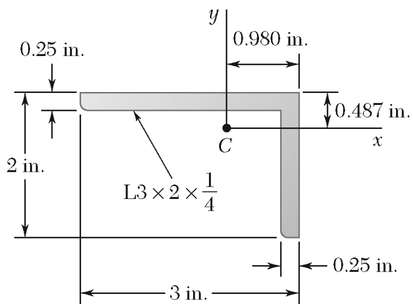
For each semicircle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $I_{x'y'} = 0$ (symmetry)

Thus $\bar{I}_{xy} = \Sigma \bar{x}\,\bar{y}A$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\overline{xy}A$
1	$\frac{\pi}{2}(6)^2 = 18\pi$	-3	$-\frac{8}{\pi}$	432
2	$\frac{\pi}{2}(6)^2 = 18\pi$	3	$\frac{8}{\pi}$	432
Σ				864

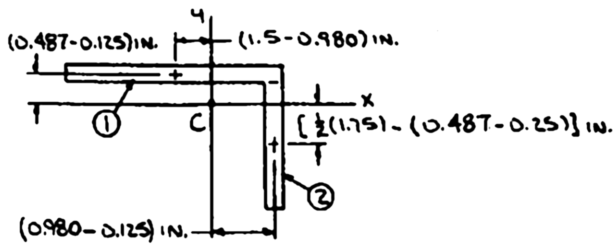
$\bar{I}_{xy} = 864 \text{ in}^4$ ◀



PROBLEM 9.74

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\overline{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each rectangle

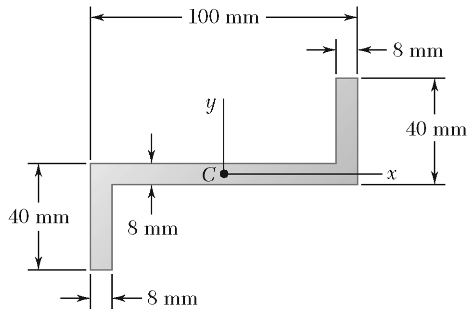
$$I_{xy} = \overline{I}_{x'y'} + \overline{x}\,\overline{y}A$$

and $\overline{I}_{x'y'} = 0 \quad \text{(symmetry)}$

Thus $\overline{I}_{xy} = \Sigma\overline{x}\,\overline{y}A$

	A, in^2	$\overline{x}$, in.	$\overline{y}$, in.	$\overline{x}\,\overline{y}A$, in^4
1	$3\times0.25=0.75$	–0.520	0.362	–0.141180
2	$0.25\times1.75=0.4375$	0.855	–0.638	–0.238652
Σ				–0.379832

$\overline{I}_{xy} = -0.380 \text{ in}^4$ ◀

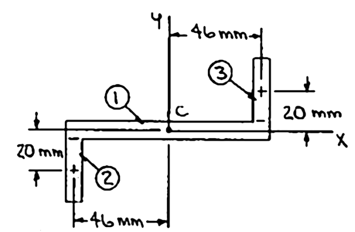


PROBLEM 9.75

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have $$I_{xy} = (\overline{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$



Now symmetry implies $$(\overline{I}_{xy})_1 = 0$$

and for the other rectangles

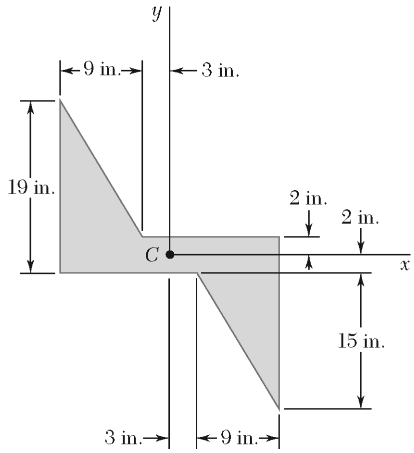
$$I_{xy} = \overline{I}_{x'y'} + \overline{x}\,\overline{y}A$$

where $$\overline{I}_{x'y'} = 0 \text{ (symmetry)}$$

Thus $$\overline{I}_{xy} = (\overline{x}\,\overline{y}A)_2 + (\overline{x}\,\overline{y}A)_3$$

	A, mm^2	$\overline{X}$, mm	$\overline{y}$, mm	$\overline{x}\,\overline{y}A$, mm^4
2	$8 \times 32 = 256$	–46	–20	235,520
3	$8 \times 32 = 256$	46	20	235,520
Σ				471,040

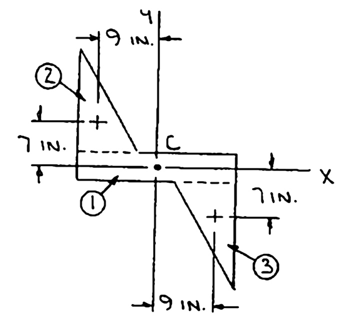
$$\overline{I}_{xy} = 471 \times 10^3 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.76

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\quad \bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

Now, symmetry implies

$$(\bar{I}_{xy})_1 = 0$$

and for each triangle $\quad I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

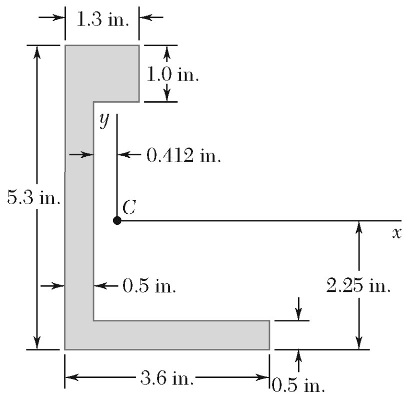
where, using the results of Sample Problem 9.6, $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$ for both triangles. Note that the sign of $\bar{I}_{x'y'}$ is unchanged because the angles of rotation are $0°$ and $180°$ for triangles 2 and 3, respectively.

Now

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
2	$\frac{1}{2}(9)(15) = 67.5$	–9	7	–4252.5
3	$\frac{1}{2}(9)(15) = 67.5$	9	–7	–4252.5
Σ				–8505

Then
$$\bar{I}_{xy} = 2\left[-\frac{1}{72}(9\text{ in.})^2\,(15\text{ in.})^2\right] - 8505\text{ in}^4$$
$$= -9011.25\text{ in}^4$$

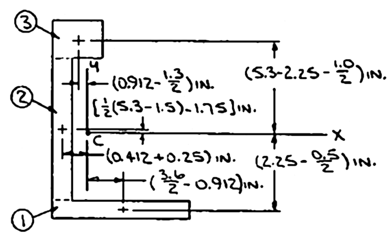
or $\quad \bar{I}_{xy} = -9010\text{ in}^4$ ◀



PROBLEM 9.77

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\qquad \bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

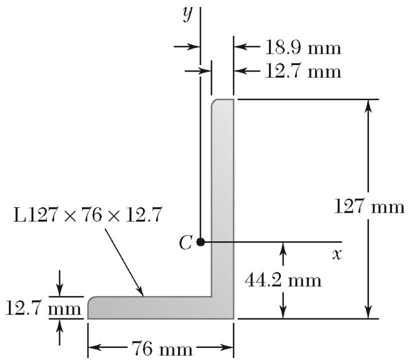
For each rectangle $\qquad I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $\qquad \bar{I}_{x'y'} = 0 \;$ (symmetry)

Thus $\qquad I_{xy} = \Sigma \bar{x}\,\bar{y}A$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
1	$3.6\times 0.5 = 1.8$	0.888	–2.00	–3.196
2	$0.5\times 3.8 = 1.9$	–0.662	0.15	–0.18867
3	$1.3\times 1.0 = 1.3$	–0.262	2.55	–0.86853
Σ				–4.25320

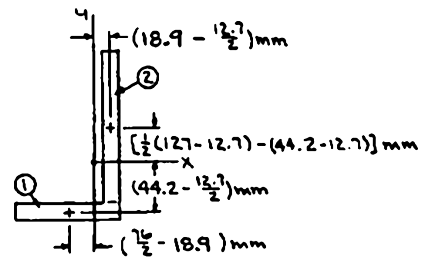
$\bar{I}_{xy} = -4.25\ \text{in}^4$ ◀



PROBLEM 9.78

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

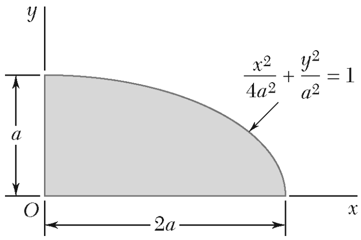
For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $\bar{I}_{x'y'} = 0$ (symmetry)

Thus $I_{xy} = \Sigma \bar{x}\,\bar{y}A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$76 \times 12.7 = 965.2$	–19.1	–37.85	697,777
2	$12.7 \times (127 - 12.7) = 1451.61$	12.55	25.65	467,284
Σ				1,165,061

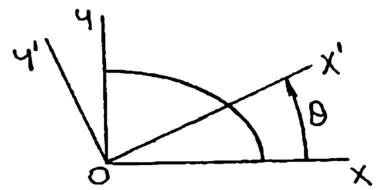
$\bar{I}_{xy} = 1.165 \times 10^6\ \text{mm}^4$ ◀



PROBLEM 9.79

Determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (a) through 45° counterclockwise, (b) through 30° clockwise.

SOLUTION



From Figure 9.12:
$$I_x = \frac{\pi}{16}(2a)(a)^3$$
$$= \frac{\pi}{8}a^4$$
$$I_y = \frac{\pi}{16}(2a)^3(a)$$
$$= \frac{\pi}{2}a^4$$

From Problem 9.67:
$$I_{xy} = \frac{1}{2}a^4$$

First note
$$\frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4$$
$$\frac{1}{2}(I_x - I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right) = -\frac{3}{16}\pi a^4$$

Now use Equations (9.18), (9.19), and (9.20).

Equation (9.18):
$$I_{x'} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy}\sin 2\theta$$
$$= \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 2\theta - \frac{1}{2}a^4 \sin 2\theta$$

Equation (9.19):
$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta + I_{xy}\sin 2\theta$$
$$= \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos 2\theta + \frac{1}{2}a^4 \sin 2\theta$$

Equation (9.20):
$$I_{x'y'} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta$$
$$= -\frac{3}{16}\pi a^4 \sin 2\theta + \frac{1}{2}a^4 \cos 2\theta$$

PROBLEM 9.79 (Continued)

(*a*) $\theta = +45°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 90° - \frac{1}{2}a^4 \sin 90°$$

or $I_{x'} = 0.482a^4$ ◀

$$I_{y'} = \frac{5}{16}\pi + \frac{3}{16}\pi a^4 \cos 90° + \frac{1}{2}a^4$$

or $I_{y'} = 1.482a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin 90° + \frac{1}{2}a^4 \cos 90°$$

or $I_{x'y'} = -0.589a^4$ ◀

(*b*) $\theta = -30°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos(-60°) - \frac{1}{2}a^4 \sin(-60°)$$

or $I_{x'} = 1.120a^4$ ◀

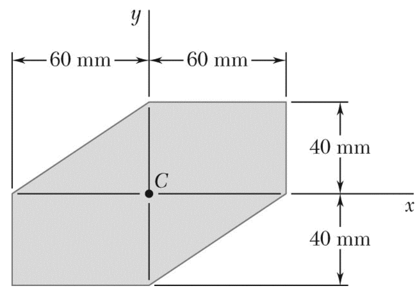
$$I_{y'} = \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos(-60°) + \frac{1}{2}a^4 \sin(-60°)$$

or $I_{y'} = 0.843a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin(-60°) + \frac{1}{2}a^4 \cos(-60°)$$

or $I_{x'y'} = 0.760a^4$ ◀

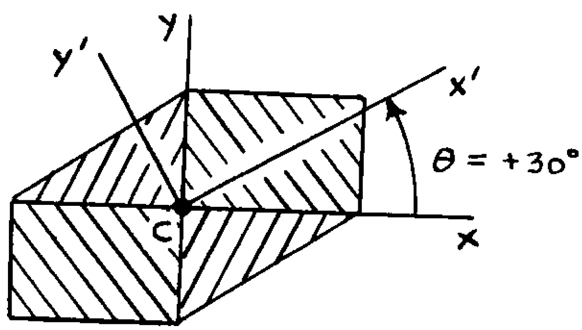
PROBLEM 9.80



Determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 30° counterclockwise.

SOLUTION

From Problem 9.72: $\bar{I}_{xy} = 2.40 \times 10^6 \text{ mm}^4$



Now, with two rectangles and two triangles:

$$\bar{I}_x = 2\left[\frac{1}{3}(60 \text{ mm})(40 \text{ mm})^3\right] + 2\left[\frac{1}{12}(60 \text{ mm})(40 \text{ mm})^3\right]$$

$$\bar{I}_x = 3.20 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 2\left[\frac{1}{3}(40 \text{ mm})(60 \text{ mm})^3\right] + 2\left[\frac{1}{12}(40 \text{ mm})(60 \text{ mm})^3\right] = 7.20 \times 10^6 \text{ mm}^4$$

Now $$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.20 + 7.20) \times 10^6 = 5.20 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(3.20 - 7.20) \times 10^6 = -2.00 \times 10^6 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), (9.20):

Eq. (9.18): $$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= [5.20 + (-2.00)\cos 60° - (2.40)\sin 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'} = 2.12 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.80 (Continued)

Eq. (9.19):

$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [5.20 - (-2.00)\cos 60° + (2.40)\sin 60°] \times 10^6 \text{ mm}^4$$

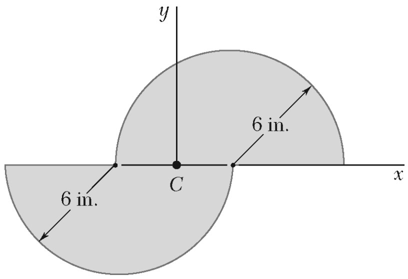
or $\bar{I}_{y'} = 8.28 \times 10^6 \text{ mm}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [(-2.00)\sin 60° + (2.40)\cos 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'y'} = -0.532 \times 10^6 \text{ mm}^4$ ◀



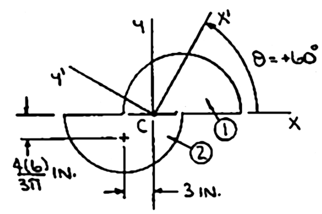
PROBLEM 9.81

Determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

SOLUTION

From Problem 9.73:

$$\bar{I}_{xy} = 864 \text{ in}^4$$



Now $$\bar{I}_x = (I_x)_1 + (I_x)_2$$

where $$(I_x)_1 = (I_x)_2 = \frac{\pi}{8}(6 \text{ in.})^4$$
$$= 162\pi \text{ in}^4$$

Then $$\bar{I}_x = 2(162\pi \text{ in}^4) = 324\pi \text{ in}^4$$

Also $$\bar{I}_y = (I_y)_1 + (I_y)_2$$

where $$(I_y)_1 = (I_y)_2 = \frac{\pi}{8}(6 \text{ in.})^4 + \left[\frac{\pi}{2}(6 \text{ in.})^2\right](3 \text{ in.})^2 = 324\pi \text{ in}^4$$

Then $$\bar{I}_y = 2(324\pi \text{ in}^4) = 648\pi \text{ in}^4$$

Now $$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 486\pi \text{ in}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(324\pi - 648\pi) = -162\pi \text{ in}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18): $$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [486\pi + (-162\pi)\cos 120° - 864 \sin 120°] \text{ in}^4$$

or $\bar{I}_{x'} = 1033 \text{ in}^4$ ◀

PROBLEM 9.81 (Continued)

Eq. (9.19):

$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [486\pi - (-162\pi)\cos 120° + 864\sin 120°]\ \text{in}^4$$

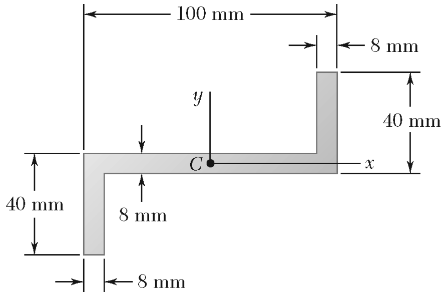
or $\bar{I}_{y'} = 2020\ \text{in}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= [(-162\pi)\sin 120° + 864\cos 120°]\ \text{in}^4$$

or $\bar{I}_{x'y'} = -873\ \text{in}^4$ ◀

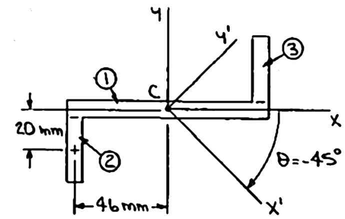


PROBLEM 9.82

Determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

SOLUTION

From Problem 9.75:



$$\bar{I}_{xy} = 471{,}040 \text{ mm}^4$$

Now $$\bar{I}_x = (\bar{I}_x)_1 + (I_x)_2 + (I_x)_3$$

where $$(\bar{I}_x)_1 = \frac{1}{12}(100 \text{ mm})(8 \text{ mm})^3$$
$$= 4266.67 \text{ mm}^4$$

$$(I_x)_2 = (I_x)_3 = \frac{1}{12}(8 \text{ mm})(32 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](20 \text{ mm})^2$$
$$= 124{,}245.33$$

Then $$\bar{I}_x = [4266.67 + 2(124{,}245.33)] \text{ mm}^4 = 252{,}757 \text{ mm}^4$$

Also $$\bar{I}_y = (\bar{I}_y)_1 + (I_y)_2 + (I_y)_3$$

where $$(\bar{I}_y)_1 = \frac{1}{12}(8 \text{ mm})(100 \text{ mm})^3 = 666{,}666.7 \text{ mm}^4$$

$$(\bar{I}_y)_2 = (I_y)_3 = \frac{1}{12}(32 \text{ mm})(8 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](46 \text{ mm})^2$$
$$= 543{,}061.3 \text{ mm}^4$$

Then $$\bar{I}_y = [666{,}666.7 + 2(543{,}061.3)] \text{ mm}^4 = 1{,}752{,}789 \text{ mm}^4$$

Now $$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252{,}757 + 1{,}752{,}789) \text{ mm}^4 = 1{,}002{,}773 \text{ mm}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(252{,}757 - 1{,}752{,}789) \text{ mm}^4 = -750{,}016 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18): $$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [1{,}002{,}773 + (-750{,}016)\cos(-90°) - 471{,}040\sin(-90°)]$$

or $\bar{I}_{x'} = 1.474 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.82 (Continued)

Eq. (9.19):

$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [1,002,773 - (-750,016)\cos(-90°) + 471,040\sin(-90°)]$$

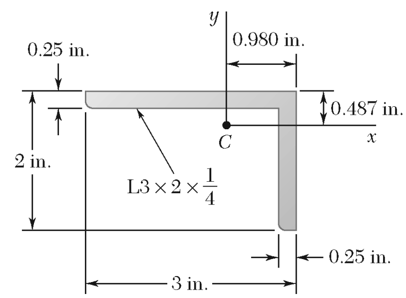
or $\bar{I}_{y'} = 0.532 \times 10^6 \text{ mm}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= [(-750,016)\sin(-90°) + 471,040\cos(-90°)]$$

or $\bar{I}_{x'y'} = 0.750 \times 10^6 \text{ mm}^4$ ◀



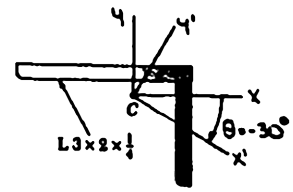
PROBLEM 9.83

Determine the moments of inertia and the product of inertia of the $L3\times 2\times \frac{1}{4}$-in. angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Figure 9.13:

$$\bar{I}_x = 0.390 \text{ in}^4$$
$$\bar{I}_y = 1.09 \text{ in}^4$$



From Problem 9.74:

$$\bar{I}_{xy} = -0.37983 \text{ in}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.390 + 1.09) \text{ in}^4 = 0.740 \text{ in}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(0.390 - 1.09) \text{ in}^4 = -0.350 \text{ in}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [0.740 + (-0.350)\cos(-60°) - (-0.37983)\sin(-60°)]$$

or $\bar{I}_{x'} = 0.236 \text{ in}^4$ ◀

Eq. (9.19):

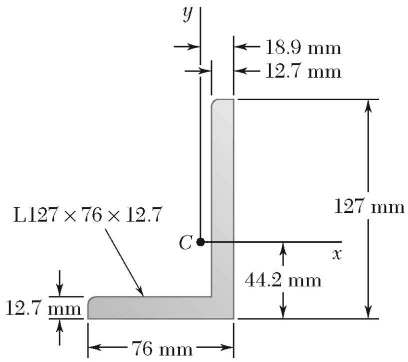
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [0.740 - (-0.350)\cos(-60°) + (-0.37983)\sin(-60°)]$$

or $\bar{I}_{y'} = 1.244 \text{ in}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$
$$= [(-0.350)\sin(-60°) + (-0.37983)\cos(-60°)]$$

or $I_{x'y'} = 0.1132 \text{ in}^4$ ◀



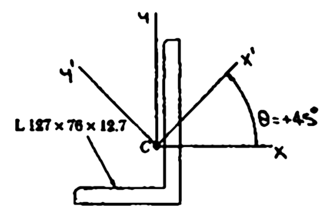
PROBLEM 9.84

Determine the moments of inertia and the product of inertia of the L127×76×12.7-mm angle cross section of Problem 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 45° counterclockwise.

SOLUTION

From Figure 9.13:

$$\bar{I}_x = 3.93\times 10^6\,\text{mm}^4$$
$$\bar{I}_y = 1.06\times 10^6\,\text{mm}^4$$



From Problem 9.78:

$$\bar{I}_{xy} = 1.165061\times 10^6\,\text{mm}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.93+1.06)\times 10^6\,\text{mm}^4$$
$$= 2.495\times 10^6\,\text{mm}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(3.93-1.06)\times 10^6\,\text{mm}^4 = 1.435\times 10^6\,\text{mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x+\bar{I}_y) + \frac{1}{2}(\bar{I}_x-\bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [2.495 + 1.435\cos 90° - 1.165061\sin 90°]\times 10^6\,\text{mm}^4$$

or $\bar{I}_{x'} = 1.330\times 10^6\,\text{mm}^4$ ◀

Eq. (9.19):

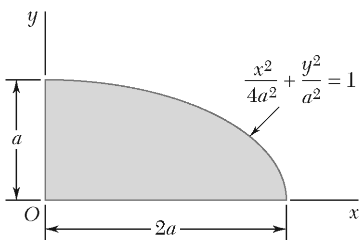
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x+\bar{I}_y) - \frac{1}{2}(\bar{I}_x-\bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [2.495 - 1.435\cos 90° + 1.165061\sin 90°]\times 10^6\,\text{mm}^4$$

or $\bar{I}_{y'} = 3.66\times 10^6\,\text{mm}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x-\bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$
$$= [(1.435\sin 90° + 1.165061\cos 90°]\times 10^6\,\text{mm}^4$$

or $\bar{I}_{x'y'} = 1.435\times 10^6\,\text{mm}^4$ ◀



PROBLEM 9.85

For the quarter ellipse of Problem 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79: $I_x = \frac{\pi}{8}a^4 \qquad I_y = \frac{\pi}{2}a^4$

Problem 9.67: $I_{xy} = \frac{1}{2}a^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= \frac{8}{3\pi} = 0.84883$$

Then $2\theta_m = 40.326° \quad \text{and} \quad 220.326°$

or $\theta_m = 20.2° \text{ and } 110.2°$ ◀

Also Eq. (9.27):

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right)$$

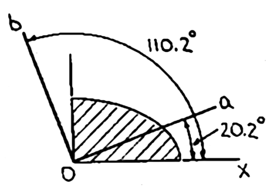
$$\pm \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

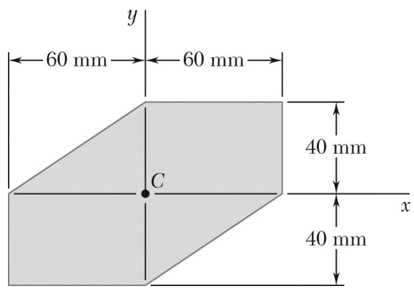
$$= (0.981{,}748 \pm 0.772{,}644)a^4$$

or $I_{\max} = 1.754a^4$ ◀

and $I_{\min} = 0.209a^4$ ◀

By inspection, the a axis corresponds to $I_{\min}$ and the b axis corresponds to $I_{\max}$.





PROBLEM 9.86

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.72.

SOLUTION

From Problem 9.80:
$$\bar{I}_x = 3.20\times10^6\,\text{mm}^4$$
$$\bar{I}_y = 7.20\times10^6\,\text{mm}^4$$

From Problem 9.72:
$$\bar{I}_{xy} = 2.40\times10^6\,\text{mm}^4$$

Now Eq. (9.25):
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x-\bar{I}_y} = -\frac{2(2.40\times10^6)}{(3.20-7.20)\times10^6} = 1.200$$

Then
$$2\theta_m = 50.194° \quad \text{and} \quad 230.194°$$

or $\theta_m = 25.1°$ and $115.1°$ ◀

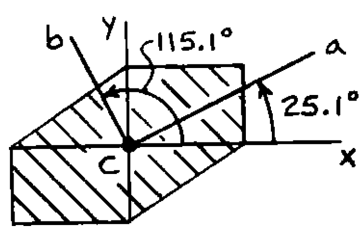
Also Eq. (9.27):
$$\bar{I}_{\max,\min} = \frac{\bar{I}_x+\bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x-\bar{I}_y}{2}\right)^2 + I_{xy}^2}$$

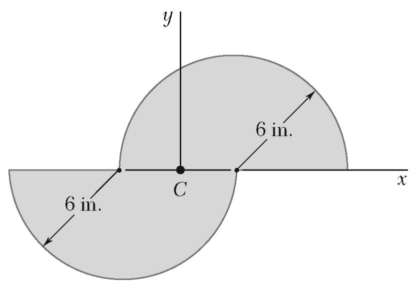
Then
$$\bar{I}_{\max,\min} = \left[\frac{3.20+7.20}{2} \pm \sqrt{\left(\frac{3.20-7.20}{2}\right)^2 + (2.40)^2}\right]\times10^6\,\text{mm}^4$$
$$= (5.20 \pm 3.1241)\times10^6\,\text{mm}^4$$

or $\bar{I}_{\max} = 8.32\times10^6\,\text{mm}^4$ ◀

and $\bar{I}_{\min} = 2.08\times10^6\,\text{mm}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.





PROBLEM 9.87

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.73.

SOLUTION

From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4 \qquad \bar{I}_y = 648\pi \text{ in}^4$

Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(864)}{324\pi - 648\pi}$$
$$= 1.69765$$

Then $2\theta_m = 59.500° \quad \text{and} \quad 239.500°$

or $\theta_m = 29.7° \quad \text{and} \quad 119.7°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

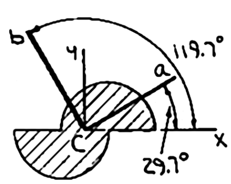
Then

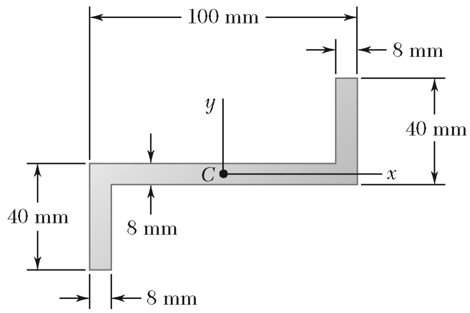
$$\bar{I}_{\text{max,min}} = \frac{324\pi + 648\pi}{2} \pm \sqrt{\left(\frac{324\pi - 648\pi}{2}\right)^2 + 864^2}$$
$$= (1526.81 \pm 1002.75) \text{ in}^4$$

or $\bar{I}_{\max} = 2530 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 524 \text{ in}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.





PROBLEM 9.88

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.75.

SOLUTION

From Problem 9.82: $\bar{I}_x = 252{,}757 \text{ mm}^4$

$\bar{I}_y = 1{,}752{,}789 \text{ mm}^4$

Problem 9.75: $\bar{I}_{xy} = 471{,}040 \text{ mm}^4$

Now Eq. (9.25):
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(471{,}040)}{252{,}757 - 1{,}752{,}789}$$
$$= 0.62804$$

Then $2\theta_m = 32.130°$ and $212.130°$

or $\theta_m = 16.07°$ and $106.1°$ ◀

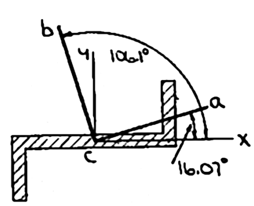
Also Eq. (9.27):
$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x - \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

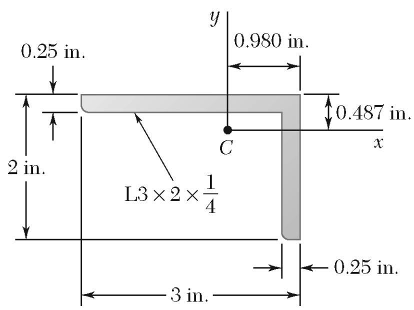
Then
$$\bar{I}_{\text{max, min}} = \frac{252{,}757 + 1{,}752{,}789}{2} \pm \sqrt{\left(\frac{252{,}757 - 1{,}752{,}789}{2}\right)^2 + 471040^2}$$
$$= (1{,}002{,}773 \pm 885{,}665) \text{ mm}^4$$

or $\bar{I}_{\text{max}} = 1.888 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\text{min}} = 0.1171 \times 10^6 \text{ mm}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\text{min}}$ and the b axis corresponds to $\bar{I}_{\text{max}}$.





PROBLEM 9.89

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The $L3\times 2\times\frac{1}{4}$-in. angle cross section of Problem 9.74.

SOLUTION

From Problem 9.83: $\bar{I}_x = 0.390 \text{ in}^4 \qquad \bar{I}_y = 1.09 \text{ in}^4$

Problem 9.74: $\bar{I}_{xy} = -0.37983 \text{ in}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-0.37983)}{0.390 - 1.09}$$
$$= -1.08523$$

Then $2\theta_m = -47.341° \text{ and } 132.659°$

or $\theta_m = -23.7° \text{ and } 66.3°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

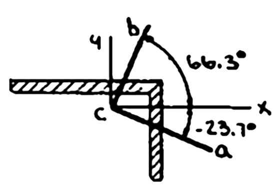
Then

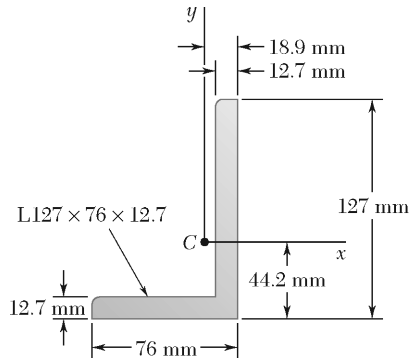
$$\bar{I}_{\text{max, min}} = \frac{0.390 + 1.09}{2} \pm \sqrt{\left(\frac{0.390 - 1.09}{2}\right)^2 + (-0.37983)^2}$$
$$= (0.740 \pm 0.51650)^2 \text{ in}^4$$

or $\bar{I}_{\text{max}} = 1.257 \text{ in}^4$ ◀

and $\bar{I}_{\text{min}} = 0.224 \text{ in}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\text{min}}$ and the b axis corresponds to $\bar{I}_{\text{max}}$.





PROBLEM 9.90

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The L127×76×12.7-mm angle cross section of Problem 9.78.

SOLUTION

From Problem 9.84: $\bar{I}_x = 3.93\times 10^6 \text{ mm}^4$

$\bar{I}_y = 1.06\times 10^6 \text{ mm}^4$

Problem 9.78: $\bar{I}_{xy} = 1.165061\times 10^6 \text{ mm}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(1.165061\times 10^6)}{(3.93-1.06)\times 10^6}$$
$$= -0.81189$$

Then $2\theta_m = -39.073°$ and $140.927°$

or $\theta_m = -19.54°$ and $70.5°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

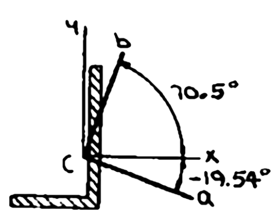
Then

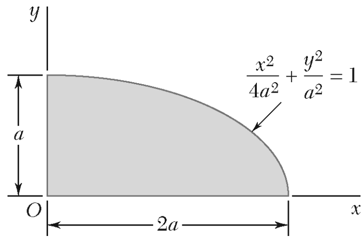
$$\bar{I}_{\text{max, min}} = \left[\frac{3.93+1.06}{2} \pm \sqrt{\left(\frac{3.93-1.06}{2}\right)^2 + 1.165061^2}\right]\times 10^6 \text{ mm}^4$$
$$= (2.495 \pm 1.84840)\times 10^6 \text{ mm}^4$$

or $\bar{I}_{\text{max}} = 4.34\times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\text{min}} = 0.647\times 10^6 \text{ mm}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\text{max}}$ and the b axis corresponds to $\bar{I}_{\text{min}}$.

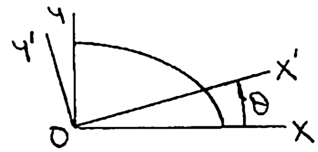




PROBLEM 9.91

Using Mohr's circle, determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (a) through 45° counterclockwise, (b) through 30° clockwise.

SOLUTION



From Problem 9.79: $$I_x = \frac{\pi}{8}a^4$$

$$I_y = \frac{\pi}{2}a^4$$

Problem 9.67: $$I_{xy} = \frac{1}{2}a^4$$

The Mohr's circle is defined by the diameter XY, where

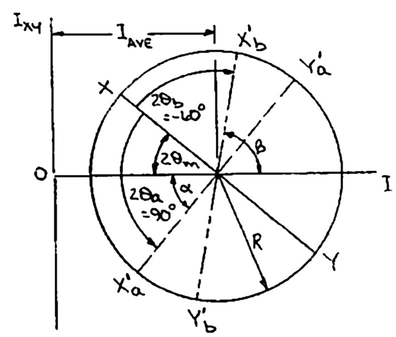
$$X\left(\frac{\pi}{8}a^4, \frac{1}{2}a^4\right) \quad \text{and} \quad Y\left(\frac{\pi}{2}a^4, -\frac{1}{2}a^4\right)$$

Now $$I_{\text{ave}} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4 = 0.98175a^4$$

and $$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} = \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

$$= 0.77264a^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

$$= -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= 0.84883$$

or $$2\theta_m = 40.326°$$

Then
$$\alpha = 90° - 40.326° = 49.674°$$
$$\beta = 180° - (40.326° + 60°) = 79.674°$$

(*a*) $\theta = +45°$:
$$I_{x'} = I_{\text{ave}} - R\cos\alpha = 0.98175a^4 - 0.77264a^4\cos 49.674°$$

or $I_{x'} = 0.482a^4$ ◀

$$I_{y'} = I_{\text{ave}} + R\cos\alpha = 0.98175a^4 + 0.77264a^4\cos 49.674°$$

or $I_{y'} = 1.482a^4$ ◀

$$I_{x'y'} = -R\sin\alpha = -0.77264a^4\sin 49.674°$$

or $I_{x'y'} = -0.589a^4$ ◀

(*b*) $\theta = -30°$:
$$I_{x'} = I_{\text{ave}} + R\cos\beta = 0.98175a^4 + 0.77264a^4\cos 79.674°$$

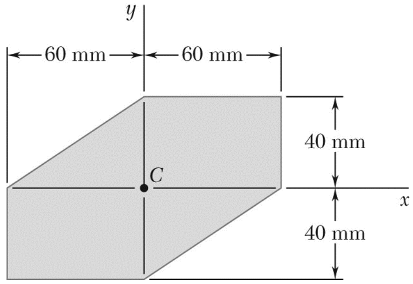
or $I_{x'} = 1.120a^4$ ◀

$$I_{y'} = I_{\text{ave}} - R\cos\beta = 0.98175a^4 - 0.77264a^4\cos 79.674°$$

or $I_{y'} = 0.843a^4$ ◀

$$I_{x'y'} = R\sin\beta = 0.77264a^4\sin 79.674°$$

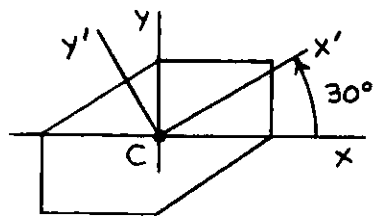
or $I_{x'y'} = 0.760a^4$ ◀



PROBLEM 9.92

Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 30° counterclockwise.

SOLUTION



From Problem 9.80: $\bar{I}_x = 3.20\times10^6\,\text{mm}^4$

$\bar{I}_y = 7.20\times10^6\,\text{mm}^4$

From Problem 9.72: $\bar{I}_{xy} = 2.40\times10^6\,\text{mm}^4$

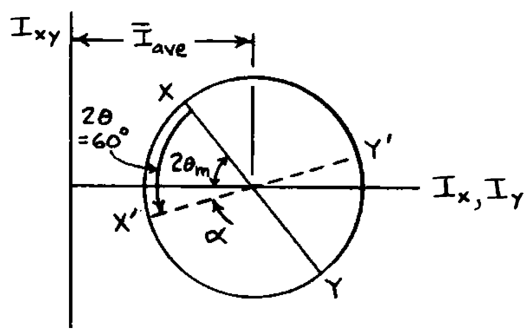
The Mohr's circle is defined by the diameter XY, where $X(3.20\times10^6,\ 2.40\times10^6)$ and $Y(7.20\times10^6,\ -2.40\times10^6)$.

Now $$I_{\text{ave}} = \frac{1}{2}(\bar{I}_x+\bar{I}_y) = \frac{1}{2}(3.20+7.20)\times10^6 = 5.20\times10^6\,\text{mm}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x-\bar{I}_y)\right]^2 + I_{xy}^2} = \left\{\sqrt{\left[\frac{1}{2}(3.20-7.20)\right]^2 + (2.40)^2}\right\}\times10^6\,\text{mm}^4$$

$$= 3.1241\times10^6\,\text{mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x-\bar{I}_y}$$

$$= -\frac{2(2.40\times10^6)}{(3.20-7.20)\times10^6}$$

$$= 1.200$$

or $$2\theta_m = 50.1944°$$

Then $$\alpha = 60° - 50.1944° = 9.8056°$$

PROBLEM 9.92 (Continued)

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = (5.20 - 3.1241\cos 9.8056°)\times 10^6$$

or $\bar{I}_{x'} = 2.12\times 10^6\ \text{mm}^4$ ◀

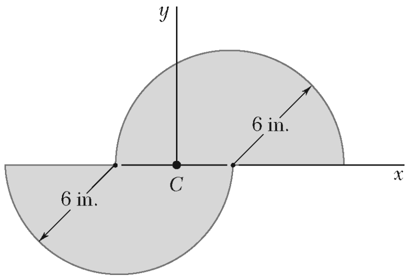
$$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = (5.20 + 3.1241\cos 9.8056°)\times 10^6$$

or $\bar{I}_{y'} = 8.28\times 10^6\ \text{mm}^4$ ◀

$$\bar{I}_{x'y'} = -R\sin\alpha = -(3.1241\times 10^6)\sin 9.8056°$$

or $\bar{I}_{x'y'} = -0.532\times 10^6\ \text{mm}^4$ ◀

PROBLEM 9.93

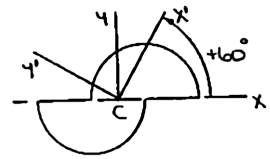


Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

SOLUTION

From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4$

$\bar{I}_y = 648\pi \text{ in}^4$



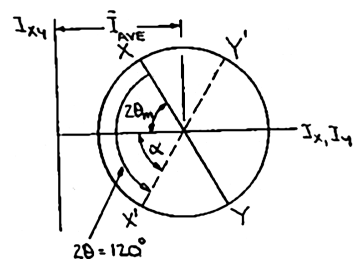
Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

The Mohr's circle is defined by the diameter XY, where $X(324\pi,\ 864)$ and $Y(648\pi,\ -864)$.

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 1526.81 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)^2 + \bar{I}_{xy}^2\right.} = \sqrt{\left[\frac{1}{2}(324\pi + 648\pi)\right]^2 + 864^2}$$
$$= 1002.75 \text{ in}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(864)}{324\pi - 648\pi}$$
$$= 1.69765$$

or $$2\theta_m = 59.500°$$

Then $$\alpha = 120° - 59.500°$$
$$= 60.500°$$

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = 1526.81 - 1002.75\cos 60.500°$$

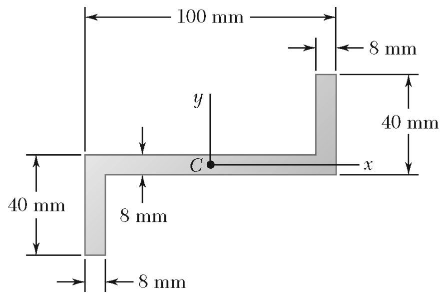
or $\bar{I}_{x'} = 1033 \text{ in}^4$ ◀

$$\bar{I}_{y} = \bar{I}_{\text{ave}} + R\cos\alpha = 1526.81 + 1002.75\cos 60.500°$$

or $\bar{I}_{y'} = 2020 \text{ in}^4$ ◀

$$\bar{I}_{x'y'} = -R\sin\alpha = -1002.75\sin 60.500°$$

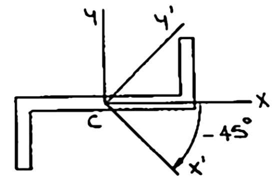
or $\bar{I}_{x'y'} = -873 \text{ in}^4$ ◀



PROBLEM 9.94

Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

SOLUTION



From Problem 9.82: $\bar{I}_x = 252,757 \text{ mm}^4$

$\bar{I}_y = 1,752,789 \text{ mm}^4$

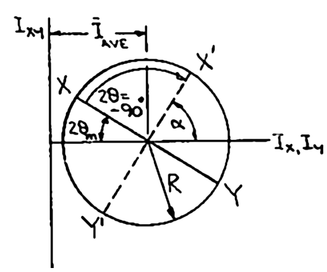
Problem 9.75: $\bar{I}_{xy} = 471,040 \text{ mm}^4$

The Mohr's circle is defined by the diameter XY, where X (252,757; 471,040) and Y (1,752,789; −471,040).

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252,757 + 1,752,789)$$
$$= 1,002,773 \text{ mm}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \sqrt{\left[\frac{1}{2}(252,757 - 1,752,789)^2\right] + 471,040^2}$$
$$= 885,665 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(471,040)}{252,757 - 1,752,789}$$
$$= 0.62804$$

or
$$2\theta_m = 32.130°$$

Then
$$\alpha = 180° - (32.130 + 90°)$$
$$= 57.870°$$

PROBLEM 9.94 (Continued)

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} + R\cos\alpha = 1{,}002{,}773 + 885{,}665\cos 57.870°$$

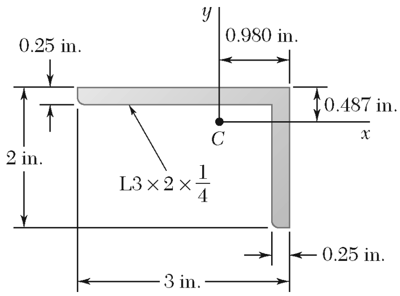
or $\bar{I}_{x'} = 1.474 \times 10^6 \text{ mm}^4$ ◀

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} - R\cos\alpha = 1{,}002{,}773 - 885{,}665\cos 57.870°$$

or $\bar{I}_{y'} = 0.532 \times 10^6 \text{ mm}^4$ ◀

$$\bar{I}_{x'y'} = R\sin\alpha = 885{,}665\sin 57.870°$$

or $\bar{I}_{x'y'} = 0.750 \times 10^6 \text{ mm}^4$ ◀



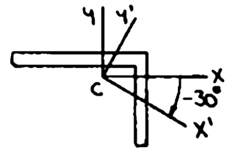
PROBLEM 9.95

Using Mohr's circle, determine the moments of inertia and the product of inertia of the $L3\times2\times\frac{1}{4}$-in. angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Problem 9.83: $\bar{I}_x = 0.390 \text{ in}^4$

$\bar{I}_y = 1.09 \text{ in}^4$



Problem 9.74:

$$\bar{I}_{xy} = -0.37983 \text{ in}^4$$

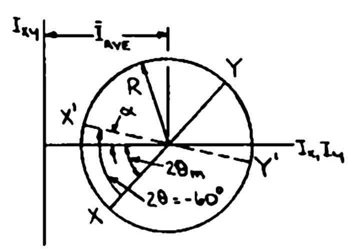
The Mohr's circle is defined by the diameter XY, where $X(0.390, -0.37983)$ and $Y(1.09, 0.37983)$.

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y)$$
$$= \frac{1}{2}(0.390 + 1.09)$$
$$= 0.740 \text{ in}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \sqrt{\left[\frac{1}{2}(0.390 - 1.09)\right]^2 + (-0.37983)^2}$$
$$= 0.51650 \text{ in}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$
$$= -\frac{2(-0.37983)}{0.390 - 1.09}$$
$$= -1.08523$$



or
$$2\theta_m = -47.341°$$

Then
$$\alpha = 60° - 47.341° = 12.659°$$

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = 0.740 - 0.51650\cos 12.659°$$

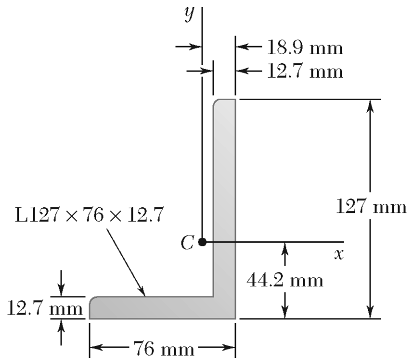
or $\bar{I}_{x'} = 0.236 \text{ in}^4$ ◀

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = 0.740 + 0.51650\cos 12.659°$$

or $\bar{I}_{y} = 1.244 \text{ in}^4$ ◀

$$\bar{I}_{x'y'} = R\sin\alpha = 0.51650\sin 12.659°$$

or $\bar{I}_{x'y'} = 0.1132 \text{ in}^4$ ◀



PROBLEM 9.96

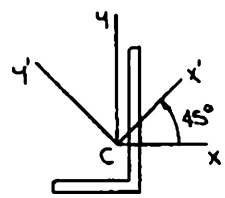
Using Mohr's circle, determine the moments of inertia and the product of inertia of the $L127 \times 76 \times 12.7$-mm angle cross section of Problem 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 45° counterclockwise.

SOLUTION

From Problem 9.84: $\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$

Problem 9.78: $\bar{I}_{xy} = 1.165061 \times 10^6 \text{ mm}^4$



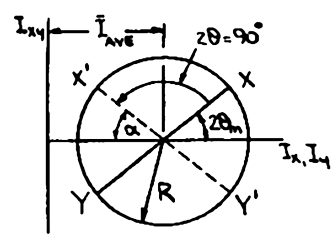
The Mohr's circle is defined by the diameter XY, where $X(3.93 \times 10^6,\ 1.165061 \times 10^6)$, $Y(1.06 \times 10^6,\ -1.165061 \times 10^6)$.

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.93 + 1.06) \times 10^6 = 2.495 \times 10^6 \text{ mm}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \left\{\sqrt{\left[\frac{1}{2}(3.93 - 1.06)\right]^2 + 1.165061^2}\right\} \times 10^6 \text{ mm}^4$$
$$= 1.84840 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(1.165061 \times 10^6)}{(3.93 - 1.06) \times 10^6}$$
$$= -0.81189$$



or
$$2\theta_m = -39.073°$$

PROBLEM 9.96 (Continued)

Then $$\alpha = 180° - (39.073° + 90°)$$
$$= 50.927°$$

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = (2.495 - 1.84840\cos 50.927°)\times 10^6$$

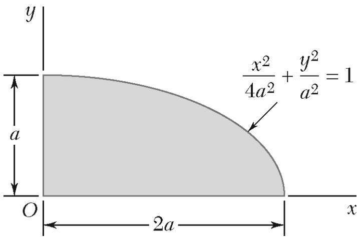
or $\bar{I}_{x'} = 1.330\times 10^6\ \text{mm}^4$ ◀

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = (2.495 + 1.84840\cos 50.927°)\times 10^6$$

or $\bar{I}_{y'} = 3.66\times 10^6\ \text{mm}^4$ ◀

$$\bar{I}_{x'y'} = R\sin\alpha = (1.84840\times 10^6)\sin 50.927°$$

or $\bar{I}_{x'y'} = 1.435\times 10^6\ \text{mm}^4$ ◀



PROBLEM 9.97

For the quarter ellipse of Problem 9.67, use Mohr's circle to determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79: $I_x = \dfrac{\pi}{8}a^4 \qquad I_y = \dfrac{\pi}{2}a^4$

Problem 9.67: $I_{xy} = \dfrac{1}{2}a^4$

The Mohr's circle is defined by the diameter XY, where

$$X\left(\frac{\pi}{8}a^4,\ \frac{1}{2}a^4\right) \quad \text{and} \quad Y\left(\frac{\pi}{2}a^4,\ -\frac{1}{2}a^4\right)$$

Now

$$I_{\text{ave}} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = 0.98175a^4$$

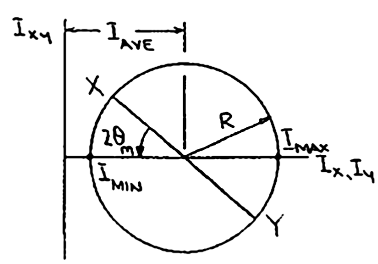
and

$$R = \sqrt{\left[\frac{1}{2}(I_x - I_y)\right]^2 + I_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

$$= 0.77264a^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

$$= -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= 0.84883$$

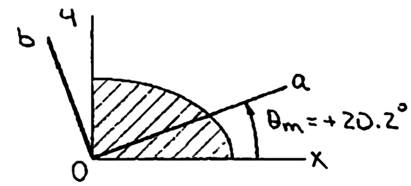
or $2\theta_m = 40.326°$

and $\theta_m = 20.2°$

The principal axes are obtained by rotating the xy axes through

20.2° counterclockwise ◀

about O.

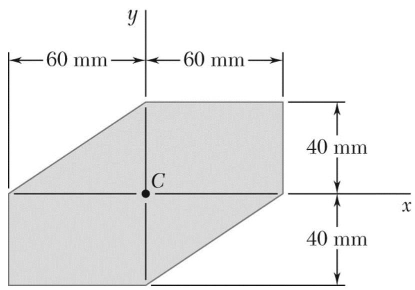


Now $$I_{\max,\min} = I_{\text{ave}} \pm R = 0.98175a^4 \pm 0.77264a^4$$

or $I_{\max} = 1.754\,a^4$ ◀

and $I_{\min} = 0.209\,a^4$ ◀

From the Mohr's circle it is seen that the a axis corresponds to $I_{\min}$ and the b axis corresponds to $I_{\max}$.



PROBLEM 9.98

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.72.

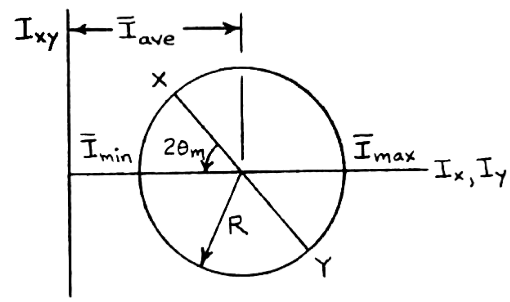
SOLUTION

From Problem 9.80: $\bar{I}_x = 3.20 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$

From Problem 9.72: $\bar{I}_{xy} = 2.40 \times 10^6 \text{ mm}^4$

The Mohr's circle is defined by the diameter XY, where $X(3.20 \times 10^6, 2.40 \times 10^6)$ and $Y(7.20 \times 10^6, -2.40 \times 10^6)$.



Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y)$$
$$= \frac{1}{2}(3.20 + 7.20) \times 10^6 \text{ mm}^4$$
$$= 5.20 \times 10^6 \text{ mm}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \left\{\sqrt{\left[\frac{1}{2}(3.20 - 7.20)\right]^2 + (2.40)^2}\right\} \times 10^6 \text{ mm}^4$$
$$= 3.1241 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(2.40 \times 10^6)}{(3.20 - 7.20) \times 10^6} = 1.200$$

or
$$2\theta_m = 50.194°$$

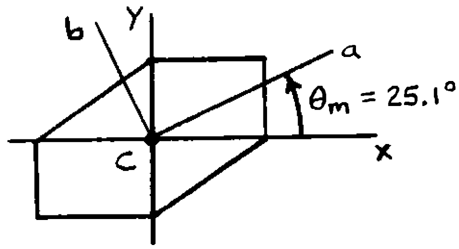
and
$$\theta_m = 25.097°$$

PROBLEM 9.98 (Continued)

The principal axes are obtained by rotating the xy axes through

25.1° counterclockwise ◀

about C.

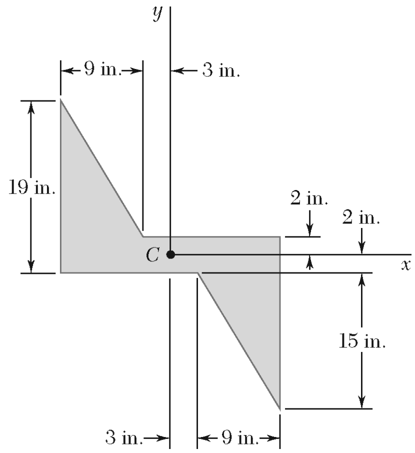


Now $$\bar{I}_{\max,\,\min} = \bar{I}_{\text{ave}} \pm R = (5.20 + 3.1241) \times 10^6$$

or $\bar{I}_{\max} = 8.32 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 2.08 \times 10^6 \text{ mm}^4$ ◀

From the Mohr's circle it is seen that the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.

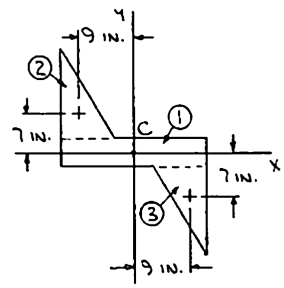


PROBLEM 9.99

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.76.

SOLUTION



From Problem 9.76:

$$\bar{I}_{xy} = -9011.25 \text{ in}^4$$

Now $\quad \bar{I}_x = (\bar{I}_x)_1 + (I_x)_2 + (I_x)_3$

where $\quad (\bar{I}_x)_1 = \frac{1}{12}(24 \text{ in.})(4 \text{ in.})^3 = 128 \text{ in}^4$

$$(I_x)_2 = (I_x)_3 = \frac{1}{36}(9 \text{ in.})(15 \text{ in.})^3 + \left[\frac{1}{2}(9 \text{ in.})(15 \text{ in.})\right](7 \text{ in.})^2$$

$$= 4151.25 \text{ in}^4$$

Then $\quad \bar{I}_x = [128 + 2(4151.25)] \text{ in}^4$

$$= 8430.5 \text{ in}^4$$

Also $\quad \bar{I}_y = (\bar{I}_y)_1 + (I_y)_2 + (I_y)_3$

where $\quad (\bar{I}_y)_1 = \frac{1}{12}(4 \text{ in.})(24 \text{ in.})^3 = 4608 \text{ in}^4$

$$(I_y)_2 = (I_y)_3 = \frac{1}{36}(15 \text{ in.})(9 \text{ in.})^3 + \left[\frac{1}{2}(9 \text{ in.})(15 \text{ in.})\right](9 \text{ in.})^2$$

$$= 5771.25 \text{ in}^4$$

Then $\quad \bar{I}_y = [4608 + 2(5771.25)] \text{ in}^4 = 16150.5 \text{ in}^4$

PROBLEM 9.99 (Continued)

The Mohr's circle is defined by the diameter XY, where $X(8430.5, \ -9011.25)$ and $Y(16150.5, \ 9011.25)$.

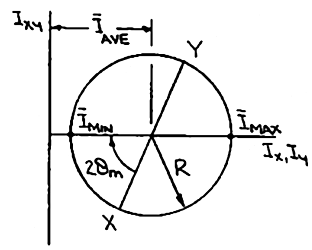
Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(8430.5 + 16150.5) = 12290.5 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(8430.5 - 16150.5)\right]^2 + (-9011.25)^2}$$

$$= 9803.17 \text{ in}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-9011.25)}{8430.5 - 16150.5}$$

$$= -2.33452$$

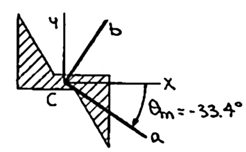
or $$2\theta_m = -66.812°$$

and $$\theta_m = -33.4°$$

The principal axes are obtained by rotating the xy axes through

33.4° clockwise ◀

about C.

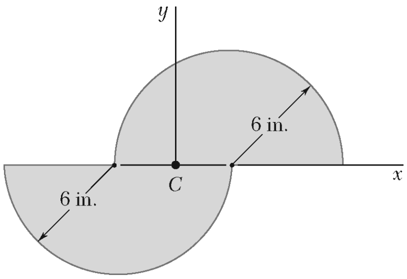


Now $$\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = 12290.5 \pm 9803.17$$

or $\bar{I}_{\text{max}} = 22.1 \times 10^3 \text{ in}^4$ ◀

and $\bar{I}_{\text{min}} = 2490 \text{ in}^4$ ◀

From the Mohr's circle it is seen that the a axis corresponds to $\bar{I}_{\text{min}}$ and the b axis corresponds to $\bar{I}_{\text{max}}$.



PROBLEM 9.100

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.73

SOLUTION

From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4 \qquad \bar{I}_y = 648\pi \text{ in}^4$

Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

The Mohr's circle is defined by the diameter XY, where $X(324\pi,\ 864)$ and $Y(648\pi,\ -864)$.

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 1526.81 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)^2\right] + I_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(324\pi - 648\pi)\right]^2 + 864^2}$$

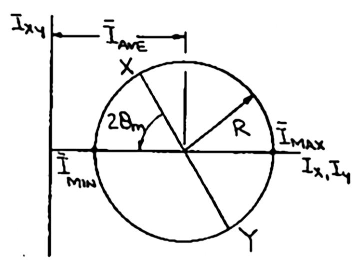
$$= 1002.75 \text{ in}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(864)}{324\pi - 648\pi}$$

$$= 1.69765$$



or $$2\theta_m = 59.4998°$$

and $$\theta_m = 29.7°$$

The principal axes are obtained by rotating the xy axes through

29.7° counterclockwise ◀

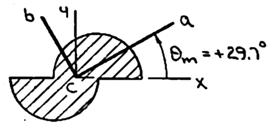
about C.

PROBLEM 9.100 (Continued)

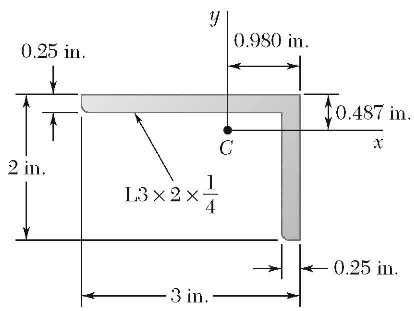
Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = 1526.81 \pm 1002.75$$

or $I_{\max} = 2530 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 524 \text{ in}^4$ ◀



From the Mohr's circle it is seen that the a axis corresponds to $I_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.



PROBLEM 9.101

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.74.

SOLUTION

From Problem 9.83: $\bar{I}_x = 0.390 \text{ in}^4$

$\bar{I}_y = 1.09 \text{ in}^4$

Problem 9.74: $\bar{I}_{xy} = -0.37983 \text{ in}^4$

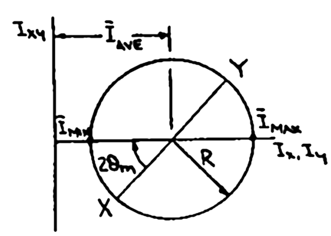
The Mohr's circle is defined by the diameter XY, where $X(0.390, -0.37983)$ and $Y(1.09, 0.37983)$.

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.390 + 1.09) = 0.740 \text{ in}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \sqrt{\left[\frac{1}{2}(0.390 - 1.09)\right]^2 + (-0.37983)^2}$$
$$= 0.51650 \text{ in}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(-0.37983)}{0.390 - 1.09}$$
$$= -1.08523$$



Then $2\theta_m = -47.341°$

and $\theta_m = -23.7°$

The principal axes are obtained by rotating the xy axes through

23.7° clockwise ◀

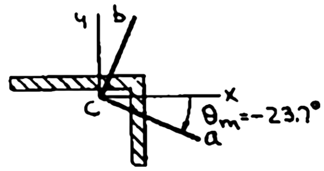
about C.

PROBLEM 9.101 (Continued)

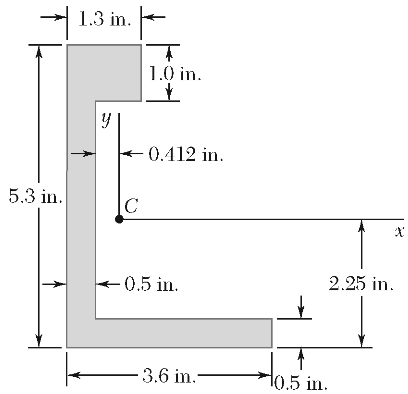
Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = 0.740 \pm 0.51650$$

or $\bar{I}_{\max} = 1.257 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 0.224 \text{ in}^4$ ◀



From the Mohr's circle it is seen that the *a* axis corresponds to $I_{\min}$ and the *b* axis corresponds to $I_{\max}$.



PROBLEM 9.102

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.77

(The moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area of Problem 9.102 were determined in Problem 9.44).

SOLUTION

From Problem 9.44: $\bar{I}_x = 18.1282 \text{ in}^4$

$\bar{I}_y = 4.5080 \text{ in}^4$

Problem 9.77: $\bar{I}_{xy} = -4.25320 \text{ in}^4$

The Mohr's circle is defined by the diameter XY, where $X(18.1282, -4.25320)$ and $Y(4.5080, 4.25320)$.

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(18.1282 + 4.5080) = 11.3181 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(18.1282 - 4.5080)\right]^2 + (-4.25320)^2}$$

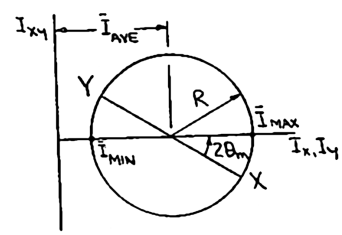
$$= 8.02915 \text{ in}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-4.25320)}{18.1282 - 4.5080}$$

$$= 0.62454$$



or $2\theta_m = 31.986°$

and $\theta_m = 15.99°$

The principal axes are obtained by rotating the xy axes through

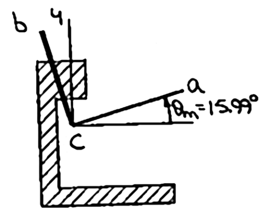
15.99° counterclockwise ◀

about C.

Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = 11.3181 \pm 8.02915$$

or $$\bar{I}_{\max} = 19.35 \text{ in}^4 \blacktriangleleft$$

and $$\bar{I}_{\min} = 3.29 \text{ in}^4 \blacktriangleleft$$



From the Mohr's circle it is seen that the *a* axis corresponds to $I_{\max}$ and the *b* axis corresponds to $I_{\min}$.

PROBLEM 9.103

The moments and product of inertia of an L4 × 3 × $\frac{1}{4}$-in. angle cross section with respect to two rectangular axes x and y through C are, respectively, $\bar{I}_x = 1.33$ in^4, $\bar{I}_y = 2.75$ in^4, and $\bar{I}_{xy} < 0$, with the minimum value of the moment of inertia of the area with respect to any axis through C being $\bar{I}_{\min} = 0.692$ in^4. Using Mohr's circle, determine (*a*) the product of inertia $\bar{I}_{xy}$ of the area, (*b*) the orientation of the principal axes, (*c*) the value of $\bar{I}_{\max}$.

SOLUTION

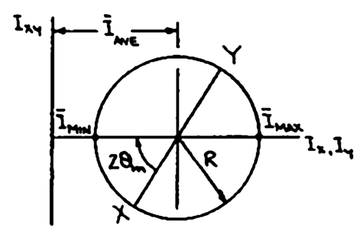
(*Note:* A review of a table of rolled-steel shapes reveals that the given values of $\bar{I}_x$ and $\bar{I}_y$ are obtained when the 4-in. leg of the angle is parallel to the x axis. Further, for $\bar{I}_{xy} < 0$, the angle must be oriented as shown.)

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(1.33 + 2.75) = 2.040 \text{ in}^4$$

and $$\bar{I}_{\min} = \bar{I}_{\text{ave}} - R \quad \text{or} \quad R = 2.040 - 0.692$$

$$= 1.348 \text{ in}^4$$

Using $\bar{I}_{\text{ave}}$ and R, the Mohr's circle is then drawn as shown; note that for the diameter XY, $X(1.33, \bar{I}_{xy})$ and $Y(2.75, |\bar{I}_{xy}|)$.



(*a*) We have $$R^2 = \left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2$$

or $$\bar{I}_{xy}^2 = 1.348^2 - \left[\frac{1}{2}(1.33 - 2.75)\right]^2$$

Solving for $\bar{I}_{xy}$ and taking the negative root (since $\bar{I}_{xy} < 0$) yields $\bar{I}_{xy} = -1.14586$ in^4.

$\bar{I}_{xy} = -1.146$ in^4 ◀

(*b*) We have $$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-1.14586)}{1.33 - 2.75}$$

$$= -1.61389$$

or $$2\theta_m = -58.217° \qquad \theta_m = -29.1°$$

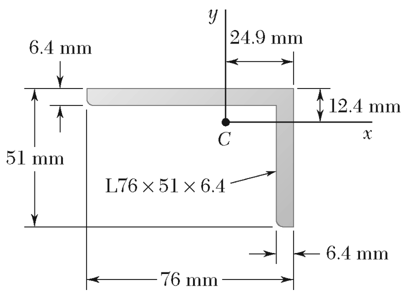
The principal axes are obtained by rotating the xy axes through

29.1° clockwise ◀

about C.

(*c*) We have $$\bar{I}_{\max} = \bar{I}_{\text{ave}} + R = 2.040 + 1.348$$

or $\bar{I}_{\max} = 3.39$ in^4 ◀



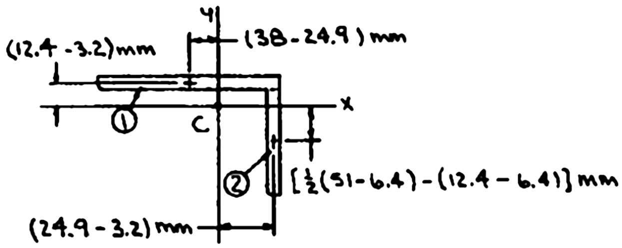
PROBLEM 9.104

Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Figure 9.13.)

SOLUTION

From Figure 9.13B: $\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$



We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$

and $\bar{I}_{x'y'} = 0$ (symmetry) $I_{xy} = \Sigma \bar{x}\bar{y}A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
1	$76 \times 6.4 = 486.4$	-13.1	9.2	-58620.93
2	$6.4 \times (51 - 6.4) = 285.44$	21.7	-16.3	-100962.98
Σ				-159583.91

$$\bar{I}_{xy} = -159584 \text{ mm}^4$$

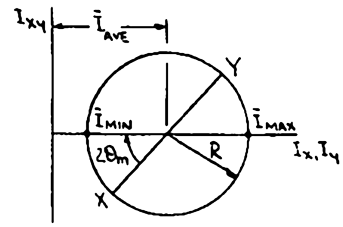
The Mohr's circle is defined by the diameter XY where $X(0.162 \times 10^6, -0.159584 \times 10^6)$ and $Y(0.454 \times 10^6, 0.159584 \times 10^6)$

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.162 + 0.454) \times 10^6$$
$$= 0.3080 \times 10^6 \text{ mm}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + I_{xy}^2} = \left\{\sqrt{\left[\frac{1}{2}(0.162 - 0.454)\right]^2 + (-0.159584)^2}\right\} \times 10^6$$

$$= 0.21629 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-0.159584 \times 10^6)}{(0.162 - 0.454) \times 10^6}$$

$$= -1.09304$$

or $$2\theta_m = -47.545$$

and $$\theta_m = -23.8°$$

The principal axes are obtained by rotating the *xy* axes through

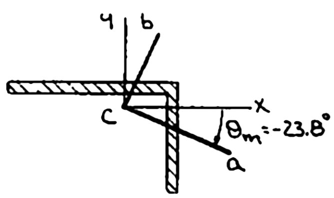
23.8° clockwise ◀

About *C*.

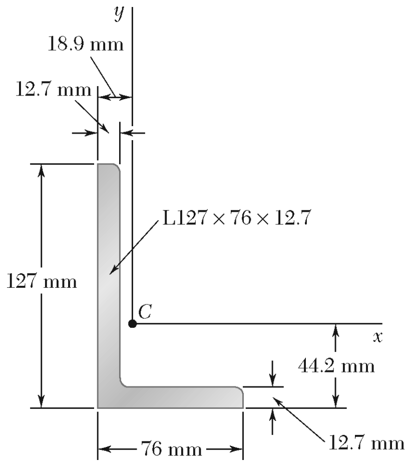
Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = (0.3080 \pm 0.21629) \times 10^6$$

or $$\bar{I}_{\max} = 0.524 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and $$\bar{I}_{\min} = 0.0917 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.



PROBLEM 9.105

Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Figure 9.13.)

SOLUTION

From Figure 9.13B:

$$\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$$
$$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

Problem 9.7B:

$$\bar{I}_{xy} = -1.165061 \times 10^6 \text{ mm}^4$$

(Note that the figure of Problem 9.105 is obtained by replacing *x* with –*x* in the figure of Problem 9.78; thus the change in sign of $\bar{I}_{xy}$.)

The Mohr's circle is defined by the diameter *XY*, where $X(3.93 \times 10^6, -1.165061 \times 10^6)$ and $Y(1.06 \times 10^6, 1.165061 \times 10^6)$.

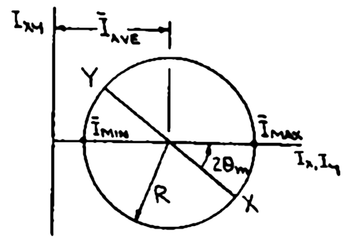
Now

$$I_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y)$$
$$= \frac{1}{2}(3.93 + 1.06) \times 10^6$$
$$= 2.495 \times 10^6 \text{ mm}^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \left\{\sqrt{\left[\frac{1}{2}(3.93 - 1.06)\right]^2 + (-1.165061)^2}\right\} \times 10^6$$
$$= 1.84840 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-1.165061\times 10^6)}{(3.93-1.06)\times 10^6}$$

$$= 0.81189$$

or $$2\theta_m = 39.073°$$

and $$\theta_m = 19.54°$$

The principal axes are obtained by rotating the xy axes through

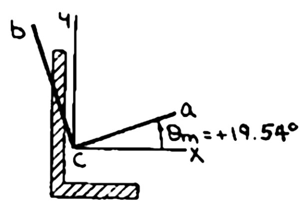
19.54° counterclockwise ◀

about C.

Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = (2.495 \pm 1.84840)\times 10^6$$

or $\bar{I}_{\max} = 4.34\times 10^6\ \text{mm}^4$ ◀

and $\bar{I}_{\min} = 0.647\times 10^6\ \text{mm}^4$ ◀



From the Mohr's circle it is seen that the a axis corresponds to $\bar{I}_{\max}$ and the b axis corresponds to $\bar{I}_{\min}$.

PROBLEM 9.106*

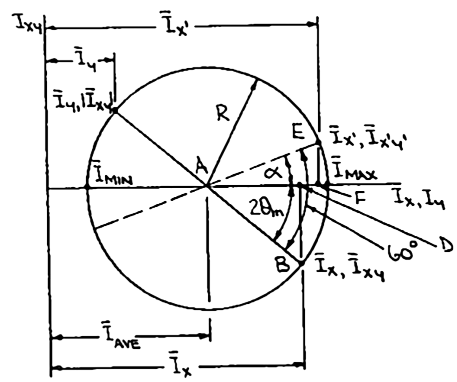
For a given area the moments of inertia with respect to two rectangular centroidal x and y axes are $\bar{I}_x = 1200\ \text{in}^4$ and $\bar{I}_y = 300\ \text{in}^4$, respectively. Knowing that after rotating the x and y axes about the centroid 30° counterclockwise, the moment of inertia relative to the rotated x axis is 1450 in^4, use Mohr's circle to determine (a) the orientation of the principal axes, (b) the principal centroidal moments of inertia.

SOLUTION

We have $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(1200 + 300) = 750\ \text{in.}^4$$

Now observe that $\bar{I}_x > \bar{I}_{\text{ave}}$, $\bar{I}_{x'} > \bar{I}_x$, and $2\theta = +60°$. This is possible only if $\bar{I}_{xy} < 0$. Therefore, assume $\bar{I}_{xy} < 0$ and (for convenience) $\bar{I}_{x'y'} > 0$. Mohr's circle is then drawn as shown.

We have $$2\theta_m + \alpha = 60°$$



Now using ΔABD: $$R = \frac{\bar{I}_x - \bar{I}_{\text{ave}}}{\cos 2\theta_m} = \frac{1200 - 750}{\cos 2\theta_m}$$
$$= \frac{450}{\cos 2\theta_m} \quad (\text{in}^4)$$

Using ΔAEF: $$R = \frac{\bar{I}_{x'} - \bar{I}_{\text{ave}}}{\cos \alpha} = \frac{1450 - 750}{\cos \alpha}$$
$$= \frac{700}{\cos \alpha} \quad (\text{in}^4)$$

Then $$\frac{450}{\cos 2\theta_m} = \frac{700}{\cos \alpha} \qquad \alpha = 60° - 2\theta_m$$

or $$9\cos(60° - 2\theta_m) = 14 \cos 2\theta_m$$

Expanding: $$9(\cos 60° \cos 2\theta_m + \sin 60° \sin 2\theta_m) = 14 \cos 2\theta_m$$

or $$\tan 2\theta_m = \frac{14 - 9\cos 60°}{9 \sin 60°} = 1.21885$$

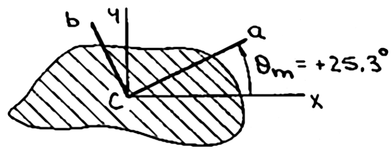
or $$2\theta_m = 50.633° \quad \text{and} \quad \theta_m = 25.3°$$

(*Note:* $2\theta_m < 60°$ implies assumption $\bar{I}_{x'y'} > 0$ is correct.)

Finally, $$R = \frac{450}{\cos 50.633°} = 709.46\ \text{in}^4$$

(a) From the Mohr's circle it is seen that the principal axes are obtained by rotating the given centroidal x and y axes through θ_m about the centroid C or

25.3° counterclockwise ◀



(*b*) We have $\bar{I}_{max,\,min} = \bar{I}_{ave} \pm R = 750 \pm 709.46$

or $\bar{I}_{max} = 1459 \text{ in}^4$ ◀

and $\bar{I}_{min} = 40.5 \text{ in}^4$ ◀

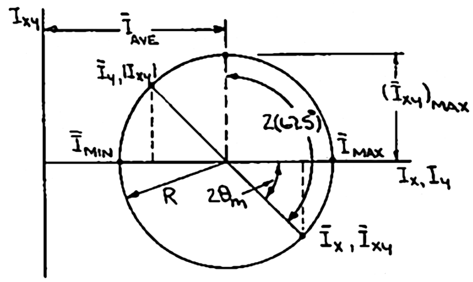
From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{max}$ and the *b* axis corresponds to $\bar{I}_{min}$.

PROBLEM 9.107

It is known that for a given area $\bar{I}_y = 48 \times 10^6$ mm^4 and $\bar{I}_{xy} = -20 \times 10^6$ mm^4, where the x and y axes are rectangular centroidal axes. If the axis corresponding to the maximum product of inertia is obtained by rotating the x axis 67.5° counterclockwise about C, use Mohr's circle to determine (a) the moment of inertia $\bar{I}_x$ of the area, (b) the principal centroidal moments of inertia.

SOLUTION

First assume $\bar{I}_x > \bar{I}_y$ and then draw the Mohr's circle as shown. (*Note:* Assuming $\bar{I}_x < \bar{I}_y$ is not consistent with the requirement that the axis corresponding to $(\bar{I}_{xy})_{\max}$ is obtained after rotating the x axis through 67.5° CCW.)



From the Mohr's circle we have

$$2\theta_m = 2(67.5°) - 90° = 45°$$

(a) From the Mohr's circle we have

$$\bar{I}_x = \bar{I}_y + 2\frac{|I_{xy}|}{\tan 2\theta_m} = 48 \times 10^6 + 2\frac{20 \times 10^6}{\tan 45°}$$

or $\bar{I}_x = 88.0 \times 10^6 \text{ mm}^4$ ◀

(b) We have

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(88.0 + 48) \times 10^6$$
$$= 68.0 \times 10^6 \text{ mm}^4$$

and

$$R = \frac{|I_{xy}|}{\sin 2\theta_m} = \frac{20 \times 10^6}{\sin 45°} = 28.284 \times 10^6 \text{ mm}^4$$

Now

$$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = (68.0 \pm 28.284) \times 10^6$$

or $\bar{I}_{\max} = 96.3 \times 10^6 \text{ mm}^4$ ◀

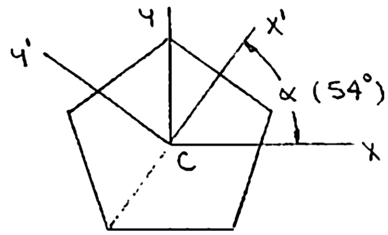
and $\bar{I}_{\min} = 39.7 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.108

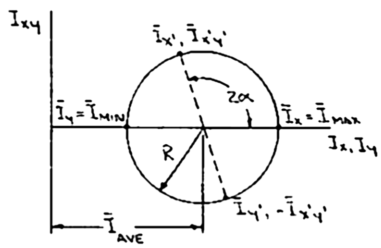
Using Mohr's circle, show that for any regular polygon (such as a pentagon) (*a*) the moment of inertia with respect to every axis through the centroid is the same, (*b*) the product of inertia with respect to every pair of rectangular axes through the centroid is zero.

SOLUTION

Consider the regular pentagon shown, with centroidal axes *x* and *y*.



Because the *y* axis is an axis of symmetry, it follows that $\bar{I}_{xy} = 0$. Since $\bar{I}_{xy} = 0$, the *x* and *y* axes must be principal axes. Assuming $\bar{I}_x = \bar{I}_{\max}$ and $\bar{I}_y = \bar{I}_{\min}$, the Mohr's circle is then drawn as shown.



Now rotate the coordinate axes through an angle α as shown; the resulting moments of inertia, $\bar{I}_{x'}$ and $\bar{I}_{y'}$, and product of inertia, $\bar{I}_{x'y'}$, are indicated on the Mohr's circle. However, the x' axis is an axis of symmetry, which implies $\bar{I}_{x'y'} = 0$. For this to be possible on the Mohr's circle, the radius *R* must be equal to zero (thus, the circle degenerates into a point). With $R = 0$, it immediately follows that

(*a*) $\bar{I}_x = \bar{I}_y = \bar{I}_{x'} = \bar{I}_{y'} = \bar{I}_{\text{ave}}$ (for all moments of inertia with respect to an axis through *C*) ◀

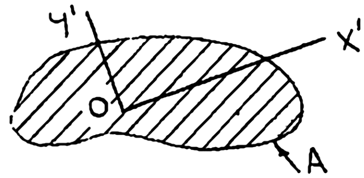
(*b*) $\bar{I}_{xy} = \bar{I}_{x'y'} = 0$ (for all products of inertia with respect to all pairs of rectangular axes with origin at *C*) ◀

PROBLEM 9.109

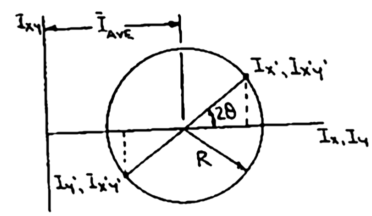
Using Mohr's circle, prove that the expression $I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the x' and y' axes, where $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ represent the moments and product of inertia, respectively, of a given area with respect to a pair of rectangular axes x' and y' through a given Point O. Also show that the given expression is equal to the square of the length of the tangent drawn from the origin of the coordinate system to Mohr's circle.

SOLUTION

First observe that for a given area A and origin O of a rectangular coordinate system, the values of I_{ave} and R are the same for all orientations of the coordinate axes. Shown below is a Mohr's circle, with the moments of inertia, $I_{x'}$ and $I_{y'}$, and the product of inertia, $I_{x'y'}$, having been computed for an arbitrary orientation of the $x'y'$ axes.



From the Mohr's circle



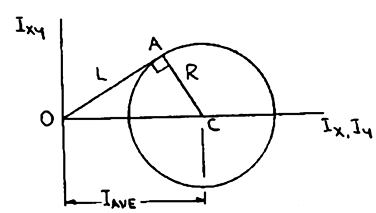
$$I_{x'} = I_{\text{ave}} + R\cos 2\theta$$
$$I_{y'} = I_{\text{ave}} - R\cos 2\theta$$
$$I_{x'y'} = R\sin 2\theta$$

Then, forming the expression $\quad I_{x'}I_{y'} - I_{x'y'}^2$

$$\begin{aligned} I_{x'}I_{y'} - I_{x'y'}^2 &= (I_{\text{ave}} + R\cos 2\theta)(I_{\text{ave}} - R\cos 2\theta) - (R\sin 2\theta)^2 \\ &= \left(I_{\text{ave}}^2 - R^2\cos^2 2\theta\right) - (R^2\sin^2 2\theta) \\ &= I_{\text{ave}}^2 - R^2 \quad \text{which is a constant} \end{aligned}$$

$I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the coordinate axes Q.E.D. ◀

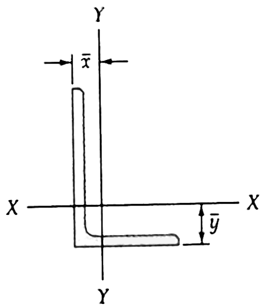
Shown is a Mohr's circle, with line $\overline{OA}$, of length L, the required tangent.



Noting that $\angle OAC$ is a right angle, it follows that

$$L^2 = I_{\text{ave}}^2 - R^2$$

or $\quad L^2 = I_{x'}I_{y'} - I_{x'y'}^2$ Q.E.D. ◀



PROBLEM 9.110

Using the invariance property established in the preceding problem, express the product of inertia I_{xy} of an area A with respect to a pair of rectangular axes through O in terms of the moments of inertia I_x and I_y of A and the principal moments of inertia $I_{\min}$ and $I_{\max}$ of A about O. Use the formula obtained to calculate the product of inertia I_{xy} of the L$3 \times 2 \times \frac{1}{4}$-in. angle cross section shown in Figure 9.13A, knowing that its maximum moment of inertia is 1.257 in^4.

SOLUTION

Consider the following two sets of moments and products of inertia, which correspond to two different orientations of the coordinate axes whole origin is at Point O.

Case 1: $I_{x'} = I_x, \quad I_{y'} = I_y, \quad I_{x'y'} = I_{xy}$

Case 2: $I_{x'} = I_{\max}, \quad I_{y'} = I_{\min}, \quad I_{x'y'} = 0$

The invariance property then requires

$$I_x I_y - I_{xy}^2 = I_{\max} I_{\min} \qquad \text{or} \qquad I_{xy} = \pm\sqrt{I_x I_y - I_{\max} I_{\min}} \blacktriangleleft$$

From Figure 9.13A: $\bar{I}_x = 1.09 \text{ in}^4$

$\bar{I}_y = 0.390 \text{ in}^4$

Using Eq. (9.21): $\bar{I}_x + \bar{I}_y = \bar{I}_{\max} + \bar{I}_{\min}$

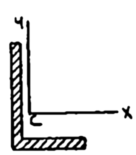
Substituting $1.09 + 0.390 = 1.257 + \bar{I}_{\min}$

or $\bar{I}_{\min} = 0.223 \text{ in}^4$

Then
$$\bar{I}_{xy} = \sqrt{(1.09)(0.390) - (1.257)(0.223)}$$
$$= \pm 0.381 \text{ in}^4$$

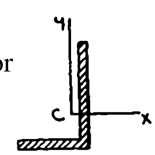
The two roots correspond to the following two orientations of the cross section.

For

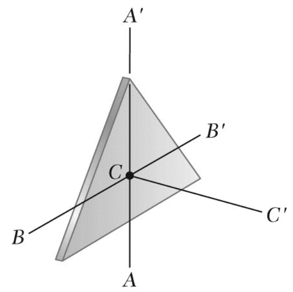


$\bar{I}_{xy} = -0.381 \text{ in}^4 \blacktriangleleft$

and for



$\bar{I}_{xy} = 0.381 \text{ in}^4 \blacktriangleleft$

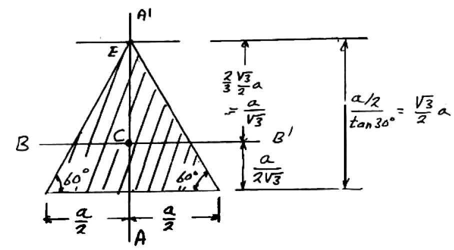


PROBLEM 9.111

A thin plate of mass m is cut in the shape of an equilateral triangle of side a. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axes AA' and BB', (b) the centroidal axis CC' that is perpendicular to the plate.

SOLUTION

$$\text{Area} = \frac{1}{2}a\left(\frac{\sqrt{3}}{2}a\right) = \frac{\sqrt{3}}{4}a^2$$



$$\text{Mass} = m = \rho V = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

(a) Axis AA':

$$I_{AA',\text{area}} = 2\left[\frac{1}{12}\left(\frac{\sqrt{3}}{2}a\right)\left(\frac{a}{2}\right)^3\right] = \frac{\sqrt{3}}{96}a^4$$

$$I_{AA',\text{ mass}} = \frac{m}{A} I_{AA',\text{ area}} = \frac{\frac{m}{\sqrt{3}}}{4}a^2\left(\frac{\sqrt{3}}{96}a^4\right) = \frac{1}{24}ma^2 \quad ◀$$

Axis BB'':

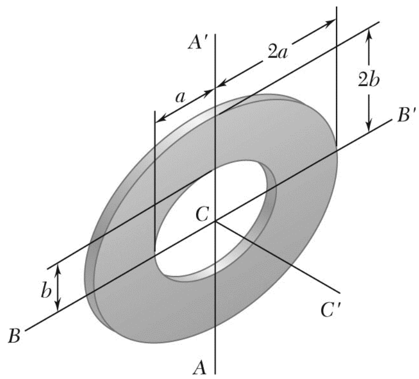
$$I_{BB',\text{ area}} = \frac{1}{36}a\left(\frac{\sqrt{3}}{2}a\right)^3 = \frac{\sqrt{3}}{96}a^4 \quad (\text{we check that } I_{AA} = I_{BB})$$

$$I_{BB',\text{mass}} = \frac{m}{A} I_{BB',\text{ area}} = \frac{\frac{m}{\sqrt{3}}}{4}a^2\left(\frac{\sqrt{3}}{96}a^4\right) = \frac{1}{24}ma^2 \quad ◀$$

(b) Eq. (9.38):

$$I_{CC'} = I_{AA'} + I_{BB'} = 2\left(\frac{1}{24}ma^2\right)$$

$$I_{CC'} = \frac{1}{12}ma^2 \quad ◀$$



PROBLEM 9.112

The elliptical ring shown was cut from a thin, uniform plate. Denoting the mass of the ring by m, determine its mass moment of inertia with respect to (a) the centroidal axis BB', (b) the centroidal axis CC' that is perpendicular to the plane of the ring.

SOLUTION

First note

$$\text{Mass} = m = \rho V = \rho t A$$
$$= \rho t \times \eta[(2a)(2b) - ab]$$
$$= 3\pi\, \rho t a b$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}}$$
$$= \frac{m}{3\pi ab} I_{\text{area}}$$

(a) Using Figure 9.12,

$$I_{BB',\text{area}} = \frac{\pi}{4}[(2a)(2b)^3 - ab^3]$$
$$= \frac{15}{4}\pi ab^3$$

Then

$$I_{BB',\text{ mass}} = \frac{m}{3\pi ab} \times \frac{15}{4}\pi ab^3$$

or $\quad I_{BB'} = \dfrac{5}{4} mb^2$ ◀

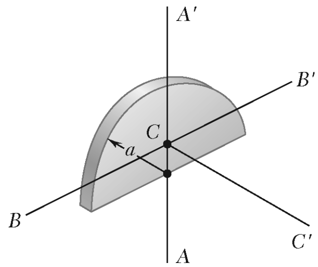
(b) Using Figure 9.12 and symmetry, we can conclude that

$$I_{AA',\text{ mass}} = \frac{5}{4} ma^2$$

Now

$$I_{CC',\text{mass}} = I_{AA',\text{ mass}} + I_{BB',\text{ mass}}$$
$$= \frac{5}{4} ma^2 + \frac{5}{4} mb^2$$

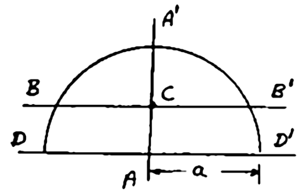
or $\quad I_{CC'} = \dfrac{5}{4} m(a^2 + b^2)$ ◀



PROBLEM 9.113

A thin semicircular plate has a radius a and a mass m. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis BB', (b) the centroidal axis CC' that is perpendicular to the plate.

SOLUTION



$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

Area: $$A = \frac{1}{2}\pi a^2$$

$$I_{AA',\text{area}} = I_{DD',\text{area}} = \frac{1}{2}\left(\frac{\pi}{4}a^4\right) = \frac{1}{8}\pi a^4$$

$$I_{AA',\text{mass}} = I_{DD',\text{mass}} = \frac{m}{A} I_{AA',\text{area}} = \frac{m}{\frac{1}{2}\pi a^2}\left(\frac{1}{8}\pi a^4\right) = \frac{1}{4}ma^2$$

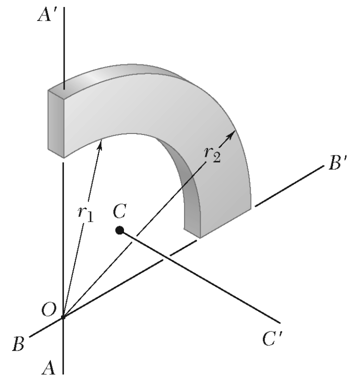
(a) $$I_{BB'} = I_{DD'} - m(AC)^2 = \frac{1}{4}ma^2 - m\left(\frac{4a}{3\pi}\right)^2$$

$$= (0.25 - 0.1801)ma^2$$

$$I_{BB'} = 0.0699ma^2 \blacktriangleleft$$

(b) Eq. (9.38): $$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{4}ma^2 + 0.0699ma^2$$

$$I_{CC'} = 0.320ma^2 \blacktriangleleft$$



PROBLEM 9.114

The quarter ring shown has a mass m and was cut from a thin, uniform plate. Knowing that $r_1 = \frac{3}{4} r_2$, determine the mass moment of inertia of the quarter ring with respect to (*a*) the axis AA', (*b*) the centroidal axis CC' that is perpendicular to the plane of the quarter ring.

SOLUTION

First note

$$\text{mass} = m = \rho V = \rho t A$$

$$= \rho t \frac{\pi}{4}\left(r_2^2 - r_1^2\right)$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}}$$

$$= \frac{m}{\frac{\pi}{4}\left(r_2^2 - r_1^2\right)} I_{\text{area}}$$

(*a*) Using Figure 9.12,

$$I_{AA',\text{area}} = \frac{\pi}{16}\left(r_2^4 - r_1^4\right)$$

Then

$$I_{AA',\text{mass}} = \frac{m}{\frac{\pi}{4}\left(r_2^2 - r_1^2\right)} \times \frac{\pi}{16}\left(r_2^4 - r_1^4\right)$$

$$= \frac{m}{4}\left(r_2^2 + r_1^2\right)$$

$$= \frac{m}{4}\left[r_2^2 + \left(\frac{3}{4} r_2\right)^2\right]$$

or $I_{AA'} = \dfrac{25}{64} m r_2^2$ ◀

(*b*) Symmetry implies

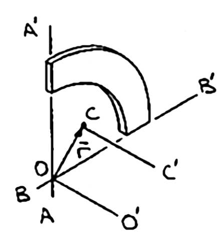
$$I_{BB',\text{mass}} = I_{AA',\text{mass}}$$

Then

$$I_{DD'} = I_{AA'} + I_{BB'}$$

$$= 2\left(\frac{25}{64} m r_2^2\right)$$

$$= \frac{25}{32} m r_2^2$$



PROBLEM 9.114 (Continued)

Now locate centroid C.

$$\bar{X}\,\Sigma A = \Sigma \bar{x} A$$

or
$$\bar{X}\left(\frac{\pi}{4}r_2^2 - \frac{\pi}{4}r_1^2\right) = \frac{4r_2}{3\pi}\left(\frac{\pi}{4}r_2^2\right) - \frac{4r_1}{3\pi}\left(\frac{\pi}{4}r_1^2\right)$$

or
$$\bar{X} = \frac{4}{3\pi}\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$$

Now
$$\bar{r} = \bar{X}\sqrt{2}$$

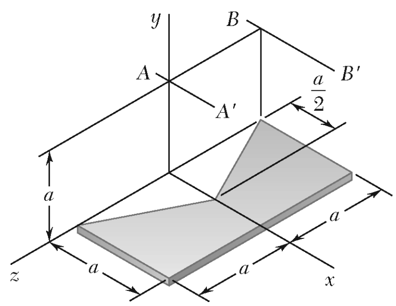
$$= \frac{4\sqrt{2}}{3\pi}\frac{r_2^3 - \left(\frac{3}{4}r_2\right)^3}{r_2^2 - \left(\frac{3}{4}r_2\right)^2}$$

$$= \frac{37\sqrt{2}}{21\pi}r_2$$

Finally,
$$I_{DD'} = I_{CC'} + m\bar{r}^2$$

or
$$\frac{25}{32}mr_2^2 = I_{CC'} + m\left(\frac{37\sqrt{2}}{21\pi}r_2\right)^2$$

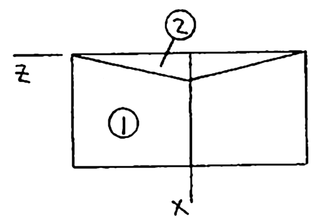
or $I_{CC'} = 0.1522mr_2^2$ ◀



PROBLEM 9.115

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its mass moment of inertia with respect to (a) the x axis, (b) the y axis.

SOLUTION



First note

$$\text{mass} = m = \rho V = \rho t A$$
$$= \rho t\left[(2a)(a) - \frac{1}{2}(2a)\left(\frac{a}{2}\right)\right]$$
$$= \frac{3}{2}\rho t a^2$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}}$$
$$= \frac{2m}{3a^2} I_{\text{area}}$$

(a) Now

$$\bar{I}_{x,\text{area}} = (I_x)_{1,\text{area}} - 2(I_x)_{2,\text{area}}$$
$$= \frac{1}{12}(a)(2a)^3 - 2\left[\frac{1}{12}\left(\frac{a}{2}\right)(a)^3\right]$$
$$= \frac{7}{12}a^4$$

Then

$$\bar{I}_{x,\text{mass}} = \frac{2m}{3a^2} \times \frac{7}{12}a^4$$

or $\bar{I}_x = \dfrac{7}{18}ma^2$ ◀

(b) We have

$$\bar{I}_{z,\text{area}} = (I_z)_{1,\text{area}} - 2(I_z)_{2,\text{area}}$$
$$= \frac{1}{3}(2a)(a)^3 - 2\left[\frac{1}{12}(a)\left(\frac{a}{2}\right)^3\right]$$
$$= \frac{31}{48}a^4$$

Then

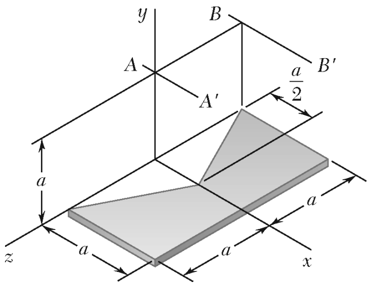
$$I_{z,\text{mass}} = \frac{2m}{3a^2} \times \frac{31}{48}a^4$$
$$= \frac{31}{72}ma^2$$

PROBLEM 9.115 (Continued)

Finally,

$$I_{y,\text{mass}} = \bar{I}_{x,\text{mass}} + I_{z,\text{mass}}$$
$$= \frac{7}{18}ma^2 + \frac{31}{72}ma^2$$
$$= \frac{59}{72}ma^2$$

or $I_y = 0.819ma^2$ ◀



PROBLEM 9.116

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its mass moment of inertia with respect to (a) the axis AA', (b) the axis BB', where the AA' and BB' axes are parallel to the x axis and lie in a plane parallel to and at a distance a above the xz plane.

SOLUTION

First note that the x axis is a centroidal axis so that

$$I = \bar{I}_{x,\text{mass}} + md^2$$

and that from the solution to Problem 9.115,

$$\bar{I}_{x,\text{mass}} = \frac{7}{18}ma^2$$

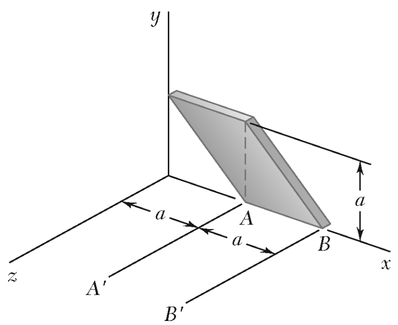
(a) We have $$I_{AA',\text{mass}} = \frac{7}{18}ma^2 + m(a)^2$$

or $I_{AA'} = 1.389ma^2$ ◀

(b) We have $$I_{BB',\text{mass}} = \frac{7}{18}ma^2 + m(a\sqrt{2})^2$$

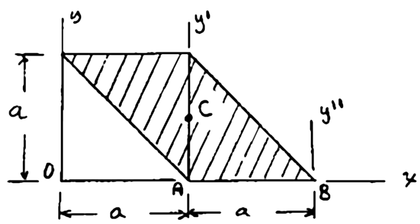
or $I_{BB'} = 2.39ma^2$ ◀

PROBLEM 9.117



A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (*a*) the x axis, (*b*) the axis BB', which is perpendicular to the plate.

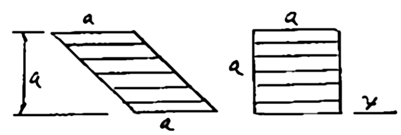
SOLUTION



$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

(*a*) Consider parallelogram as made of horizontal strips and slide strips to form a square since distance from each strip to x axis is unchanged.



$$I_{x,\text{area}} = \frac{1}{3} a^4$$

$$I_{x,\text{mass}} = \frac{m}{A} I_{x,\text{area}} = \frac{m}{a^2}\left(\frac{1}{3} a^4\right) \qquad I_x = \frac{1}{3} m a^2 \blacktriangleleft$$

(*b*) For centroidal axis y':

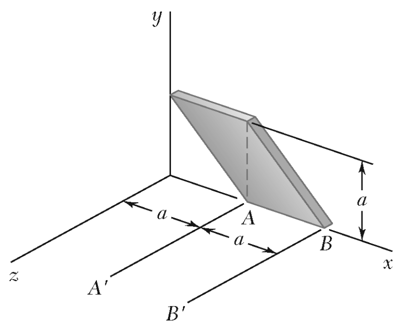
$$\bar{I}_{y',\text{area}} = 2\left[\frac{1}{12} a^4\right] = \frac{1}{6} a^4$$

$$\bar{I}_{y',\text{mass}} = \frac{m}{A} I_{y',\text{area}} = \frac{m}{a^2}\left(\frac{1}{6} a^4\right) = \frac{1}{6} m a^2$$

$$I_{y''} = \bar{I}_{y'} + m a^2 = \frac{1}{6} m a^2 + m a^2 = \frac{7}{6} m a^2$$

For axis $BB' \perp$ to plate, Eq. (9.38):

$$I_{BB'} = I_x + I_{y''} = \frac{1}{3} m a^2 + \frac{7}{6} m a^2 \qquad I_{BB'} = \frac{3}{2} m a^2 \blacktriangleleft$$



PROBLEM 9.118

A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (*a*) the y axis, (*b*) the axis AA', which is perpendicular to the plate.

SOLUTION

See sketch of solution of Problem 9.117.

(*a*) From part *b* of solution of Problem 9.117:

$$\bar{I}_{y'} = \frac{1}{6}ma^2$$

$$I_y = \bar{I}_{y'} + ma^2 = \frac{1}{6}ma^2 + ma^2 \qquad I_y = \frac{7}{6}ma^2 \blacktriangleleft$$

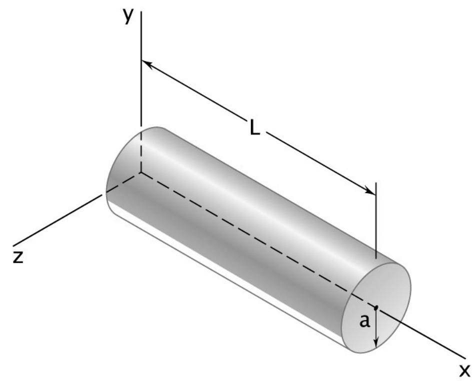
(*b*) From solution of Problem 9.115:

$$\bar{I}_{y'} = \frac{1}{6}ma^2 \quad \text{and} \quad I_x = \frac{1}{3}ma^2$$

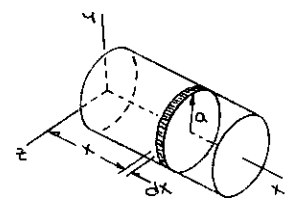
Eq. (9.38):

$$I_{AA'} = \bar{I}_{y'} + I_x$$

$$= \frac{1}{6}ma^2 + \frac{1}{3}ma^2 \qquad I_{AA'} = \frac{1}{2}ma^2 \blacktriangleleft$$

	**PROBLEM 9.119** Determine by direct integration the mass moment of inertia with respect to the z axis of the right circular cylinder shown, assuming that it has a uniform density and a mass m.

SOLUTION



For the cylinder: $m = \rho V = \rho\pi a^2 L$

For the element shown: $dm = \rho\pi a^2 dx$

$$= \frac{m}{L} dx$$

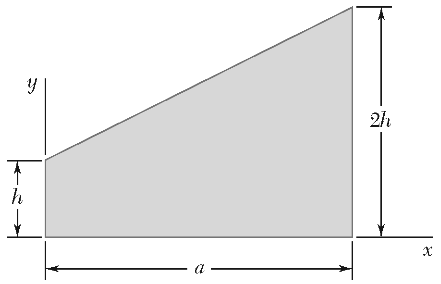
and $dI_z = dI_{z'} + x^2 dm$

$$= \frac{1}{4}a^2 dm + x^2 dm$$

Then $I_z = \int dI_z = \int_0^L \left(\frac{1}{4}a^2 + x^2\right)\left(\frac{m}{L}dx\right)$

$$= \frac{m}{L}\left[\frac{1}{4}a^2 x + \frac{1}{3}x^3\right]_0^L$$

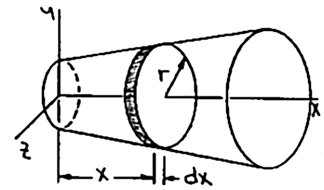
$$I_z = \frac{1}{12}m(3a^2 + 4L^2) \blacktriangleleft$$



PROBLEM 9.120

The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m. Using direct integration, express the mass moment of inertia of the solid with respect to the x axis in terms of m and h.

SOLUTION



We have $$y = \frac{2h-h}{a}x + h$$

so that $$r = \frac{h}{a}(x+a)$$

For the element shown:

$$dm = \rho\pi r^2 dx \quad dI_x = \frac{1}{2}r^2 dm$$

$$= \rho\pi\left[\frac{h}{a}(x+a)\right]^2 dx$$

Then $$m = \int dm = \int_0^a \rho\pi\frac{h^2}{a^2}(x+a)^2 dx = \frac{1}{3}\rho\pi\frac{h^2}{a^2}[(x+a)^3]_0^a$$

$$= \frac{1}{3}\rho\pi\frac{h^2}{a^2}(8a^3 - a^3) = \frac{7}{3}\rho\pi a h^2$$

Now $$I_x = \int dI_x = \int \frac{1}{2}r^2(\rho\pi r^2 dx) = \frac{1}{2}\rho\pi\int_0^a\left[\frac{h}{a}(x+a)\right]^4 dx$$

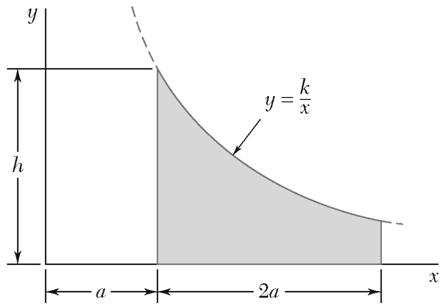
$$= \frac{1}{2}\rho\pi \times \frac{1}{5}\frac{h^4}{a^4}[(x+a)^5]_0^a = \frac{1}{10}\rho\pi\frac{h^4}{a^4}(32a^5 - a^5)$$

$$= \frac{31}{10}\rho\pi a h^4$$

From above: $$\rho\pi a h^2 = \frac{3}{7}m$$

Then $$I_x = \frac{31}{10}\left(\frac{3}{7}m\right)h^2 = \frac{93}{70}mh^2$$

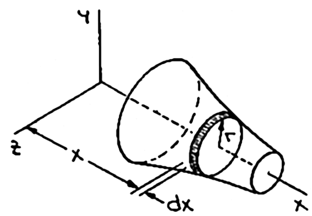
or $I_x = 1.329\, mh^2$ ◀



PROBLEM 9.121

The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m. Determine by direct integration the mass moment of inertia of the solid with respect to (a) the x axis, (b) the y axis. Express your answers in terms of m and the dimensions of the solid.

SOLUTION



We have at (a, h): $\quad h = \dfrac{k}{a}$

or $\quad k = ah$

For the element shown:

$$r = 4$$

$$dm = \rho\pi r^2 dx$$

$$= \rho\pi\left(\frac{ah}{x}\right)^2 dx$$

Then

$$m = \int dm = \int_a^{3a} \rho\pi\left(\frac{ah}{x}\right)^2 dx$$

$$= \rho\pi a^2 h^2\left[-\frac{1}{x}\right]_a^{3a}$$

$$= \rho\pi a^2 h^2\left[-\frac{1}{3a} - \left(-\frac{1}{a}\right)\right] = \frac{2}{3}\rho\pi a h^2$$

(a) For the element:

$$dI_x = \frac{1}{2}r^2 dm = \frac{1}{2}\rho\pi r^4 dx$$

Then

$$I_x = \int dI_x = \int_a^{3a}\frac{1}{2}\rho\pi\left(\frac{ah}{x}\right)^4 dx = \frac{1}{2}\rho\pi a^4 h^4\left[-\frac{1}{3}\frac{1}{x^3}\right]_a^{3a}$$

$$= -\frac{1}{6}\rho\pi a^4 h^4\left[\left(\frac{1}{3a}\right)^3 - \left(\frac{1}{a}\right)^3\right] = \frac{1}{6}\times\frac{26}{27}\rho\pi a h^4$$

$$= \frac{1}{6}\times\frac{2}{3}\rho\pi a h^2 \times \frac{13}{9}h^2 = \frac{13}{54}mh^2$$

or $\quad I_x = 0.241\, mh^2$ ◀

PROBLEM 9.121 (Continued)

(*b*) For the element: $dI_y = dI_{y'} + x^2 dm$

$$= \frac{1}{4} r^2 dm + x^2 dm$$

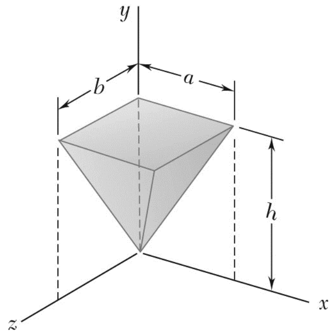
Then $$I_y = \int dI_y = \int_a^{3a} \left[\frac{1}{4} \left(\frac{ah}{x} \right)^2 + x^2 \right] \rho \pi \left(\frac{ah}{x} \right)^2 dx$$

$$= \rho \pi a^2 h^2 \int_a^{3a} \left(\frac{1}{4} \frac{a^2 h^2}{x^4} + 1 \right) dx = \rho \pi a^2 h^2 \left[-\frac{1}{12} \frac{a^2 h^2}{x^3} + x \right]_a^{3a}$$

$$= \left(\frac{3}{2} m \right) a \left\{ \left[-\frac{1}{12} \frac{a^2 h^2}{(3a)^3} + 3a \right] - \left[-\frac{1}{12} \frac{a^2 h^2}{(a)^3} + a \right] \right\}$$

$$= \frac{3}{2} ma \left(\frac{1}{12} \times \frac{26}{27} \frac{h^2}{a} + 2a \right) = m \left(\frac{13}{108} h^2 + 3a^2 \right)$$

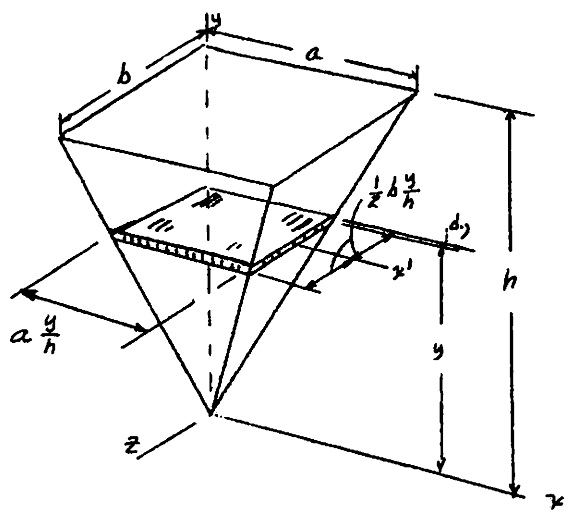
or $I_y = m(3a^2 + 0.1204h^2)$ ◀



PROBLEM 9.122

Determine by direct integration the mass moment of inertia with respect to the x axis of the pyramid shown, assuming that it has a uniform density and a mass m.

SOLUTION



For element shown:

$$dm = \rho dV = \rho\left(a\frac{y}{h}\right)\left(b\frac{y}{h}\right)dy = \rho\frac{ab}{h^2}y^2dy$$

$$dI_{x'} = \frac{1}{12}\left(b\frac{y}{h}\right)^2 dm = \frac{1}{12}\frac{b^2}{h^2}y^2\left(\rho\frac{ab}{h^2}\right)y^2dy = \frac{1}{12}\rho\frac{ab^3}{h^4}y^4dy$$

Parallel-axis theorem

$$dI_x = dI_{x'} + d^2dm \qquad \text{where} \quad d^2 = y^2 + \left(\frac{1}{2}b\frac{y}{h}\right)^2$$

$$dI_x = \frac{1}{12}\rho\frac{ab^3}{h^4}y^4dy + \left[y^2 + \frac{1}{4}\frac{b^2}{h^2}y^2\right]\rho\frac{ab}{h^2}y^2dy$$

$$= \left(\frac{\rho}{3}\frac{ab^3}{h^4} + \rho\frac{ab}{h^2}\right)y^4dy$$

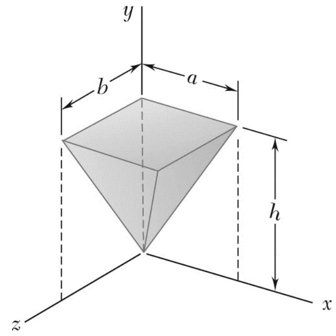
$$I_x = \int dI_x = \left(\frac{\rho}{3}\frac{ab^3}{h^4} + \rho\frac{ab}{h^2}\right)\int_0^h y^4dy = \frac{\rho ab^3h}{15} + \frac{\rho abh^3}{5}$$

PROBLEM 9.122 (Continued)

For pyramid, $$m = \rho v = \frac{1}{3}\rho abh$$

Thus: $$I_x = \left(\frac{1}{3}\rho abh\right)\left(\frac{b^2}{5} + \frac{3h^2}{5}\right)$$

$$I_x = \frac{1}{5}m(b^2 + 3h^2) \blacktriangleleft$$

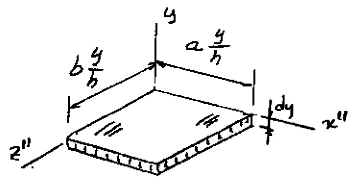


PROBLEM 9.123

Determine by direct integration the mass moment of inertia with respect to the x axis of the pyramid shown, assuming that it has a uniform density and a mass m.

SOLUTION

See figure of solution of Problem 9.122 for element shown



$$dm = \rho\left(b\frac{y}{h}\right)\left(a\frac{y}{h}\right)dy$$

$$dm = \rho\frac{ab}{h^2}y^2dy$$

For thin plate: $$dI_y = dI_{x''} + dI_{z''}$$

$$dI_y = \frac{1}{3}\left(b\frac{y}{h}\right)^2 dm + \frac{1}{3}\left(a\frac{y}{h}\right)^2 dm = \frac{1}{3h^2}(b^2+a^2)y^2dm$$

$$= \frac{1}{3h^2}(a^2+b^2)y^2\left(\rho\frac{ab}{h^2}y^2dy\right) = \frac{\rho ab}{3h^4}(a^2+b^2)y^4dy$$

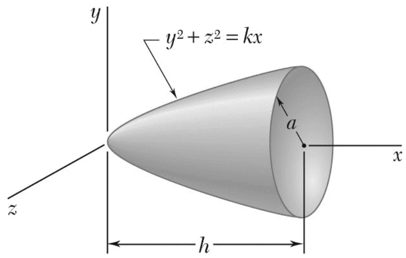
$$I_y = \int dI_y = \frac{\rho ab}{3h^4}(a^2+b^2)\int_0^h y^4dy = \frac{\rho ab}{15}(a^2+b^2)h$$

For pyramid, $$m = \rho v = \frac{1}{3}\rho abh$$

$$I_y = \left(\frac{1}{3}\rho abh\right)\frac{1}{5}(a^2+b^2)$$

$$I_y = \frac{1}{5}m(a^2+b^2) \blacktriangleleft$$

PROBLEM 9.124

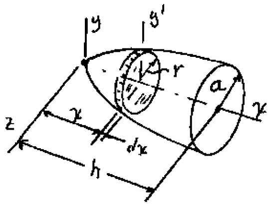


Determine by direct integration the mass moment of inertia with respect to the y axis of the paraboloid shown, assuming that it has a uniform density and a mass m.

SOLUTION

$r^2 = y^2 + z^2 = kx$: at $x = h$, $r = a$; $a^2 = kh$; $k = \dfrac{a^2}{h}$

Thus: $r^2 = \dfrac{a^2}{h}x$



$$dm = \rho\pi r^2 dx = \rho\pi \frac{a^2}{h} xdy$$

$$m = \int_0^h dm = \rho\pi \frac{a^2}{h}\int_0^h xdx; \quad m = \frac{1}{2}\rho\pi a^2 h$$

$$dI_y = dI_{y'} + mx^2 = \frac{1}{4}r^2 dm + x^2 dm = \left(\frac{1}{4}\frac{a^2}{h}x + x^2\right)dm$$

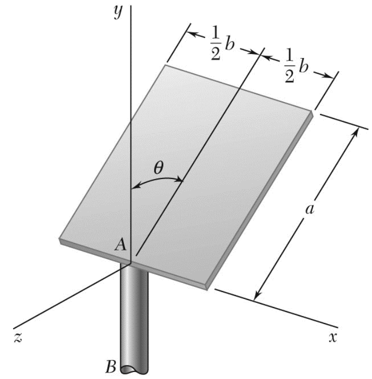
$$I_y = \int_0^a I_y = \int_0^a \left(\frac{1}{4}\frac{a^2}{h}x + x^2\right)\rho\pi\frac{a^2}{h}xdy = \rho\pi\frac{a^2}{h}\int_0^h \left(\frac{1}{4}\frac{a^2}{h}x^2 + x^3\right)dx$$

$$= \rho\pi\frac{a^2}{h}\left(\frac{1}{4}\frac{a^2}{h}\frac{h^3}{3} + \frac{h^4}{4}\right)$$

$$I_y = \frac{1}{12}\rho\pi a^2 h(a^2 + 3h^2) \quad \triangleleft$$

Recall: $m = \dfrac{1}{2}\rho\pi a^2 h$; $\quad I_y = \left(\dfrac{1}{2}\rho\pi a^2 h\right)\left(\dfrac{1}{6}\right)(a^2 + 3h^2)$

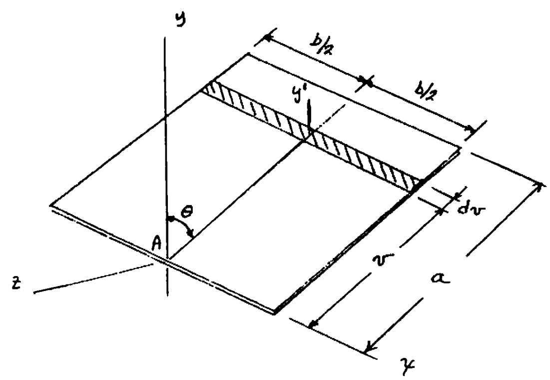
$$I_y = \frac{1}{6}m(a^2 + 3h^2) \quad \blacktriangleleft$$



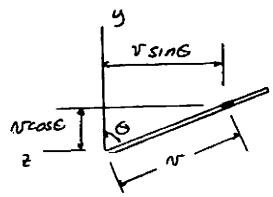
PROBLEM 9.125

A thin rectangular plate of mass m is welded to a vertical shaft AB as shown. Knowing that the plate forms an angle θ with the y axis, determine by direct integration the mass moment of inertia of the plate with respect to (a) the y axis, (b) the z axis.

SOLUTION



Projection on yz plane



Mass of plate: $m = \rho tab$

(a) For element shown: $dm = \rho btdv$

$$d\bar{I}_y = d\bar{I}_{y'} + (v\sin\theta)^2 dm = \frac{1}{12}b^2 dm + v^2\sin^2\theta dm$$

$$= \left(\frac{1}{12}b^2 + v^2\sin^2\theta\right)\rho btdv$$

$$I_y = \int dI_y = \rho bt\int_0^a \left(\frac{1}{12}b^2 + v^2\sin^2\theta\right)dv$$

$$= \rho bt\left|\frac{1}{12}b^2 v + \frac{v^3}{3}\sin\theta\right|_0^a = \rho bt\left(\frac{1}{12}ab^2 + \frac{a^3}{3}\sin^2\theta\right)$$

$$= (\rho tab)\frac{1}{12}(b^2 + 4a^2\sin^2\theta)$$

$$I_y = \frac{1}{12}m(b^2 + 4a^2\sin^2\theta) \blacktriangleleft$$

(b)

$$dI_z = d\overline{I}_{z'} + (v\cos\theta)^2\,dm = \frac{1}{12}b^2 dm + v^2\cos^2\theta\,dm$$

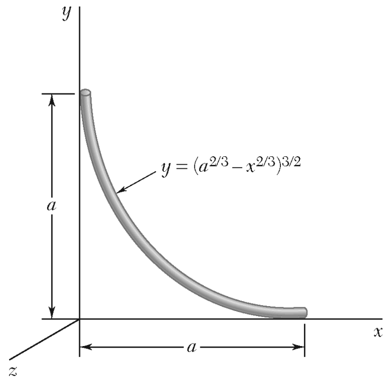
$$= \left(\frac{1}{12}b^2 + v^2\cos^2\theta\right)\rho bt\,dv$$

$$I_z = \int dI_z = \rho bt\int_0^a \left(\frac{1}{12}b^2 + v^2\cos^2\theta\right)dv$$

$$= \rho bt\left|\frac{1}{12}b^2 v + \frac{v^3}{3}\cos^2\theta\right|_0^a = \rho bt\left(\frac{1}{12}ab^2 + \frac{a^3}{3}\cos^2\theta\right)$$

$$= (\rho tab)\frac{1}{12}(b^2 + 4a^2\cos^2\theta)$$

$$I_z = \frac{1}{12}m(b^2 + 4a^2\cos^2\theta) \blacktriangleleft$$



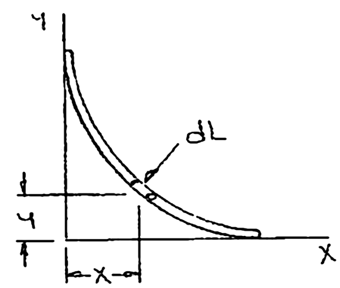
PROBLEM 9.126*

A thin steel wire is bent into the shape shown. Denoting the mass per unit length of the wire by m', determine by direct integration the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION

First note $$\frac{dy}{dx} = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$$

Then $$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^{-2/3}(a^{2/3} - x^{2/3})$$
$$= \left(\frac{a}{x}\right)^{2/3}$$



For the element shown: $$dm = m'dL = m'\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= m'\left(\frac{a}{x}\right)^{1/3} dx$$

Then $$m = \int dm = \int_0^a m'\frac{a^{1/3}}{x^{1/3}}dx = \frac{3}{2}m'a^{1/3}\left[x^{2/3}\right]_0^a = \frac{3}{2}m'a$$

Now $$I_x = \int y^2 dm = \int_0^a (a^{2/3} - x^{2/3})^3 \left(m'\frac{a^{1/3}}{x^{1/3}}dx\right)$$
$$= m'a^{1/3}\int_0^a \left(\frac{a^2}{x^{1/3}} - 3a^{4/3}x^{1/3} + 3a^{2/3}x - x^{5/3}\right)dx$$
$$= m'a^{1/3}\left[\frac{3}{2}a^2x^{2/3} - \frac{9}{4}a^{4/3}x^{4/3} + \frac{3}{2}a^{2/3}x^2 - \frac{3}{8}x^{8/3}\right]_0^a$$
$$= m'a^3\left(\frac{3}{2} - \frac{9}{4} + \frac{3}{2} - \frac{3}{8}\right) = \frac{3}{8}m'a^3$$

or $$I_x = \frac{1}{4}ma^2 \blacktriangleleft$$

Symmetry implies $$I_y = \frac{1}{4}ma^2 \blacktriangleleft$$

PROBLEM 9.126* (Continued)

Alternative solution:

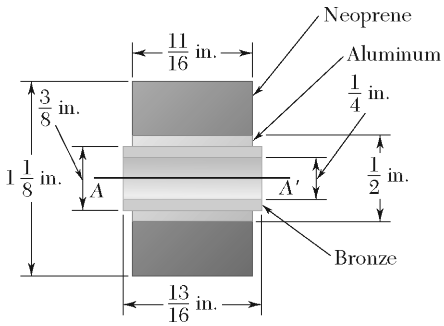
$$I_y = \int x^2 dm = \int_0^a x^2 \left(m' \frac{a^{1/3}}{x^{1/3}} dx \right) = m' a^{1/3} \int_0^a x^{5/3} dx$$

$$= m' a^{1/3} \times \frac{3}{8} \left[x^{8/3} \right]_0^a = \frac{3}{8} m' a^3$$

$$= \frac{1}{4} ma^2$$

Also $$I_z = \int (x^2 + y^2)\, dm = I_y + I_x$$

or $$I_z = \frac{1}{2} ma^2 \blacktriangleleft$$



PROBLEM 9.127

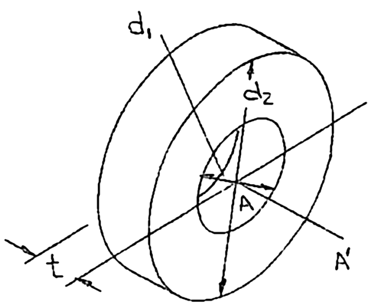
Shown is the cross section of an idler roller. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The specific weight of bronze is 0.310 lb/in^3; of aluminum, 0.100 lb/in^3; and of neoprene, 0.0452 lb/in^3.)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4}\left(d_2^2 - d_1^2\right) = \frac{\pi}{4}\rho t\left(d_2^2 - d_1^2\right)$$

and, using Figure 9.28, that



$$\begin{aligned}
I_{AA'} &= \frac{1}{2}m_2\left(\frac{d_2}{2}\right)^2 - \frac{1}{2}m_1\left(\frac{d_1}{2}\right)^2 \\
&= \frac{1}{8}\left[\left(\rho t \times \frac{\pi}{4}d_2^2\right)d_2^2 - \left(\rho t \times \frac{\pi}{4}d_1^2\right)d_1^2\right] \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^4 - d_1^4\right) \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^2 - d_1^2\right)\left(d_2^2 + d_1^2\right) \\
&= \frac{1}{8}m\left(d_1^2 + d_2^2\right)
\end{aligned}$$

Now treat the roller as three concentric rings and, working from the bronze outward, we have

$$\begin{aligned}
m = \frac{\pi}{4} \times \frac{1}{32.2}\,\text{ft/s}^2 \Bigg\{ &(0.310\ \text{lb/in}^3)\left(\frac{13}{16}\text{in.}\right)\left[\left(\frac{3}{8}\right)^2 - \left(\frac{1}{4}\right)^2\right]\text{in}^2 \\
&+ (0.100\ \text{lb/in}^3)\left(\frac{11}{16}\text{in.}\right)\left[\left(\frac{1}{2}\right)^2 - \left(\frac{3}{8}\right)^2\right]\text{in}^2 \\
&+ (0.0452\ \text{lb/in}^3)\left(\frac{11}{16}\text{in.}\right)\left[\left(1\frac{1}{8}\right)^2 - \left(\frac{1}{2}\right)^2\right]\text{in}^2 \Bigg\} \\
= {} &(479.96 + 183.41 + 769.80) \times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft} \\
= {} &1.4332 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}
\end{aligned}$$

PROBLEM 9.127 (Continued)

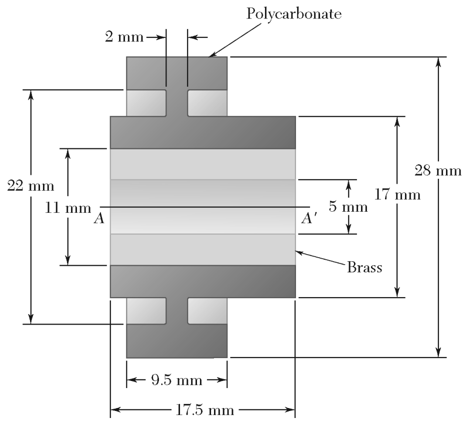
and
$$I_{AA'} = \frac{1}{8}\left\{(479.96)\left[\left(\frac{1}{4}\right)^2 + \left(\frac{3}{8}\right)^2\right] + (183.41)\left[\left(\frac{3}{8}\right)^2 + \left(\frac{1}{2}\right)^2\right]\right.$$
$$\left. + (769.80)\left[\left(1\frac{1}{8}\right)^2 + \left(\frac{1}{2}\right)^2\right]\right\} \times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft} \times \text{in}^2 \times \frac{1\ \text{ft}^2}{144\ \text{in}^2}$$
$$= (84.628 + 62.191 + 1012.78) \times 10^{-9}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$= 1.15960 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_{AA'} = 1.160 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

Now
$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{1.15960 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2}{1.4332 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}} = 809.09 \times 10^{-6}\ \text{ft}^2$$

Then
$$k_{AA'} = 28.445 \times 10^{-3}\ \text{ft}$$

or $k_{AA'} = 0.341\ \text{in.}$ ◀



PROBLEM 9.128

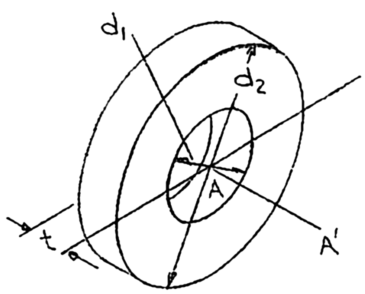
Shown is the cross section of a molded flat-belt pulley. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The density of brass is 8650 kg/m^3 and the density of the fiber-reinforced polycarbonate used is 1250 kg/m^3.)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4}\left(d_2^2 - d_1^2\right)$$

and, using Figure 9.28, that



$$\begin{aligned}
I_{AA'} &= \frac{1}{2} m_2 \left(\frac{d_2}{2}\right)^2 - \frac{1}{2} m_1 \left(\frac{d_1}{2}\right)^2 \\
&= \frac{1}{8}\left[\left(\rho t \times \frac{\pi}{4} d_2^2\right) d_2^2 - \left(\rho t \times \frac{\pi}{4} d_1^2\right) d_1^2\right] \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^4 - d_1^4\right) \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^2 - d_1^2\right)\left(d_2^2 + d_1^2\right) \\
&= \frac{1}{8} m\left(d_1^2 + d_2^2\right)
\end{aligned}$$

Now treat the pulley as four concentric rings and, working from the brass outward, we have

$$\begin{aligned}
m = \frac{\pi}{4}\{&8650 \text{ kg/m}^3 \times (0.0175 \text{ m}) \times (0.011^2 - 0.005^2) \text{ m}^2 \\
&+ 1250 \text{ kg/m}^3 [(0.0175 \text{ m}) \times (0.017^2 - 0.011^2) \text{ m}^2 \\
&+ (0.002 \text{ m}) \times (0.022^2 - 0.017^2) \text{ m}^2 \\
&+ (0.0095 \text{ m}) \times (0.028^2 - 0.022^2) \text{ m}^2]\} \\
= {} & (11.4134 + 2.8863 + 0.38288 + 2.7980) \times 10^{-3} \text{ kg} \\
= {} & 17.4806 \times 10^{-3} \text{ kg}
\end{aligned}$$

PROBLEM 9.128 (Continued)

and

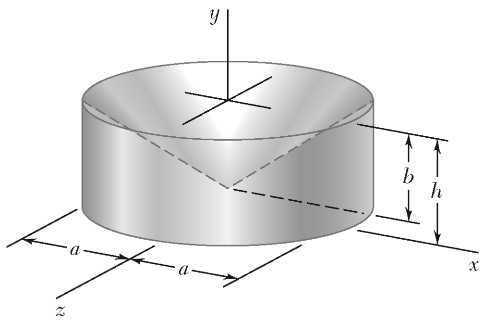
$$I_{AA'} = \frac{1}{8}[(11.4134)(0.005^2 + 0.011^2) + (2.8863)(0.011^2 + 0.017^2)$$
$$+ (0.38288)(0.017^2 + 0.022^2)$$
$$+ (2.7980)(0.022^2 + 0.028^2)] \times 10^{-3}\ \text{kg} \times \text{m}^2$$
$$= (208.29 + 147.92 + 37.00 + 443.48) \times 10^{-9}\ \text{kg} \cdot \text{m}^2$$
$$= 836.69 \times 10^{-9}\ \text{kg} \cdot \text{m}^2$$

or $I_{AA'} = 837 \times 10^{-9}\ \text{kg} \cdot \text{m}^2$ ◀

Now

$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{836.69 \times 10^{-9}\ \text{kg} \cdot \text{m}^2}{17.4806 \times 10^{-3}\ \text{kg}} = 47.864 \times 10^{-6}\ \text{m}^2$$

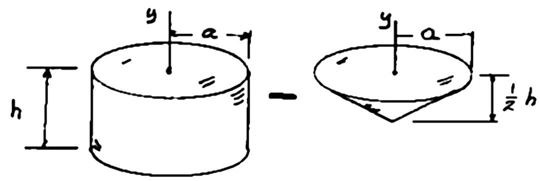
or $k_{AA'} = 6.92$ mm ◀



PROBLEM 9.129

The machine part shown is formed by machining a conical surface into a circular cylinder. For $b = \frac{1}{2}h$, determine the mass moment of inertia and the radius of gyration of the machine part with respect to the y axis.

SOLUTION



Mass:

$$m_{\text{cyl}} = \rho\pi a^2 h \qquad m_{\text{cone}} = \frac{1}{3}\rho\pi a^2 \frac{h}{2} = \frac{1}{6}\rho\pi a^2 h$$

$$I_y: \quad I_{\text{cyl}} = \frac{1}{2}m_{\text{cyl}}a^2 \qquad I_{\text{cone}} = \frac{3}{10}m_{\text{cone}}a^2$$

$$= \frac{1}{2}\rho\pi a^4 h \qquad = \frac{1}{20}\rho\pi a^4 h$$

For entire machine part:

$$m = m_{\text{cyl}} - m_{\text{cone}} = \rho\pi a^2 h - \frac{1}{6}\rho\pi a^2 h = \frac{5}{6}\rho\pi a^2 h$$

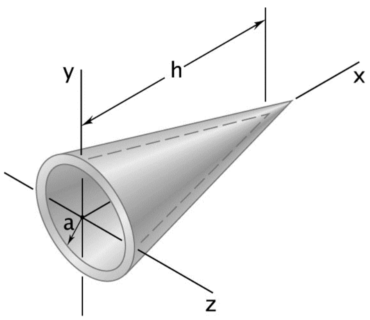
$$I_y = I_{\text{cyl}} - I_{\text{cone}} = \frac{1}{2}\rho\pi a^4 h - \frac{1}{20}\rho\pi a^4 h = \frac{9}{20}\rho\pi a^4 h$$

or

$$I_y = \left(\frac{5}{6}\rho\pi a^2 h\right)\left(\frac{6}{5}\right)\left(\frac{9}{20}\right)a^2 \qquad I_y = \frac{27}{50}ma^2 \blacktriangleleft$$

Then

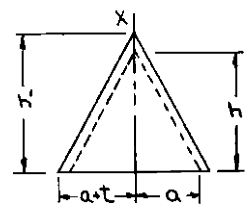
$$k_y^2 = \frac{I}{m} = \frac{27}{50}a^2 \qquad k_y = 0.735a \blacktriangleleft$$



PROBLEM 9.130

Given the dimensions and the mass m of the thin conical shell shown, determine the mass moment of inertia and the radius of gyration of the shell with respect to the x axis. (*Hint:* Assume that the shell was formed by removing a cone with a circular base of radius a from a cone with a circular base of radius $a + t$, where t is the thickness of the wall. In the resulting expressions, neglect terms containing t^2, t^3, etc. Do not forget to account for the difference in the heights of the two cones.)

SOLUTION



First note $$\frac{h'}{a+t} = \frac{h}{a}$$

or $$h' = \frac{h}{a}(a+t)$$

For a cone of height H whose base has a radius r, have

$$I_x = \frac{3}{10}mr^2$$

$$m = \rho V = \rho \times \frac{\pi}{3} r^2 H$$

Then $$I_x = \frac{3}{10}\left(\frac{\pi}{3}\rho r^2 H\right) r^2$$

$$= \frac{\pi}{10}\rho r^4 H$$

Now following the hint have

$$m_{\text{shell}} = m_{\text{outer}} - m_{\text{inner}} = \frac{\pi}{3}\rho[(a+t)^2 h' - a^2 h]$$

$$= \frac{\pi}{3}\rho\left[(a+t)^2 = \frac{b}{a}(a+t) - a^2 h\right]$$

$$= \frac{\pi}{3}\rho a^2 h\left[\left(1+\frac{t}{a}\right)^3 - 1\right] = \frac{\pi}{3}\rho a^2 h\left(1 + 3\frac{t}{a} + \cdots - 1\right)$$

Neglecting the t^2 and t^3 terms obtain $\quad m_{\text{shell}} = \pi\rho aht$

PROBLEM 9.130 (Continued)

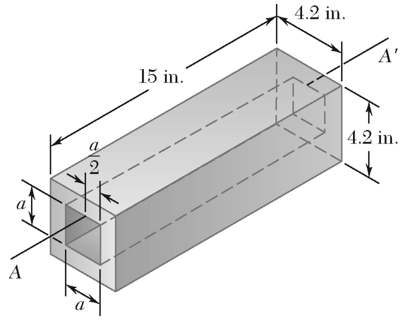
Also
$$(I_x)_{\text{shell}} = (I_x)_{\text{outer}} - (I_x)_{\text{inner}}$$
$$= \frac{\pi}{10}\rho[(a+t)^4 h' - a^4 h]$$
$$= \frac{\pi}{10}\rho\left[(a+t)^4 \times \frac{h}{a}(a+t) - a^4 h\right]$$
$$= \frac{\pi}{10}\rho a^4 h\left[\left(1+\frac{t}{a}\right)^5 - 1\right] = \frac{\pi}{10}\rho a^4 h\left(1 + 5\frac{t}{a} + \cdots - 1\right)$$

Neglecting t^2 and higher order terms, obtain
$$(I_x)_{\text{shell}} = \frac{\pi}{2}\rho a^3 ht$$

Now
$$k_x^2 = \frac{I_x}{m} = \frac{\frac{1}{2}ma^2}{m}$$

or $I_x = \dfrac{1}{2}ma^2$ ◀

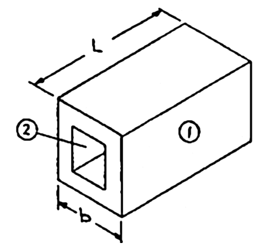
or $k_x = \dfrac{a}{\sqrt{2}}$ ◀



PROBLEM 9.131

A square hole is centered in and extends through the aluminum machine component shown. Determine (*a*) the value of *a* for which the mass moment of inertia of the component with respect to the axis *AA*′, which bisects the top surface of the hole, is maximum, (*b*) the corresponding values of the mass moment of inertia and the radius of gyration with respect to the axis *AA*′. (The specific weight of aluminum is 0.100 lb/in^3.)

SOLUTION



First note $$m_1 = \rho V_1 = \rho b^2 L$$

and $$m_2 = \rho V_2 = \rho a^2 L$$

(*a*) Using Figure 9.28 and the parallel-axis theorem, we have

$$\begin{aligned} I_{AA'} &= (I_{AA'})_1 - (I_{AA'})_2 \\ &= \left[\frac{1}{12}m_1(b^2+b^2) + m_1\left(\frac{a}{2}\right)^2\right] \\ &\quad - \left[\frac{1}{12}m_2(a^2+a^2) + m_2\left(\frac{a}{2}\right)^2\right] \\ &= (\rho b^2 L)\left(\frac{1}{6}b^2 + \frac{1}{4}a^2\right) - (\rho a^2 L)\left(\frac{5}{12}a^2\right) \\ &= \frac{\rho L}{12}(2b^4 + 3b^2a^2 - 5a^4) \end{aligned}$$

Then $$\frac{dI_{AA'}}{da} = \frac{\rho L}{12}(6b^2 a - 20a^3) = 0$$

or $$a = 0 \qquad \text{and} \qquad a = b\sqrt{\frac{3}{10}}$$

Also $$\frac{d^2 I_{AA'}}{da^2} = \frac{\rho L}{12}(6b^2 - 60a^2) = \frac{1}{2}\rho L(b^2 - 10a^2)$$

Now for $a = 0$, $$\frac{d^2 I_{AA'}}{da^2} > 0$$

and for $a = b\sqrt{\frac{3}{10}}$, $$\frac{d^2 I_{AA'}}{da^2} < 0$$

$(I_{AA'})_{\max}$ occurs when $$a = b\sqrt{\frac{3}{10}}$$

Then $$a = (4.2\text{ in.})\sqrt{\frac{3}{10}}$$

or $a = 2.30$ in. ◀

(*b*) From part *a*:

$$(I_{AA'})_{\max} = \frac{\rho L}{12}\left[2b^4 + 3b^2\left(b\sqrt{\frac{3}{10}}\right)^2 - 5\left(b\sqrt{\frac{3}{10}}\right)^4\right] = \frac{49}{240}\rho L b^4$$

$$= \frac{49}{240}\frac{\gamma_{AL}}{g}Lb^4 = \frac{49}{240}\times\frac{0.100\text{ lb/in}^3}{32.2\text{ ft/s}^2}\times(15\text{ in.})(4.2\text{ in.})^4\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

or $(I_{AA'})_{\max} = 20.6\times10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

Now $$k_{AA'}^2 = \frac{(I_{AA'})_{\max}}{m}$$

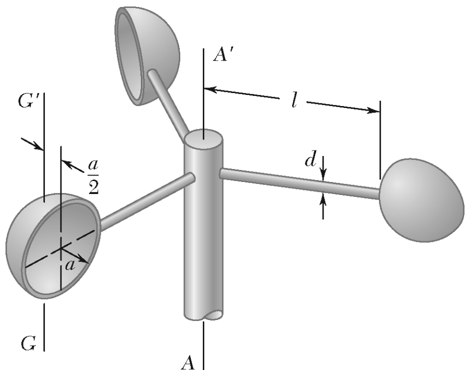
where $$m = m_1 - m_2 = \rho L(b^2 - a^2)$$

$$= \rho L\left[b^2 - \left(b\sqrt{\frac{3}{10}}\right)^2\right]$$

$$= \frac{7}{10}\rho L b^2$$

Then $$k_{AA'}^2 = \frac{\frac{49}{240}\rho L b^4}{\frac{7}{10}\rho L b^2} = \frac{7}{24}b^2 = \frac{7}{24}(4.2\text{ in.})^2$$

or $k_{AA'} = 2.27$ in. ◀

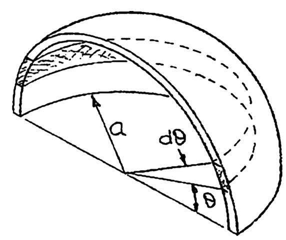


PROBLEM 9.132

The cups and the arms of an anemometer are fabricated from a material of density ρ. Knowing that the mass moment of inertia of a thin, hemispherical shell of mass m and thickness t with respect to its centroidal axis GG' is $5ma^2/12$, determine (*a*) the mass moment of inertia of the anemometer with respect to the axis AA', (*b*) the ratio of a to l for which the centroidal moment of inertia of the cups is equal to 1 percent of the moment of inertia of the cups with respect to the axis AA'.

SOLUTION

(*a*) First note $$m_{\text{arm}} = \rho V_{\text{arm}} = \rho \times \frac{\pi}{4} d^2 l$$



and $$dm_{\text{cup}} = \rho dV_{\text{cup}}$$
$$= \rho[(2\pi a\cos\theta)(t)(a\,d\theta)]$$

Then $$m_{\text{cup}} = \int dm_{\text{cup}} = \int_0^{\pi/2} 2\pi\rho a^2 t\cos\theta\, d\theta$$
$$= 2\pi\rho a^2 t[\sin\theta]_0^{\pi/2}$$
$$= 2\pi\rho a^2 t$$

Now $$(I_{AA'})_{\text{anem}} = (I_{AA'})_{\text{cups}} + (I_{AA'})_{\text{arms}}$$

Using the parallel-axis theorem and assuming the arms are slender rods, we have

$$(I_{AA'})_{\text{anem}} = 3\left[(I_{GG'})_{\text{cup}} + m_{\text{cup}} d_{AG}^2\right] + 3\left[\bar{I}_{\text{arm}} + m_{\text{arm}} d_{AG_{\text{arm}}}\right]$$
$$= 3\left\{\frac{5}{12} m_{\text{cup}} a^2 + m_{\text{cup}}\left[(l+a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} + 3\left[\frac{1}{2} m_{\text{arm}} l^2 + m_{\text{arm}}\left(\frac{l}{2}\right)^2\right]$$
$$= 3m_{\text{cup}}\left(\frac{5}{3}a^2 + 2la + l^2\right) + m_{\text{arm}} l^2$$
$$= 3(2\pi\rho a^2 t)\left(\frac{5}{3}a^2 + 2la + l^2\right) + \left(\frac{\pi}{4}\rho d^2 l\right)(l^2)$$

or $$(I_{AA'})_{\text{anem}} = \pi\rho l^2\left[6a^2 t\left(\frac{5}{3}\frac{a^2}{l^2} + 2\frac{a}{l} + 1\right) + \frac{d^2 l}{4}\right] \blacktriangleleft$$

PROBLEM 9.132 (Continued)

(*b*) We have $$\frac{(I_{GG'})_{\text{cup}}}{(I_{AA'})_{\text{cup}}} = 0.01$$

or $$\frac{5}{12}m_{\text{cup}}a^2 = 0.01m_{\text{cup}}\left(\frac{5}{3}a^2 + 2la + l^2\right) \quad \text{(from part } a\text{)}$$

Now let $\zeta = \dfrac{a}{l}$.

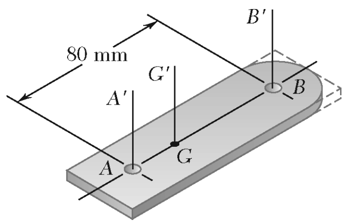
Then $$5\zeta^2 = 0.12\left(\frac{5}{3}\zeta^2 + 2\zeta + 1\right)$$

or $$40\zeta^2 - 2\zeta - 1 = 0$$

Then $$\zeta = \frac{2 \pm \sqrt{(-2)^2 - 4(40)(-1)}}{2(40)}$$

or $$\zeta = 0.1851 \quad \text{and} \quad \zeta = -0.1351$$

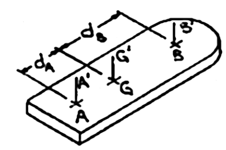
$$\frac{a}{l} = 0.1851 \blacktriangleleft$$



PROBLEM 9.133

After a period of use, one of the blades of a shredder has been worn to the shape shown and is of mass 0.18 kg. Knowing that the mass moments of inertia of the blade with respect to the AA' and BB' axes are $0.320\ \text{g}\cdot\text{m}^2$ and $0.680\ \text{g}\cdot\text{m}^2$, respectively, determine (*a*) the location of the centroidal axis GG', (*b*) the radius of gyration with respect to axis GG'.

SOLUTION



(*a*) We have $\qquad d_B = (0.08 - d_A)\ \text{m}$

and, using the parallel axis

$$I_{AA'} = \bar{I}_{GG'} + md_A^2$$
$$I_{BB'} = \bar{I}_{GG'} + md_B^2$$

Then
$$I_{BB'} - I_{AA'} = m\left(d_B^2 - d_A^2\right)$$
$$= m\left[(0.08 - d_A)^2 - d_A^2\right]$$
$$= m(0.0064 - 0.16d_A)$$

Substituting:
$$(0.68 - 0.32)\times 10^{-3}\ \text{kg}\cdot\text{m}^2 = 0.18\ \text{kg}(0.0064 - 0.16d_A)\ \text{m}^2$$

or $\qquad d_A = 27.5\ \text{mm}$ ◀

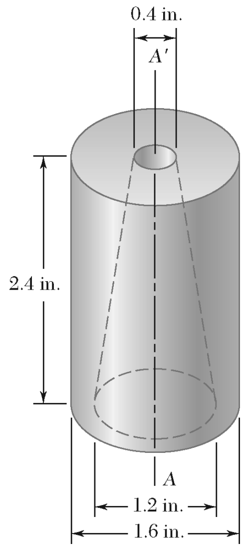
to the right of A

(*b*) We have $\qquad I_{AA'} = I_{GG'} + md_A^2$

or
$$I_{GG'} = 0.32\times 10^{-3}\ \text{kg}\cdot\text{m}^2 - 0.18\ \text{kg}\cdot(0.0275\ \text{m})^2$$
$$= 0.183875\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Then
$$k_{GG'}^2 = \frac{I_{GG'}}{m} = \frac{0.183875\times 10^{-3}\ \text{kg}\cdot\text{m}^2}{0.18\ \text{kg}} = 1.02153\times 10^{-3}\ \text{m}^2$$

or $\qquad k_{GG'} = 32.0\ \text{mm}$ ◀

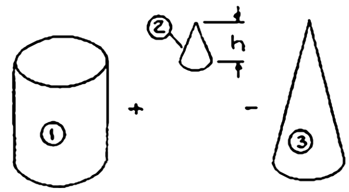


PROBLEM 9.134

Determine the mass moment of inertia of the 0.9-lb machine component shown with respect to the axis AA'.

SOLUTION

First note that the given shape can be formed adding a small cone to a cylinder and then removing a larger cone as indicated.



Now $$\frac{h}{0.4} = \frac{h+2.4}{1.2} \quad \text{or} \quad h = 1.2 \text{ in.}$$

The weight of the body is given by

$$W = mg = g(m_1 + m_2 - m_3) = \rho g(V_1 + V_2 - V_3)$$

or $$0.9 \text{ lb} = \rho \times 32.2 \text{ ft/s}^2$$

$$\left[\pi(0.8)^2(2.4) + \frac{\pi}{3}(0.2)^2(1.2) - \frac{\pi}{3}(0.6)^2(3.6)\right] \text{in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= \rho \times 32.2 \text{ ft/s}^2 (2.79253 + 0.02909 - 0.78540) \times 10^{-3} \text{ ft}^3$$

or $$\rho = 13.7266 \text{ lb}\cdot\text{s}^2/\text{ft}^4$$

Then $$m_1 = (13.7266)(2.79253 \times 10^{-3}) = 0.038332 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_2 = (13.7266)(0.02909 \times 10^{-3}) = 0.000399 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_3 = (13.7266)(0.78540 \times 10^{-3}) = 0.010781 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Finally, using Figure 9.28, we have

$$\begin{aligned}I_{AA'} &= (I_{AA'})_1 + (I_{AA'})_2 - (I_{AA'})_3\\ &= \frac{1}{2}m_1a_1^2 + \frac{3}{10}m_2a_2^2 - \frac{3}{10}m_3a_3^2\\ &= \left[\frac{1}{2}(0.038332)(0.8)^2 + \frac{3}{10}(0.000399)(0.2)^2 - \frac{3}{10}(0.010781)(0.6)^2\right](\text{lb}\cdot\text{s}^2/\text{ft})\times\text{in}^2\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2\\ &= (85.1822 + 0.0333 - 8.0858)\times 10^{-6}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2\end{aligned}$$

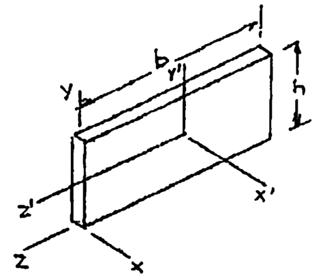
or $I_{AA'} = 77.1\times 10^{-6}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

To the instructor:

The following formulas for the mass moment of inertia of thin plates and a half cylindrical shell are derived at this time for use in the solutions of Problems 9.135 through 9.140.

Thin rectangular plate

$$(I_x)_m = (\bar{I}_{x'})_m + md^2$$
$$= \frac{1}{12}m(b^2+h^2) + m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$
$$= \frac{1}{3}m(b^2+h^2)$$

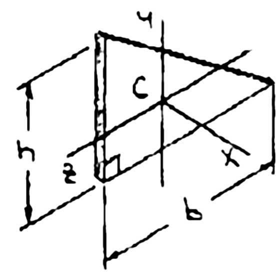


$$(I_y)_m = (\bar{I}_{y'})_m + md^2$$
$$= \frac{1}{12}mb^2 + m\left(\frac{b}{2}\right)^2 = \frac{1}{3}mb^2$$

$$I_z = (\bar{I}_{z'})_m + md^2$$
$$= \frac{1}{12}mh^2 + m\left(\frac{h}{2}\right)^2 = \frac{1}{3}mh^2$$

Thin triangular plate

We have $$m = \rho V = \rho\left(\frac{1}{2}bht\right)$$



and $$\bar{I}_{z,\text{area}} = \frac{1}{36}bh^3$$

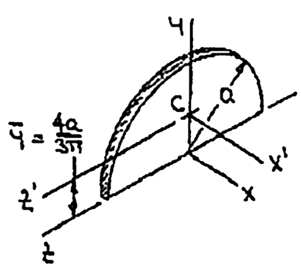
Then $$\bar{I}_{z,\text{mass}} = \rho t I_{z,\text{area}}$$
$$= \rho t \times \frac{1}{36}bh^3$$
$$= \frac{1}{18}mh^2$$

Similarly, $$\bar{I}_{y,\text{mass}} = \frac{1}{18}mb^2$$

Now $$\bar{I}_{x,\text{mass}} = \bar{I}_{y,\text{mass}} + \bar{I}_{z,\text{mass}} = \frac{1}{18}m(b^2+h^2)$$

Thin semicircular plate

We have $$m = \rho V = \rho\left(\frac{\pi}{2}a^2 t\right)$$



and $$\bar{I}_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{8}a^4$$

(Continued)

Then $$\bar{I}_{y,\text{mass}} = I_{z,\text{mass}} = \rho t \bar{I}_{y,\text{area}}$$
$$= \rho t \times \frac{\pi}{8} a^4$$
$$= \frac{1}{4} ma^2$$

Now $$I_{x,\text{mass}} = \bar{I}_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} ma^2$$

Also $$I_{x,\text{mass}} = \bar{I}_{x',\text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{x',\text{mass}} = m\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$$

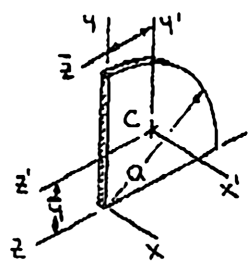
and $$I_{z,\text{mass}} = \bar{I}_{z',\text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{z',\text{mass}} = m\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2$$

$$\bar{y} = \bar{z} = \frac{4a}{3\pi}$$

Thin Quarter-Circular Plate

We have $$m = \rho V = \rho\left(\frac{\pi}{4} a^2 t\right)$$

and $$I_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{16} a^4$$



Then $$I_{y,\text{mass}} = I_{z,\text{mass}} = \rho t I_{y,\text{area}}$$
$$= \rho t \times \frac{\pi}{16} a^4$$
$$= \frac{1}{4} ma^2$$

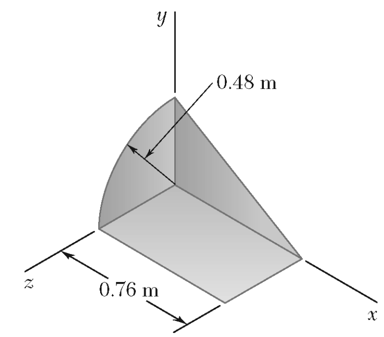
Now $$I_{x,\text{mass}} = I_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} ma^2$$

Also $$I_{x,\text{mass}} = I_{x',\text{mass}} + m(\bar{y}^2 + \bar{z}^2)$$

or $$\bar{I}_{x',\text{mass}} = m\left(\frac{1}{2} - \frac{32}{9\pi^2}\right)a^2$$

and $$I_{y,\text{mass}} = I_{y',\text{mass}} + m\bar{z}^2$$

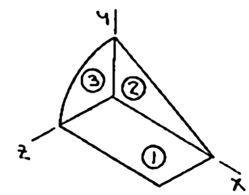
or $$\bar{I}_{y',\text{mass}} = m\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2$$



PROBLEM 9.135

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{ST}V = \rho_{ST}tA$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.76 \times 0.48)\ \text{m}^2$$
$$= 5.72736\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{1}{2} \times 0.76 \times 0.48\right)\text{m}^2$$
$$= 2.86368\ \text{kg}$$

$$m_3 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{\pi}{4} \times 0.48^2\right)\text{m}^2$$
$$= 2.84101\ \text{kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$
$$= \left[\frac{1}{12}(5.72736\ \text{kg})(0.48\ \text{m})^2 + (5.72736\ \text{kg})\left(\frac{0.48}{2}\ \text{m}\right)^2\right]$$
$$+ \left[\frac{1}{18}(2.86368\ \text{kg})(0.48\ \text{m})^2 + (2.86368\ \text{kg})\left(\frac{0.48}{3}\ \text{m}\right)^2\right] + \left[\frac{1}{2}(2.84101\ \text{kg})(0.48\ \text{m})^2\right]$$
$$= [(0.109965 + 0.329896) + (0.036655 + 0.073310) + (0.327284)]\ \text{kg}\cdot\text{m}^2$$
$$= (0.439861 + 0.109965 + 0.327284)\ \text{kg}\cdot\text{m}^2$$

or $I_x = 0.877\ \text{kg}\cdot\text{m}^2$ ◀

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

$$= \left\{\frac{1}{12}(5.72736\text{ kg})(0.76^2 + 0.48^2)\text{ m}^2 + (5.72736\text{ kg})\left[\left(\frac{0.76}{2}\right)^2 + \left(\frac{0.48}{2}\right)^2\right]\text{m}^2\right\}$$

$$+\left[\frac{1}{18}(2.86368\text{ kg})(0.76\text{ m})^2 + (2.86368\text{ kg})\left(\frac{0.76}{3}\text{ m}\right)^2\right] + \left[\frac{1}{4}(2.84101\text{ kg})(0.48\text{ m})^2\right]$$

$$= [(0.385642 + 1.156927) + (0.091892 + 0.183785) + (0.163642)]\text{ kg}\cdot\text{m}^2$$

$$= (1.542590 + 0.275677 + 0.163642)\text{ kg}\cdot\text{m}^2$$

or $\quad I_y = 1.982\text{ kg}\cdot\text{m}^2$ ◀

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3$$

$$= \left[\frac{1}{12}(5.72736\text{ kg})(0.76\text{ m})^2 + (5.72736\text{ kg})\left(\frac{0.76}{2}\text{ m}\right)^2\right]$$

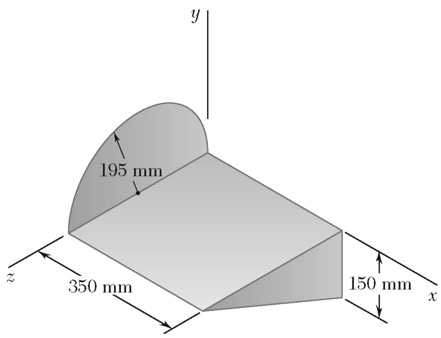
$$+\left\{\frac{1}{18}(2.86368\text{ kg})(0.76^2 + 0.48^2)\text{ m}^2 + (2.86368\text{ kg})\left[\left(\frac{0.76}{3}\right)^2 + \left(\frac{0.48}{3}\right)^2\right]\text{m}^2\right\}$$

$$+\left[\frac{1}{4}(2.84101\text{ kg})(0.48\text{ m})^2\right]$$

$$I_z = [(0.275677 + 0.827031) + (0.128548 + 0.257095) + (0.163642)]\text{ kg}\cdot\text{m}^2$$

$$= (1.102708 + 0.385643 + 0.163642)\text{ kg}\cdot\text{m}^2$$

or $\quad I_z = 1.652\text{ kg}\cdot\text{m}^2$ ◀



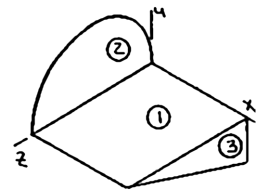
PROBLEM 9.136

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{ST}V = \rho_{ST}tA$$



Then

$$m_1 = (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.35 \times 0.39) \text{ m}^2$$
$$= 2.14305 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{2} \times 0.195^2\right)\text{m}^2 = 0.93775 \text{ kg}$$

$$m_3 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{1}{2} \times 0.39 \times 0.15\right)\text{m}^2 = 0.45923 \text{ kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

$$= \left[\frac{1}{12}(2.14305 \text{ kg})(0.39 \text{ m})^2 + (2.14305 \text{ kg})\left(\frac{0.39}{2} \text{ m}\right)^2\right]$$

$$+ \left\{\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.93775 \text{ kg})(0.195 \text{ m})^2 + (0.93775 \text{ kg})\left[\left(\frac{4 \times 0.195}{3\pi}\right)^2 + (0.195)^2\right]\text{m}^2\right\}$$

$$+ \left\{\frac{1}{18}(0.45923 \text{ kg})[(0.39)^2 + (0.15)^2]\text{ m}^2 + (0.45923 \text{ kg})\left[\left(\frac{0.39}{3}\right)^2 + \left(\frac{0.15}{3}\right)^2\right]\text{m}^2\right\}$$

$$= [(0.027163 + 0.081489) + (0.011406 + 0.042081) + (0.004455 + 0.008909)] \text{ kg} \cdot \text{m}^2$$

$$= (0.108652 + 0.053487 + 0.013364) \text{ kg} \cdot \text{m}^2$$

$$= 0.175503 \text{ kg} \cdot \text{m}^2$$

or $\quad I_x = 175.5 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.136 (Continued)

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

$$= \left\{\frac{1}{12}(2.14305\text{ kg})[(0.35)^2 + (0.39)^2]\text{ m}^2 + (2.14305\text{ kg})\left[\left(\frac{0.35}{2}\right)^2 + \left(\frac{0.39}{2}\right)^2\right]\text{ m}^2\right\}$$

$$+\left[\frac{1}{4}(0.93775\text{ kg})(0.195\text{ m})^2 + (0.93775\text{ kg})(0.195\text{ m})^2\right]$$

$$+\left\{\frac{1}{18}(0.45923\text{ kg})(0.39\text{ m})^2 + (0.45923\text{ kg})\left[(0.35)^2 + \left(\frac{0.39}{3}\right)^2\right]\text{ m}^2\right\}$$

$$= [(0.049040 + 0.147120) + (0.008914 + 0.035658)$$

$$+ (0.003880 + 0.064017)]\text{ kg}\cdot\text{m}^2$$

$$= (0.196160 + 0.044572 + 0.067897)\text{ kg}\cdot\text{m}^2$$

$$= 0.308629\text{ kg}\cdot\text{m}^2$$

or $\quad I_y = 309\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3$$

$$= \left[\frac{1}{12}(2.14305\text{ kg})(0.35\text{ m})^2 + (2.14305\text{ kg})\left(\frac{0.35}{2}\text{ m}\right)^2\right]$$

$$+\left[\frac{1}{4}(0.93775\text{ kg})(0.195\text{ m})^2\right]$$

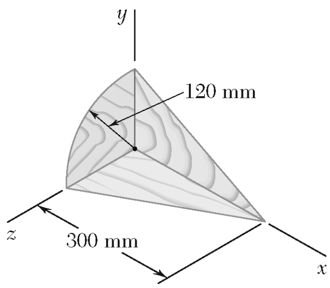
$$+\left\{\frac{1}{18}(0.45923\text{ kg})(0.15\text{ m})^2 + (0.45923\text{ kg})\left[(0.35)^2 + \left(\frac{0.15}{3}\right)^2\right]\text{ m}^2\right\}$$

$$= [(0.021877 + 0.065631) + 0.008914) + (0.000574 + 0.057404)]\text{ kg}\cdot\text{m}^2$$

$$= (0.087508 + 0.008914 + 0.057978)\text{ kg}\cdot\text{m}^2$$

$$= 0.154400\text{ kg}\cdot\text{m}^2$$

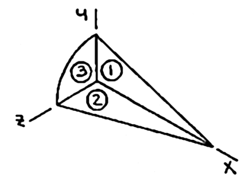
or $\quad I_z = 154.4\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM 9.137

A subassembly for a model airplane is fabricated from three pieces of 1.5-mm plywood. Neglecting the mass of the adhesive used to assemble the three pieces, determine the mass moment of inertia of the subassembly with respect to each of the coordinate axes. (The density of the plywood is $780\ \text{kg/m}^3$.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \rho t A$$

Then

$$m_1 = m_2 = (780\ \text{kg/m}^3)(0.0015\ \text{m})\left(\frac{1}{2}\times 0.3\times 0.12\right)\text{m}^2$$

$$= 21.0600\times 10^{-3}\ \text{kg}$$

$$m_3 = (780\ \text{kg/m}^3)(0.0015\ \text{m})\left(\frac{\pi}{4}\times 0.12^2\right)\text{m}^2 = 13.2324\times 10^{-3}\ \text{kg}$$

Using the equations derived above and the parallel-axis theorem, we have

$$(I_x)_1 = (I_x)_2$$

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

$$= 2\left[\frac{1}{18}(21.0600\times 10^{-3}\ \text{kg})(0.12\ \text{m})^2 + (21.0600\times 10^{-3}\ \text{kg})\left(\frac{0.12}{3}\ \text{m}\right)^2\right]$$

$$+\left[\frac{1}{2}(13.2324\times 10^{-3}\ \text{kg})(0.12\ \text{m})^2\right]$$

$$= [2(16.8480 + 33.6960) + (95.2733)]\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$= [2(50.5440) + (95.2733)]\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

or $\quad I_x = 196.4\times 10^{-6}\ \text{kg}\cdot\text{m}^2$ ◀

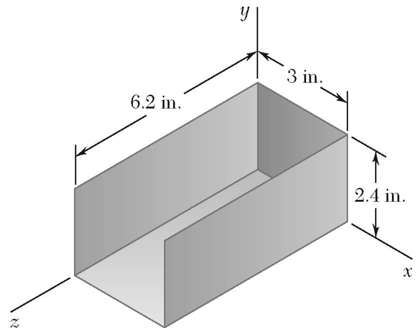
PROBLEM 9.137 (Continued)

$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
&= \left[\frac{1}{18}(21.0600\times 10^{-3}\text{ kg})(0.3\text{ m})^2 + (21.0600\text{ kg})\left(\frac{0.3}{3}\text{m}\right)^2\right] \\
&\quad + \left\{\frac{1}{18}(21.0600\times 10^{-3}\text{ kg})[(0.3)^2 + (0.12)^2]\text{ m}^2\right. \\
&\quad \left. + (21.0600\times 10^{-3}\text{ kg})\left[\left(\frac{0.3}{3}\right)^2 + \left(\frac{0.12}{3}\right)^2\right]\text{m}^2\right\} \\
&\quad + \left[\frac{1}{4}(13.2324\times 10^{-3}\text{ kg})(0.12\text{ m})^2\right] \\
&= [(105.300 + 210.600) + (122.148 + 244.296) \\
&\quad + (47.637)]\times 10^{-6}\text{ kg}\cdot\text{m}^2 \\
&= (315.900 + 366.444 + 47.637)\times 10^{-6}\text{ kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_y = 730\times 10^{-6}\text{ kg}\cdot\text{m}^2$ ◀

Symmetry implies $I_y = I_z$

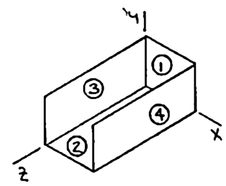
$I_z = 730\times 10^{-6}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM 9.138

The cover for an electronic device is formed from sheet aluminum that is 0.05 in. thick. Determine the mass moment of inertia of the cover with respect to each of the coordinate axes. (The specific weight of aluminum is 0.100 lb/in^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \frac{\gamma}{g} tA$$

Then
$$m_1 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (3 \times 2.4) \text{ in}^2 = 1.11801 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (3 \times 6.2) \text{ in}^2 = 2.88820 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = m_4 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.4 \times 6.2) \text{ in}^2 = 2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$(I_x)_3 = (I_x)_4$$

$$\begin{aligned}
I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 \\
&= \left[\frac{1}{12}(1.11801 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(2.4 \text{ in.})^2\right. \\
&\quad \left. + (1.11801 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{2.4}{2} \text{ in.}\right)^2\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2 \\
&\quad + \left[\frac{1}{12}(2.88820 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(6.2 \text{ in.})^2\right. \\
&\quad \left. + (2.88820 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{6.2}{2} \text{ in.}\right)^2\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2 \\
&\quad + 2\left\{\frac{1}{12}(2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left[(2.4)^2 + (6.2)^2\right] \text{in}^2\right. \\
&\quad \left. + (2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left[\left(\frac{2.4}{2}\right)^2 + \left(\frac{6.2}{2}\right)^2\right] \text{in}^2\right\}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2
\end{aligned}$$

$$= [(3.7267 + 11.1801) + (64.2491 + 192.7472) + 2(59.1011 + 177.3034)] \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= [14.9068 + 256.9963 + 2(236.4045)] \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

or $I_x = 745 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4$$

$$= \left[\frac{1}{12}(1.11801 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(3 \text{ in.})^2 + (1.11801 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{3}{2} \text{ in.}\right)^2\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left\{\frac{1}{12}(2.88820 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})[(3)^2 + (6.2)^2] \text{ in}^2 + (2.88820 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left[\left(\frac{3}{2}\right)^2 + \left(\frac{6.2}{2}\right)^2\right] \text{in}^2\right\}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left[\frac{1}{12}(2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(6.2 \text{ in.})^2 + (2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{6.2}{2} \text{ in.}\right)^2\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left\{\frac{1}{12}(2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(6.2 \text{ in.})^2 + (2.31056 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left[(3)^2 + \left(\frac{6.2}{2}\right)^2\right] \text{in}^2\right\}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$= [(5.8230 + 17.4689) + (79.2918 + 237.8754) + (51.3993 + 154.1978) + (51.3993 + 298.6078)] \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

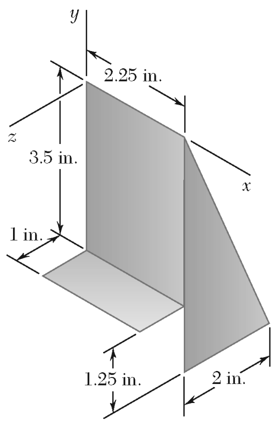
$$= (23.2919 + 317.1672 + 205.5971 + 350.0071) \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

or $I_y = 896 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4 \\
&= \left\{\frac{1}{12}(1.11801\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(3)^2 + (2.4)^2]\ \text{in}^2 \right. \\
&\quad \left. + (1.11801\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{3}{2}\right)^2 + \left(\frac{2.4}{2}\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \left[\frac{1}{12}(2.88820\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(3\ \text{in.})^2 \right. \\
&\quad \left. + (2.88820\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{3}{2}\ \text{in.}\right)^2\right]\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \left[\frac{1}{12}(2.31056\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.4\ \text{in.})^2 \right. \\
&\quad \left. + (2.31056\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\ \text{ft})\left(\frac{2.4}{2}\text{in.}\right)^2\right]\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \left\{\frac{1}{12}(2.31056\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.4\ \text{in.})^2 \right. \\
&\quad \left. + (2.31056\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(3)^2 + \left(\frac{2.4}{2}\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&= [(9.5497 + 28.6490) + (15.0427 + 45.1281) \\
&\quad + (7.7019 + 23.1056) + (7.7019 + 167.5156)]\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\
&= (38.1987 + 60.1708 + 30.8075 + 175.2175)\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2
\end{aligned}
$$

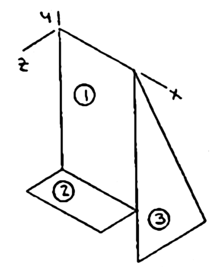
or $I_z = 304\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.139



A framing anchor is formed of 0.05-in.-thick galvanized steel. Determine the mass moment of inertia of the anchor with respect to each of the coordinate axes. (The specific weight of galvanized steel is 470 lb/ft^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \frac{\gamma_{\text{G.S.}}}{g} tA$$

Then

$$m_1 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.25 \times 3.5) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.25 \times 1) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times \left(\frac{1}{2} \times 2 \times 4.75\right) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 2006.14 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$
$$= \left[\frac{1}{12}(3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})(3.5 \text{ in.})^2\right.$$
$$\left. + (3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{3.5}{2} \text{ in}^2\right)\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$
$$+ \left\{\frac{1}{12}(950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})(1 \text{ in.})^2\right.$$
$$\left. + (950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})\left[(3.5)^2 + \left(\frac{1}{2}\right)^2\right] \text{ in}^2\right\}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+\left\{\frac{1}{18}(2006.14\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})[(4.75)^2+(2)^2]\text{ in}^2\right.$$

$$\left.+(2006.14\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2}{3}\times4.75\right)^2+\left(\frac{1}{3}\times2\right)^2\right]\text{ in}^2\right\}\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$=[(23.578+70.735)+(0.550+82.490)$$

$$+(20.559+145.894)]\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=(94.313+83.040+166.453)\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_x=344\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_y=(I_y)_1+(I_y)_2+(I_y)_3$$

$$=\left[\frac{1}{12}(3325.97\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})(2.25\text{ in.})^2\right.$$

$$\left.+(3325.97\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})\left(\frac{2.25}{2}\text{ in.}\right)^2\right]\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+\left\{\frac{1}{12}(950.28\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})[(2.25)^2+(1)^2]\text{ in}^2\right.$$

$$\left.+(950.28\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]\text{ in}^2\right\}\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+\left\{\frac{1}{18}(2006.14\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})(2\text{ in.})^2\right.$$

$$\left.+(2006.14\times10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft})\left[(2.25)^2+\left(\frac{1}{3}\times2\right)^2\right]\text{ in}^2\right\}\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$=[(9.744+29.232)+(3.334+10.002)$$

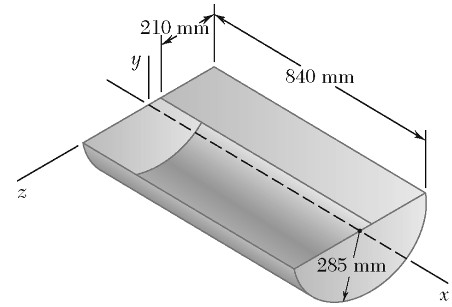
$$+(3.096+76.720)]\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=(38.976+13.336+79.816)\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_y=132.1\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
&= \left\{\frac{1}{12}(3325.97\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.25)^2 + (3.5)^2]\ \text{in}^2 \right. \\
&\quad + (3325.97\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left.\left[\left(\frac{2.25}{2}\right)^2 + \left(\frac{3.5}{2}\right)^2\right]\ \text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \left\{\frac{1}{12}(950.28\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.25\ \text{in.})^2 \right. \\
&\quad + (950.28\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left.\left[\left(\frac{2.25}{2}\right)^2 + (3.5)^2\right]\ \text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \left\{\frac{1}{18}(2006.14\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(4.75\ \text{in.})^2 \right. \\
&\quad + (2006.14\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left.\left[(2.25)^2 + \left(\frac{2}{3}\times 4.75\right)^2\right]\ \text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&= [(33.322 + 99.967) + (2.784 + 89.192) \\
&\quad + (17.463 + 210.231)]\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\
&= (133.289 + 91.976 + 227.694)\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2
\end{aligned}
$$

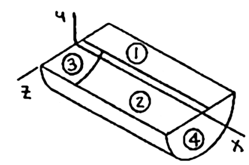
or $\quad I_z = 453\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.140*

A farmer constructs a trough by welding a rectangular piece of 2-mm-thick sheet steel to half of a steel drum. Knowing that the density of steel is $7850\ \text{kg/m}^3$ and that the thickness of the walls of the drum is 1.8 mm, determine the mass moment of inertia of the trough with respect to each of the coordinate axes. Neglect the mass of the welds.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} tA$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.84 \times 0.21)\ \text{m}^2$$
$$= 2.76948\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)(0.0018\ \text{m})(\pi \times 0.285 \times 0.84)\ \text{m}^2$$
$$= 10.62713\ \text{kg}$$

$$m_3 = m_4 = (7850\ \text{kg/m}^3)(0.0018\ \text{m})\left(\frac{\pi}{2} \times 0.285^2\right)\text{m}^2$$
$$= 1.80282\ \text{kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 through 4, we have

$$(I_x)_3 = (I_x)_4$$
$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4$$
$$= \left[\frac{1}{12}(2.76948\ \text{kg})(0.21\ \text{m})^2 + (2.76948\ \text{kg})\left(0.285 - \frac{0.21}{2}\right)^2 \text{m}^2\right]$$
$$+ [(10.62713\ \text{kg})(0.285\ \text{m})^2] + 2\left[\frac{1}{2}(1.80282\ \text{kg})(0.285\ \text{m})^2\right]$$
$$= [(0.01018 + 0.08973) + (0.86319) + 2(0.07322)]\ \text{kg}\cdot\text{m}^2$$
$$= [(0.09991 + 0.86319 + 2(0.07322)]\ \text{kg}\cdot\text{m}^2$$

or $\quad I_x = 1.110\ \text{kg}\cdot\text{m}^2$ ◀

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4$$

$$= \left\{ \frac{1}{12}(2.76948\text{ kg})[(0.84)^2 + (0.21)^2]\text{ m}^2 + (2.76948\text{ kg})\left[\left(\frac{0.84}{2}\right)^2 + \left(0.285 - \frac{0.21}{2}\right)^2\right]\text{m}^2 \right\}$$

$$+ \left\{ \frac{1}{12}(10.62713\text{ kg})[(0.84)^2 + 6(0.285)^2]\text{ m}^2 + (10.62713\text{ kg})\left(\frac{0.84}{2}\text{ m}\right)^2 \right\}$$

$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2\right]$$

$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2 + (1.80282\text{ kg})(0.84\text{ m})^2\right]$$

$$= [(0.17302 + 0.57827) + (1.05647 + 1.87463) + (0.03661) + (0.03661 + 1.27207)]\text{ kg}\cdot\text{m}^2$$

$$= (0.75129 + 2.93110 + 0.03661 + 1.30868)\text{ kg}\cdot\text{m}^2$$

or $\quad I_y = 5.03\text{ kg}\cdot\text{m}^2$ ◀

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4$$

$$= \left[\frac{1}{12}(2.76948\text{ kg})(0.84\text{ m})^2 + (2.76948\text{ kg})\left(\frac{0.84}{2}\text{ m}\right)^2\right]$$

$$+ \left\{ \frac{1}{12}(10.62713\text{ kg})[(0.84)^2 + 6(0.285)^2]\text{ m}^2 + (10.62713\text{ kg})\left(\frac{0.84}{2}\text{ m}\right)^2 \right\}$$

$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2\right]$$

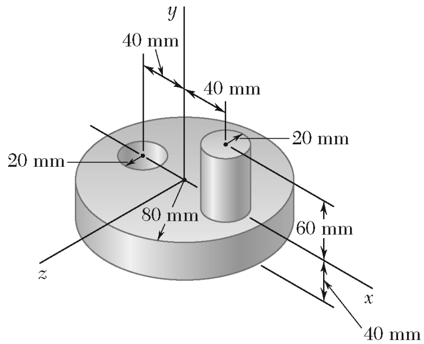
$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2 + (1.80282\text{ kg})(0.84\text{ m})^2\right]$$

$$= [(0.16285 + 0.48854) + (1.05647 + 1.87463) + (0.03661) + (0.03661 + 1.27207)]\text{ kg}\cdot\text{m}^2$$

$$= (0.65139 + 2.93110 + 0.03661 + 1.30868)\text{ kg}\cdot\text{m}^2$$

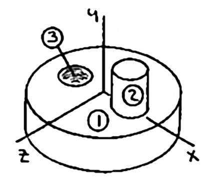
or $\quad I_z = 4.93\text{ kg}\cdot\text{m}^2$ ◀

PROBLEM 9.141



The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (*a*) the *x* axis, (*b*) the *y* axis, (*c*) the *z* axis. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)(\pi(0.08 \text{ m})^2(0.04 \text{ m})] = 6.31334 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.06 \text{ m})] = 0.59188 \text{ kg}$$

$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.04 \text{ m})] = 0.39458 \text{ kg}$$

Using Figure 9.28 and the parallel-axis theorem, we have

(*a*) $$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$

$$= \left\{\frac{1}{12}(6.31334 \text{ kg})[3(0.08)^2 + (0.04)^2] \text{ m}^2 + (6.31334 \text{ kg})(0.02 \text{ m})^2\right\}$$

$$+ \left\{\frac{1}{12}(0.59188 \text{ kg})[3(0.02)^2 + (0.06)^2] \text{ m}^2 + (0.59188 \text{ kg})(0.03 \text{ m})^2\right\}$$

$$- \left\{\frac{1}{12}(0.39458 \text{ kg})[3(0.02)^2 + (0.04)^2] \text{ m}^2 + (0.39458 \text{ kg})(0.02 \text{ m})^2\right\}$$

$$= [(10.94312 + 2.52534) + (0.23675 + 0.53269) - (0.09207 + 0.15783)] \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

$$= (13.46846 + 0.76944 - 0.24990) \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

$$= 13.98800 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

or $I_x = 13.99 \times 10^{-3} \text{ kg}\cdot\text{m}^2$ ◀

(*b*)

$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3$$

$$= \left[\frac{1}{2}(6.31334 \text{ kg})(0.08 \text{ m})^2\right]$$

$$+ \left[\frac{1}{2}(0.59188 \text{ kg})(0.02 \text{ m})^2 + (0.59188 \text{ kg})(0.04 \text{ m})^2\right]$$

$$- \left[\frac{1}{2}(0.39458 \text{ kg})(0.02 \text{ m}^2) + (0.39458 \text{ kg})(0.04 \text{ m})^2\right]$$

$$= [(20.20269) + (0.11838 + 0.94701)$$

$$- (0.07892 + 0.63133)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (20.20269 + 1.06539 - 0.71025) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_y = 20.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

(*c*)

$$I_z = (I_z)_1 + (I_z)_2 - (I_z)_3$$

$$= \left\{\frac{1}{12}(6.31334 \text{ kg})[3(0.08)^2 + (0.04)^2] \text{ m}^2 + (6.31334 \text{ kg})(0.02 \text{ m})^2\right\}$$

$$+ \left\{\frac{1}{12}(0.59188 \text{ kg})[3(0.02)^2 + (0.06)^2] \text{ m}^2 + (0.59188 \text{ kg})[(0.04)^2 + (0.03)^2] \text{ m}^2\right\}$$

$$- \left\{\frac{1}{12}(0.39458 \text{ kg})[3(0.02)^2 + (0.04)^2] \text{ m}^2 + (0.03958 \text{ kg})[(0.04)^2 + (0.02)^2] \text{ m}^2\right\}$$

$$= [(10.94312 + 2.52534) + (0.23675 + 1.47970)$$

$$- (0.09207 + 0.78916)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (13.46846 + 1.71645 - 0.88123) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

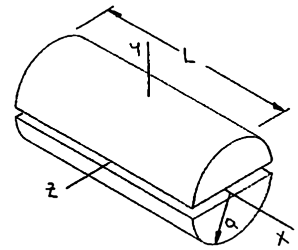
$$= 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_z = 14.30 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

To the Instructor:

The following formulas for the mass of inertia of a semicylinder are derived at this time for use in the solutions of Problems 9.142 through 9.145.

From Figure 9.28:



Cylinder $$(I_x)_{\text{cyl}} = \frac{1}{2} m_{\text{cyl}} a^2$$

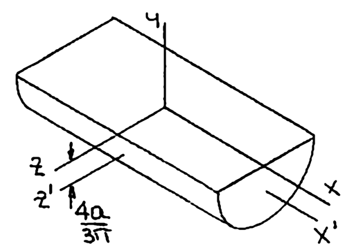
$$(I_y)_{\text{cyl}} = (I_z)_{\text{cyl}} = \frac{1}{12} m_{\text{cyl}} (3a^2 + L^2)$$

Symmetry and the definition of the mass moment of inertia $(I = \int r^2 dm)$ imply

$$(I)_{\text{semicylinder}} = \frac{1}{2}(I)_{\text{cylinder}}$$

$$(I_x)_{\text{sc}} = \frac{1}{2}\left(\frac{1}{2} m_{\text{cyl}} a^2\right)$$

and $$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{2}\left[\frac{1}{12} m_{\text{cyl}} (3a^2 + L^2)\right]$$



However, $$m_{\text{sc}} = \frac{1}{2} m_{\text{cyl}}$$

Thus, $$(I_x)_{\text{sc}} = \frac{1}{2} m_{\text{sc}} a^2$$

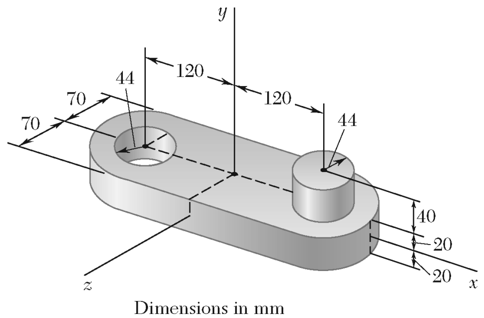
and $$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{12} m_{\text{sc}} (3a^2 + L^2)$$

Also, using the parallel axis theorem find

$$\bar{I}_{x'} = m_{\text{sc}}\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$$

$$\bar{I}_{z'} = m_{\text{sc}}\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2 + \frac{1}{12}L^2\right]$$

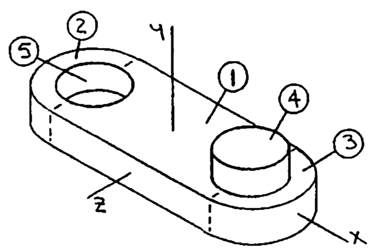
where x' and z' are centroidal axes.



PROBLEM 9.142

Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to the x and y axes. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)(0.24 \times 0.04 \times 0.14) \text{ m}^3$$
$$= 10.5504 \text{ kg}$$

$$m_2 = m_3 = (7850 \text{ kg/m}^3)\left[\frac{\pi}{2}(0.07)^2 \times 0.04\right] \text{m}^3 = 2.41683 \text{ kg}$$

$$m_4 = m_5 = (7850 \text{ kg/m}^3)[\pi(0.044)^2 \times (0.04)] \text{ m}^3 = 1.90979 \text{ kg}$$

Using Figure 9.28 for components 1, 4, and 5 and the equations derived above (before the solution to Problem 9.144) for a semicylinder, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 - (I_x)_5 \quad \text{where} \quad (I_x)_2 = (I_x)_3$$
$$= \left[\frac{1}{12}(10.5504 \text{ kg})(0.04^2 + 0.14^2) \text{ m}^2\right] + 2\left\{\frac{1}{12}(2.41683 \text{ kg})[3(0.07 \text{ m})^2 + (0.04 \text{ m})^2]\right\}$$
$$+ \left\{\frac{1}{12}(1.90979 \text{ kg})[3(0.044 \text{ m})^2 + (0.04 \text{ m})^2] + (1.90979 \text{ kg})(0.04 \text{ m})^2\right\}$$
$$- \left\{\frac{1}{12}(1.90979 \text{ kg})[3(0.044 \text{ m})^2 + (0.04 \text{ m})^2]\right\}$$
$$= [(0.0186390) + 2(0.0032829) + (0.0011790 + 0.0030557) - (0.0011790)] \text{ kg} \cdot \text{m}^2$$
$$= 0.0282605 \text{ kg} \cdot \text{m}^2$$

or $\quad I_x = 28.3 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.142 (Continued)

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 - (I_y)_5$$

where $$(I_y)_2 = (I_y)_3 \quad (I_y)_4 = |(I_y)_5|$$

Then $$I_y = \left[\frac{1}{12}(10.5504 \text{ kg})(0.24^2 + 0.14^2) \text{ m}^2\right]$$

$$+2\left[(2.41683 \text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.07 \text{ m}^2) + (2.41683 \text{ kg})\left(0.12 + \frac{4\times 0.07}{3\pi}\right)^2 \text{ m}^2\right]$$

$$= [(0.0678742) + 2(0.0037881 + 0.0541678)] \text{ kg}\cdot\text{m}^2$$

$$= 0.1837860 \text{ kg}\cdot\text{m}^2$$

or $I_y = 183.8\times 10^{-3} \text{ kg}\cdot\text{m}^2$ ◀

Also $$m = m_1 + m_2 + m_3 + m_4 - m_5 \text{ where } m_2 = m_3,\ \ m_4 = |m_5|$$

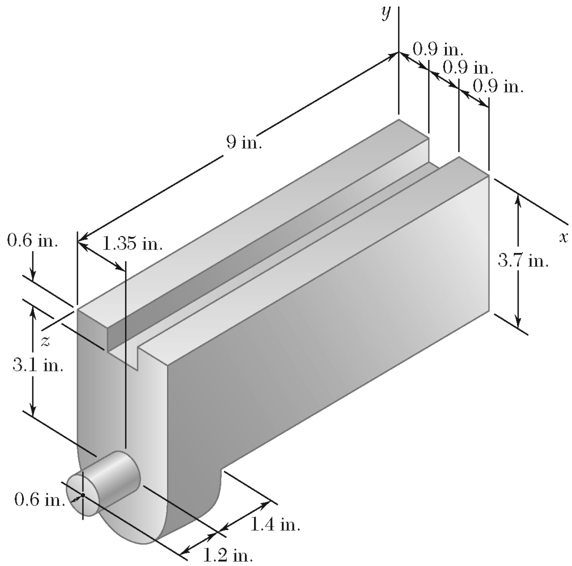
$$= (10.5504 + 2\times 2.41683) \text{ kg} = 15.38406 \text{ kg}$$

Then $$k_x^2 = \frac{I_x}{m} = \frac{0.0282605 \text{ kg}\cdot\text{m}^2}{15.38406 \text{ kg}}$$

or $k_x = 42.9 \text{ mm}$ ◀

and $$k_y^2 = \frac{I_y}{m} = \frac{0.1837860 \text{ kg}\cdot\text{m}^2}{15.38406 \text{ kg}}$$

or $k_y = 109.3 \text{ mm}$ ◀

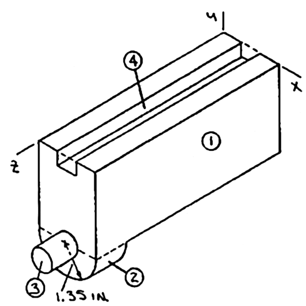


PROBLEM 9.143

Determine the mass moment of inertia of the steel machine element shown with respect to the y axis. (The specific weight of steel is 490 lb/ft^3.)

SOLUTION

First compute the mass of each component. We have



$$m = \rho_{ST} V = \frac{\gamma_{ST}}{g} V$$

Then

$$m_1 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times (2.7 \times 3.7 \times 9) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 791.780 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times \left(\frac{\pi}{2} \times 1.35^2 \times 1.4\right) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 35.295 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times (\pi \times 0.6^2 \times 1.2) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 11.952 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

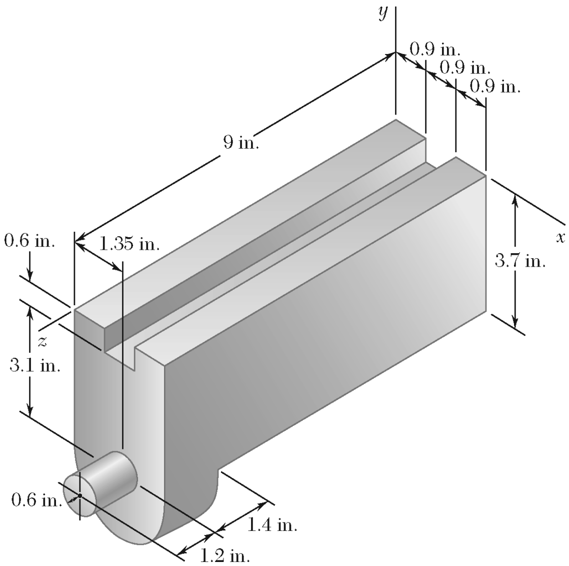
$$m_4 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times (0.9 \times 0.6 \times 9) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3 = 42.799 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

The mass moments of inertia are now computed using Figure 9.28 (components 1, 3, and 4) and the equations derived above (component 2).

Find: I_y

We have
$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 - (I_y)_4$$
$$= \left\{\frac{1}{12}(791.780\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.7)^2+(9)^2]\ \text{in}^2\right.$$
$$\left.+(791.780\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.7}{2}\right)^2+\left(\frac{9}{2}\right)^2\right]\text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$+\left\{\frac{1}{12}(35.295\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[3(1.35)^2+(1.4)^2]\ \text{in}^2\right.$$
$$\left.+(35.295\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(1.35)^2+\left(9-\frac{1.4}{2}\right)^2\right]\text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$+\left\{\frac{1}{12}(11.952\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[3(0.6)^2+(1.2)^2]\ \text{in}^2\right.$$
$$\left.+(11.952\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(1.35)^2+\left(9+\frac{1.2}{2}\right)^2\ \text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$-\left\{\frac{1}{12}(42.799\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(0.9)^2+(9)^2]\ \text{in}^2\right.$$
$$\left.+(42.799\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(1.35)^2+\left(\frac{9}{2}\right)^2\right]\text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$=[(40.4550+121.3650)+(0.1517+17.3319)$$
$$+(0.0174+7.8005)-(2.0263+6.5603)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$=(161.8200+17.4836+7.8179-8.5866)\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_y = 0.1785\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

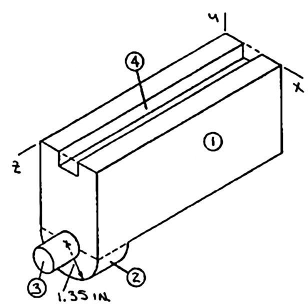


PROBLEM 9.144

Determine the mass moment of inertia of the steel machine element shown with respect to the z axis. (The specific weight of steel is 490 lb/ft^3.)

SOLUTION

First compute the mass of each component. We have



$$m = \rho_{ST}V = \frac{\gamma_{ST}}{g}V$$

Then

$$m_1 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times (2.7 \times 3.7 \times 9) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 791.780 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times \left(\frac{\pi}{2} \times 1.35^2 \times 1.4\right) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 35.295 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times (\pi \times 0.6^2 \times 1.2) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 11.952 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_4 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times (0.9 \times 0.6 \times 9) \text{ in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3 = 42.799 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

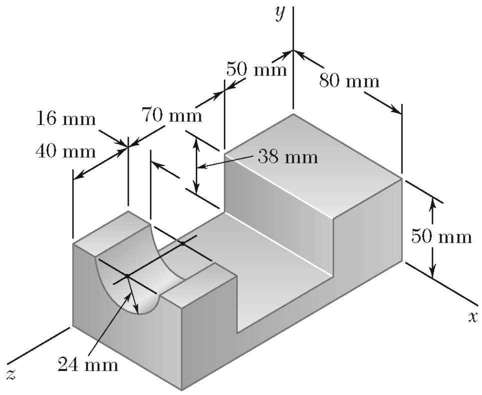
The mass moments of inertia are now computed using Figure 9.28 (components 1, 3, and 4) and the equations derived above (component 2).

PROBLEM 9.144 (Continued)

Find: I_z

We have
$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 - (I_z)_4$$
$$= \left\{ \frac{1}{12}(791.780\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.7)^2 + (3.7)^2]\ \text{in}^2 \right.$$
$$\left. + (791.780\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.7}{2}\right)^2 + \left(\frac{3.7}{2}\right)^2\right] \text{in}^2 \right\} \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$+ \left\{ (35.295\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(1.35\ \text{in.})^2 \right.$$
$$\left. +(35.295\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(1.35)^2 + \left(3.7 + \frac{4\times 1.35}{3\pi}\right)^2\right] \text{in}^2 \right\} \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$+ \left\{ \frac{1}{12}(11.952\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(0.6\ \text{in.})^2 \right.$$
$$\left. +(11.952\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(1.35)^2 + (3.7)^2]\ \text{in}^2 \right\} \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$- \left\{ \frac{1}{12}(42.799\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})[(0.9)^2 + (0.6)^2]\ \text{in}^2 \right.$$
$$\left. +(42.799\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(1.35)^2 + \left(\frac{0.6}{2}\right)^2\right] \text{in}^2 \right\} \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$
$$I_z = [(9.6132 + 28.8395) + (0.1429 + 4.9219)$$
$$+ (0.0149 + 1.2875) - (0.0290 + 0.5684)] \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$= (38.4527 + 5.0648 + 1.3024 - 0.5974) \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

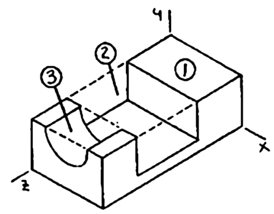
or $I_z = 0.0442\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.145

Determine the mass moment of inertia of the steel fixture shown with respect to (*a*) the *x* axis, (*b*) the *y* axis, (*c*) the *z* axis. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then $$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.160) \text{ m}^3$$
$$= 5.02400 \text{ kg}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \text{m}^3 = 0.28410 \text{ kg}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

(*a*) $I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$

$$= \left\{\frac{1}{12}(5.02400 \text{ kg})[(0.05)^2 + (0.16)^2] \text{ m}^2 + (5.02400 \text{ kg})\left[\left(\frac{0.05}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right] \text{m}^2\right\}$$

$$- \left\{\frac{1}{12}(1.67048 \text{ kg})[(0.038)^2 + (0.07)^2] \text{ m}^2 + (1.67048 \text{ kg})\left[\left(0.05 - \frac{0.038}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right] \text{m}^2\right\}$$

$$- \left\{(0.28410 \text{ kg})\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)(0.024)^2 + \frac{1}{12}(0.04)^2\right] \text{m}^2\right.$$

$$\left. + (0.28410 \text{ kg})\left[\left(0.05 - \frac{4 \times 0.024}{3\pi}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right] \text{m}^2\right\}$$

$$= [(11.7645 + 35.2936) - (0.8831 + 13.6745) - (0.0493 + 6.0187)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (47.0581 - 14.5576 - 6.0680) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 26.4325 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_x = 26.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

(*b*) $I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.16)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.07)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(0.28410\text{ kg})[3(0.024)^2 + (0.04)^2]\text{ m}^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right]\text{m}^2\right\}$$

$$= [(13.3973 + 40.1920) - (1.5730 + 14.7420) - (0.0788 + 6.0229)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (53.5893 - 16.3150 - 6.1017) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 31.1726 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_y = 31.2 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

(*c*) $I_z = (I_z)_1 - (I_z)_2 - (I_z)_3$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.05)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.05}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.038)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{0.038}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{(0.28410\text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.024\text{ m})^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{4\times 0.024}{3\pi}\right)^2\right]\text{m}^2\right\}$$

$$= [(3.7261 + 11.1784) - (1.0919 + 4.2781) - (0.0523 + 0.9049)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (14.9045 - 5.3700 - 0.9572) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

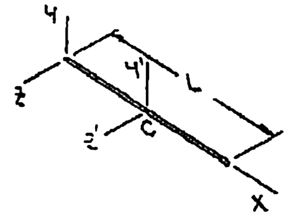
$$= 8.5773 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_z = 8.58 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

To the instructor:

The following formulas for the mass moment of inertia of wires are derived or summarized at this time for use in the solutions of Problems 9.146 through 9.148.

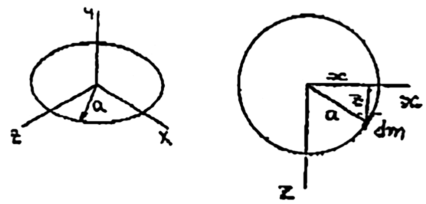
Slender Rod



$$I_x = 0 \qquad \bar{I}_{y'} = \bar{I}_{z'} = \frac{1}{12}mL^2 \text{ (Figure 9.28)}$$

$$I_y = I_z = \frac{1}{3}mL^2 \text{ (Sample Problem 9.9)}$$

Circle



We have $\qquad \bar{I}_y = \int r^2 dm = ma^2$

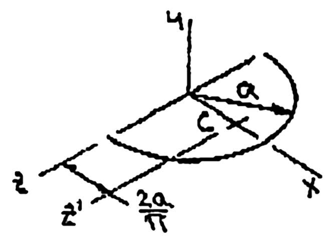
Now $\qquad \bar{I}_y = \bar{I}_x + \bar{I}_z$

And symmetry implies $\qquad \bar{I}_x = \bar{I}_z$

$$\bar{I}_x = \bar{I}_z = \frac{1}{2}ma^2$$

Semicircle

Following the above arguments for a circle, We have

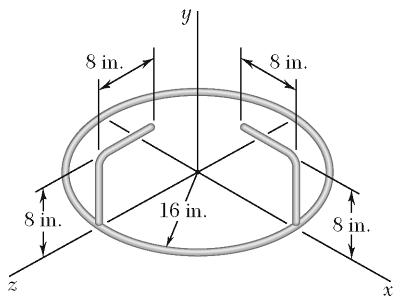


$$\bar{I}_x = I_z = \frac{1}{2}ma^2 \qquad I_y = ma^2$$

Using the parallel-axis theorem

$$I_z = \bar{I}_{z'} + m\bar{x}^2 \qquad x = \frac{2a}{\pi}$$

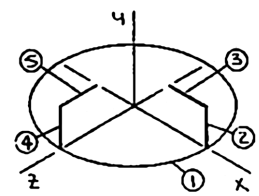
or $\qquad I_{z'} = m\left(\frac{1}{2} - \frac{4}{\pi^2}\right)a^2$



PROBLEM 9.146

Aluminum wire with a weight per unit length of 0.033 lb/ft is used to form the circle and the straight members of the figure shown. Determine the mass moment of inertia of the assembly with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g}\left(\frac{W}{L}\right)_{\text{AL}} L$$

Then

$$m_1 = \frac{1}{32.2\ \text{ft/s}^2} \times 0.033\ \text{lb/ft} \times (2\pi \times 16\ \text{in.}) \times \frac{1\ \text{ft}}{12\ \text{in.}}$$

$$= 8.5857 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_2 = m_3 = m_4 = m_5 = \frac{1}{32.2\ \text{ft/s}^2} \times 0.033\ \text{lb/ft} \times 8\ \text{in.} \times \frac{1\ \text{ft}}{12\ \text{in.}}$$

$$= 0.6832\ \text{lb}\cdot\text{s}^2/\text{ft}$$

Using the equations given above and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5$$

$$= \left[\frac{1}{2}(8.5857 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(16\ \text{in.})^2\right] \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+ \left[\frac{1}{3}(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2\right] \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+ [0 + (0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2] \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+ \left[\frac{1}{12}(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2 + (0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(4^2 + 16^2)\ \text{in}^2\right] \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+ \left[\frac{1}{12}(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2 + (0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8^2 + 12^2)\ \text{in}^2\right] \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$= [(7.6315) + (0.1012) + (0.3036) + (0.0253 + 1.2905) + (0.0253 + 0.9868)] \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= (7.6315 + 0.1012 + 0.3036) + 1.3158 + 1.0121) \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 10.3642 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_x = 10.36 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.146 (Continued)

$$(I_y)_2 = (I_y)_4, \qquad (I_y)_3 = (I_y)_5$$

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5$$

$$= [(8.5857\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(16\text{ in.})^2]\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+2[0 + (0.6832\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(16\text{ in.})^2]\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+2\left[\frac{1}{12}(0.6832\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(8\text{ in.})^2 + (0.6832\text{ lb}\cdot\text{s}^2/\text{ft})(12\text{ in.})^2\right]\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

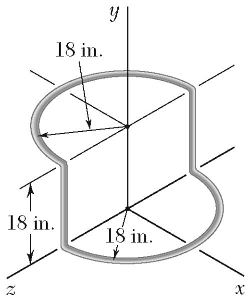
$$= [(15.2635) + 2(1.2146) + 2(0.0253 + 0.6832)]\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= [15.2635 + 2(1.2146) + 2(0.7085)]\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 19.1097\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_y = 19.11\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

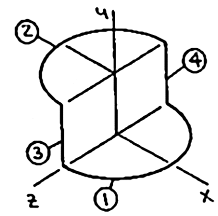
Symmetry implies $I_x = I_z$ $\qquad$ $I_z = 10.36\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.147

The figure shown is formed of $\frac{1}{8}$-in.-diameter steel wire. Knowing that the specific weight of the steel is 490 lb/ft^3, determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \frac{\gamma_{\text{ST}}}{g} AL$$

Then

$$m_1 = m_2 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8}\text{ in.}\right)^2\right] \times (\pi \times 18 \text{ in.}) \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= 6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = m_4 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} = \left[\frac{\pi}{4}\left(\frac{1}{8}\text{in.}\right)^2\right] \times 18 \text{ in.} \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= 1.9453 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Using the equations given above and the parallel-axis theorem, we have

$$(I_x)_3 = (I_x)_4$$

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4$$

$$= \left[\frac{1}{2}(6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(18 \text{ in.})^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left[\frac{1}{2}(6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(18 \text{ in.})^2 + (6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(18 \text{ in.})^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ 2\left[\frac{1}{12}(1.9453 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(18 \text{ in.})^2 + (6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(9^2 + 18^2) \text{ in}^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$= [(6.8751) + (6.8751 + 13.1502) + 2(0.3647 + 5.4712)] \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= [6.8751 + 20.6252 + 2(5.8359)] \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= 39.1721 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

or $I_x = 0.0392 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀

$$(I_y)_1 = (I_y)_2, \quad (I_y)_3 = (I_y)_4$$

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4$$

$$= 2[(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\,2[(0+1.9453\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$= [2(13.7502)+2(4.3769)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 36.2542\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \qquad \text{or} \quad I_y = 0.0363\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

$$(I_z)_3 = (I_z)_4$$

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4$$

$$= \left[\frac{1}{2}(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left\{(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{1}{2}-\frac{4}{\pi^2}\right)(18\ \text{in.})^2\right.$$

$$\left.+(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2\times18}{\pi}\right)^2+(18)^2\right]\text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

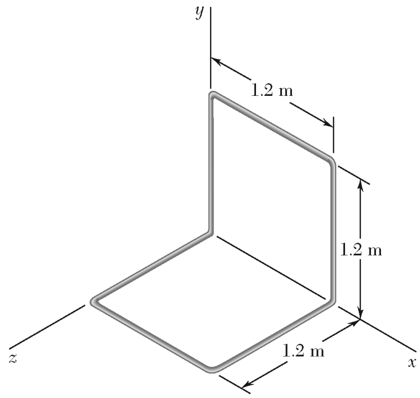
$$+\,2\left[\frac{1}{3}(1.9453\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$= [(6.8751)+(1.3024+19.3229)+2(1.4590)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= [6.8751+20.6253+2(1.4590)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

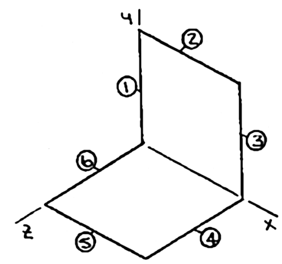
$$= 30.4184\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \qquad \text{or} \quad I_z = 0.0304\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

PROBLEM 9.148



A homogeneous wire with a mass per unit length of 0.056 kg/m is used to form the figure shown. Determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



First compute the mass m of each component. We have

$$m = (m/L)L$$
$$= 0.056 \text{ kg/m} \times 1.2 \text{ m}$$
$$= 0.0672 \text{ kg}$$

Using the equations given above and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5 + (I_x)_6$$

Now $\quad (I_x)_1 = (I_x)_3 = (I_x)_4 = (I_x)_6 \quad \text{and} \quad (I_x)_2 = (I_x)_5$

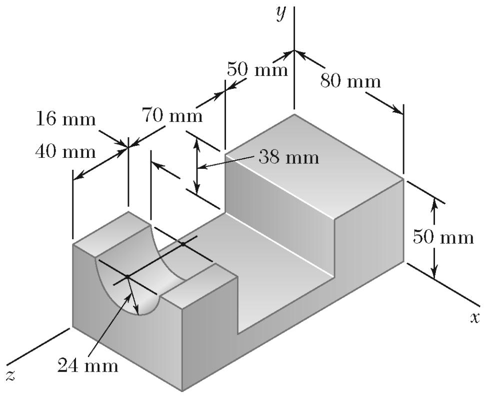
Then
$$I_x = 4\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + 2[0 + (0.0672 \text{ kg})(1.2 \text{ m})^2]$$
$$= [4(0.03226) + 2(0.09677)] \text{ kg}\cdot\text{m}^2$$
$$= 0.32258 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \quad I_x = 0.323 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5 + (I_y)_6$$

Now $\quad (I_y)_1 = 0, \quad (I_y)_2 = (I_y)_6, \quad \text{and} \quad (I_y)_4 = (I_y)_5$

Then
$$I_y = 2\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + [0 + (0.0672 \text{ kg})(1.2 \text{ m})^2]$$
$$+ 2\left[\frac{1}{12}(0.0672 \text{ kg})(1.2 \text{ m})^2 + (0.0672 \text{ kg})(1.2^2 + 0.6^2) \text{ m}^2\right]$$
$$= [2(0.03226) + (0.09677) + 2(0.00806 + 0.12096)] \text{ kg}\cdot\text{m}^2$$
$$= [2(0.03226) + (0.09677) + 2(0.12902)] \text{ kg}\cdot\text{m}^2$$
$$= 0.41933 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \quad I_y = 0.419 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

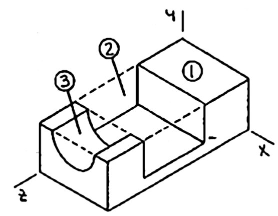
Symmetry implies $\quad I_y = I_z \qquad I_z = 0.419 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$



PROBLEM 9.149

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel fixture shown. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V$$

Then
$$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.16) \text{ m}^3 = 5.02400 \text{ kg}$$
$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$
$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \text{m}^3 = 0.28410 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{y}_2 = \left(0.05 - \frac{0.038}{2}\right) \text{m} = 0.031 \text{ m}$$
$$\bar{y}_3 = \left(0.05 - \frac{4 \times 0.024}{3\pi}\right) \text{m} = 0.039814 \text{ m}$$

and then

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\,\bar{y}$, kg·m^2	$m\bar{y}\,\bar{z}$, kg·m^2	$m\bar{z}\,\bar{x}$, kg·m^2
1	5.02400	0.04	0.025	0.08	5.0240×10^{-3}	10.0480×10^{-3}	16.0768×10^{-3}
2	1.67048	0.04	0.031	0.085	2.0714×10^{-3}	4.4017×10^{-3}	5.6796×10^{-3}
3	0.28410	0.04	0.039814	0.14	0.4524×10^{-3}	1.5836×10^{-3}	1.5910×10^{-3}

Finally,

$$I_{xy} = (I_{xy})_1 - (I_{xy})_2 - (I_{xy})_3 = [(0 + 5.0240) - (0 + 2.0714) - (0 + 0.4524)] \times 10^{-3}$$
$$= 2.5002 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_{xy} = 2.50 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

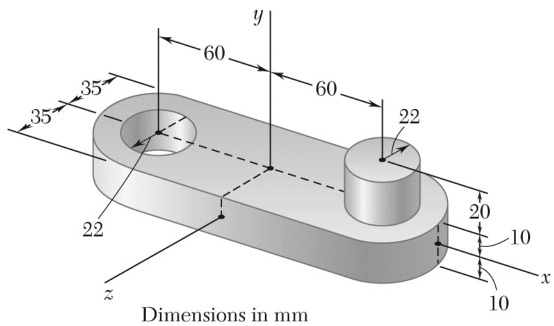
PROBLEM 9.149 (Continued)

$$I_{yz} = (I_{yz})_1 - (I_{yz})_2 - (I_{yz})_3 = [(0+10.0480)-(0+4.4017)-(0+1.5836)]\times 10^{-3}$$
$$= 4.0627\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

or $\quad I_{yz} = 4.06\times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$$I_{zx} = (I_{zx})_1 - (I_{zx})_2 - (I_{zx})_3 = [(0+16.0768)-(0+5.6796)-(0+1.5910)]\times 10^{-3}$$
$$= 8.8062\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

or $\quad I_{zx} = 8.81\times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.150

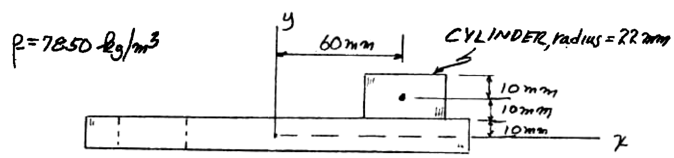
Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel machine element shown. (The density of steel is 7850 kg/m^3.)

SOLUTION

Since the machine element is symmetrical with respect to the xy plane, $I_{yz} = I_{zx} = 0$ ◀

Also, $I_{xy} = 0$

for all components except the cylinder, since these are symmetrical with respect to the xz plane.

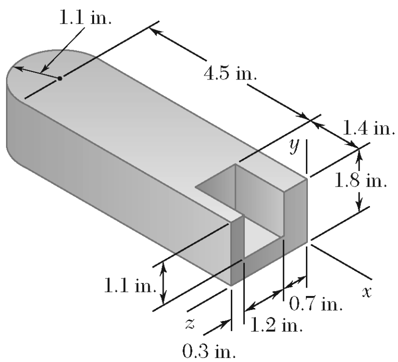


For the cylinder

$$m = \rho V = (7850\text{ kg/m}^3)\pi(0.022\text{ m})^2(0.02\text{ m}) = 0.2387\text{ kg}$$

$$I'_{xy} = \bar{I}_{x'y'} + m\bar{x}\,\bar{y} = 0 + (0.2387\text{ kg})(0.06\text{ m})(0.02\text{ m}) = +286.4\times10^{-6}\text{ kg}\cdot\text{m}^2$$

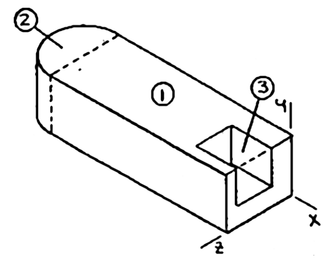
$$I_{xy} = 286\times10^{-6}\text{ kg}\cdot\text{m}^2 \quad ◀$$



PROBLEM 9.151

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \frac{\gamma_{\text{AL}}}{g} V$$

Then

$$m_1 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (5.9 \times 1.8 \times 2.2) \text{ in}^3$$

$$= 72.5590 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{2}(1.1)^2 \times 1.8\right] \text{ in}^3$$

$$= 10.6248 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (1.4 \times 1.1 \times 1.2) \text{ in}^3$$

$$= 5.7391 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\overline{I}_{x'y'}, \overline{I}_{y'z'}$, and $\overline{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\overline{x}_2 = -\left(5.9 + \frac{4 \times 1.1}{3\pi}\right) \text{ in.}$$

$$= -6.36685 \text{ in.}$$

$$\overline{y}_3 = \left(1.8 - \frac{1.1}{2}\right) \text{ in.}$$

$$= 1.25 \text{ in.}$$

PROBLEM 9.151 (Continued)

and then

	m, lb $\cdot$ s^2/ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, lb $\cdot$ ft $\cdot$ s^2	$m\bar{y}\,\bar{z}$, lb $\cdot$ ft $\cdot$ s^2	$m\bar{z}\,\bar{x}$, lb $\cdot$ ft $\cdot$ s^2
1	72.5590×10^{-3}	$-\dfrac{2.95}{12}$	$\dfrac{0.9}{12}$	$\dfrac{1.1}{12}$	-1.33781×10^{-3}	0.49884×10^{-3}	-1.63510×10^{-3}
2	10.6248×10^{-3}	$-\dfrac{6.36685}{12}$	$\dfrac{0.9}{12}$	$\dfrac{1.1}{12}$	-0.42279×10^{-3}	0.07305×10^{-3}	-0.51674×10^{-3}
3	5.7391×10^{-3}	$-\dfrac{0.7}{12}$	$\dfrac{1.25}{12}$	$\dfrac{1.3}{12}$	-0.03487×10^{-3}	0.06476×10^{-3}	-0.03627×10^{-3}

Finally,
$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 - (I_{xy})_3$$
$$= [(0-1.33781) + (0-0.42279) - (0-0.03487)]\times10^{-3}$$

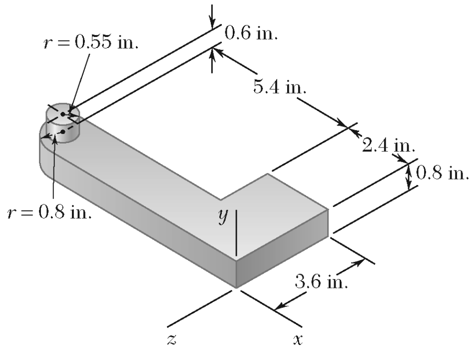
or $I_{xy} = -1.726\times10^{-3}$ lb $\cdot$ ft $\cdot$ s^2 ◀

$$I_{yz} = (I_{yz})_1 + (I_{yz})_2 - (I_{yz})_3$$
$$= [(0+0.49884) + (0+0.07305) - (0+0.06476)]\times10^{-3}$$

or $I_{yz} = 0.507\times10^{-3}$ lb $\cdot$ ft $\cdot$ s^2 ◀

$$I_{zx} = (I_{zx})_1 + (I_{zx})_2 - (I_{zx})_3$$
$$= [(0-1.63510) + (0-0.51674) - (0-0.3627)]\times10^{-3}$$

or $I_{zx} = -2.12\times10^{-3}$ lb $\cdot$ ft $\cdot$ s^2 ◀



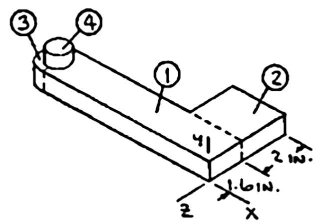
PROBLEM 9.152

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{AL} V = \frac{\gamma_{AL}}{g} V$$



Then
$$m_1 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times (7.8 \times 0.8 \times 1.6)\ \text{in}^3$$
$$= 31.0062 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$
$$m_2 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times (2.4 \times 0.8 \times 2)\ \text{in}^3$$
$$= 11.9255 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$
$$m_3 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times \left[\frac{\pi}{2}(0.8)^2 \times 0.8\right]\ \text{in}^3 = 2.4977\ \text{lb} \cdot \text{s}^2/\text{ft}$$
$$m_4 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times [\pi(0.55)^2 \times 0.6]\ \text{in}^3 = 1.7708\ \text{lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{x}_3 = -\left(7.8 + \frac{4 \times 0.8}{3\pi}\right)\ \text{in.} = -8.13953\ \text{in.}$$

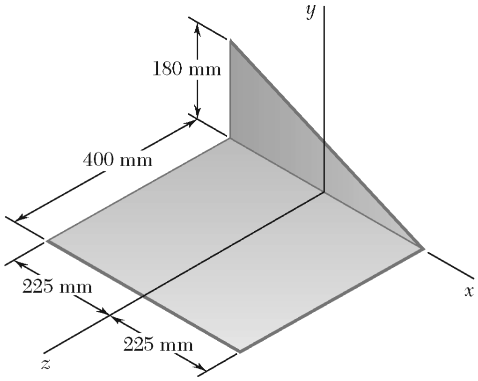
and then

	m, lb·s²/ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, lb·ft·s²	$m\bar{y}\,\bar{z}$, lb·ft·s²	$m\bar{z}\,\bar{x}$, lb·ft·s²
1	31.0062×10^{-3}	$-\frac{3.9}{12}$	$\frac{0.4}{12}$	$-\frac{0.8}{12}$	-335.901×10^{-6}	-68.903×10^{-6}	671.801×10^{-6}
2	11.9255×10^{-3}	$-\frac{1.2}{12}$	$\frac{0.4}{12}$	$-\frac{2.6}{12}$	-39.752×10^{-6}	-86.129×10^{-6}	258.386×10^{-6}
3	2.4977×10^{-3}	$-\frac{8.13953}{12}$	$\frac{0.4}{12}$	$-\frac{0.8}{12}$	-56.473×10^{-6}	-5.550×10^{-6}	112.945×10^{-6}
4	1.7708×10^{-3}	$-\frac{7.8}{12}$	$\frac{1.1}{12}$	$-\frac{0.8}{12}$	-105.511×10^{-6}	-10.822×10^{-6}	76.735×10^{-6}
Σ					-537.637×10^{-6}	-171.404×10^{-6}	1119.867×10^{-6}

Then $$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) \quad \text{or} \quad I_{xy} = -538\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \ ◀$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) \quad \text{or} \quad I_{yz} = -171.4\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \ ◀$$

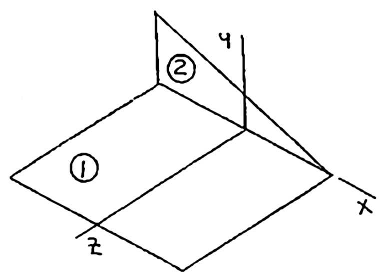
$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \quad \text{or} \quad I_{zx} = 1120\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \ ◀$$



PROBLEM 9.153

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component.

We have $\quad m = \rho_{\text{ST}} V = \rho_{\text{ST}} tA$

Then
$$m_1 = (7850\text{ kg/m}^3)[(0.002)(0.4)(0.45)]\text{m}^3$$
$$= 2.8260\text{ kg}$$
$$m_2 = 7850\text{ kg/m}^3\left[(0.002)\left(\frac{1}{2}\times 0.45\times 0.18\right)\right]\text{m}^3$$
$$= 0.63585\text{ kg}$$

Now observe that

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = (\bar{I}_{z'x'})_1 = 0$$
$$(\bar{I}_{y'z'})_2 = (\bar{I}_{z'x'})_2 = 0$$

From Sample Problem 9.6: $\quad (\bar{I}_{x'y'})_{2,\text{area}} = -\frac{1}{72} b_2^2 h_2^2$

Then $\quad (\bar{I}_{x'y'})_2 = \rho_{\text{ST}} t (I_{x'y'})_{2,\text{area}} = \rho_{\text{ST}} t\left(-\frac{1}{72} b_2^2 h_2^2\right) = -\frac{1}{36} m_2 b_2 h_2$

Also $\quad \bar{x}_1 = \bar{y}_1 = \bar{z}_2 = 0 \qquad \bar{x}_2 = \left(-0.225 + \frac{0.45}{3}\right)\text{m} = -0.075\text{ m}$

Finally,
$$I_{xy} = \Sigma(\bar{I}_{xy} + m\bar{x}\,\bar{y}) = (0+0) + \left[-\frac{1}{36}(0.63585\text{ kg})(0.45\text{ m})(0.18\text{ m}) + (0.63585\text{ kg})(-0.075\text{ m})\left(\frac{0.18\text{ m}}{3}\right)\right]$$
$$= (-1.43066\times10^{-3} - 2.8613\times10^{-3})\text{ kg}\cdot\text{m}^2$$

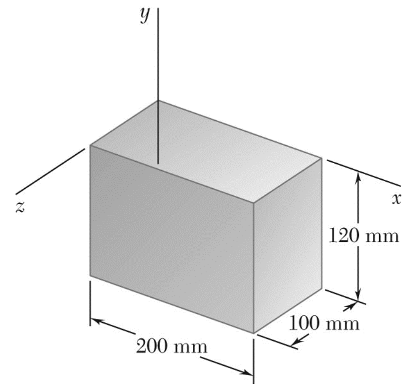
or $I_{xy} = -4.29\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

and
$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\,\bar{y}\,\bar{z}) = (0+0) + (0+0) = 0$$

or $I_{yz} = 0$ ◀

$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\,\bar{z}\,\bar{x}) = (0+0) + (0+0) = 0$$

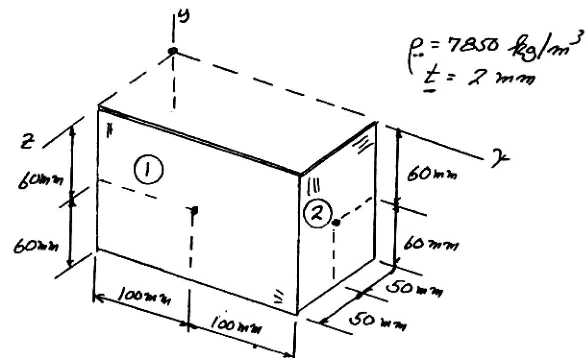
or $I_{zx} = 0$ ◀



PROBLEM 9.154

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



$$m_1 = \rho V = 7850 \text{ kg/m}^3 (0.120 \text{ m})(0.200 \text{ m})(0.002 \text{ m}) = 0.3768 \text{ kg}$$

$$m_2 = \rho V = 7850 \text{ kg/m}^3 (0.120 \text{ m})(0.100 \text{ m})(0.002 \text{ m}) = 0.1884 \text{ kg}$$

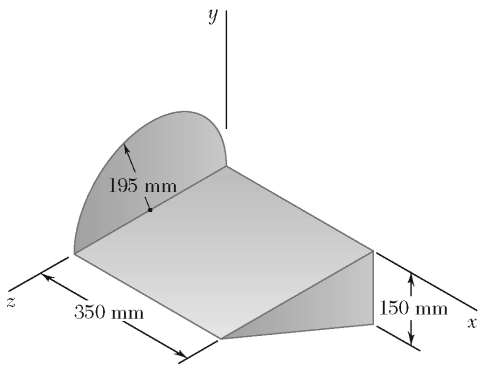
For each panel the centroidal product of inertia is zero with respect to each pair of coordinate axes.

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\bar{y}$	$m\bar{y}\bar{z}$	$m\bar{z}\bar{x}$
1	0.3768	0.1	−0.06	0.1	-2.261×10^{-3}	-2.261×10^{-3}	$+3.768\times10^{-3}$
2	0.1884	0.2	−0.06	0.05	-2.261×10^{-3}	-0.565×10^{-3}	$+1.884\times10^{-3}$
Σ					-4.522×10^{-3}	-2.826×10^{-3}	$+5.652\times10^{-3}$

$$I_{xy} = -4.52\times10^{-3} \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = -2.83\times10^{-3} \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

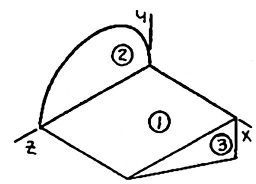
$$I_{zx} = +5.65\times10^{-3} \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM 9.155

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{ST} V = \rho_{ST} t A$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.35 \times 0.39)\ \text{m}^2$$
$$= 2.14305\ \text{kg}$$
$$m_2 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{\pi}{2} \times 0.195^2\right)\ \text{m}^2$$
$$= 0.93775\ \text{kg}$$
$$m_3 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{1}{2} \times 0.39 \times 0.15\right)\ \text{m}^2$$
$$= 0.45923\ \text{kg}$$

Now observe that because of symmetry the centroidal products of inertia of components 1 and 2 are zero and

$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0$$

Also

$$(\bar{I}_{y'z'})_{3,\text{mass}} = \rho_{ST} t (\bar{I}_{y'z'})_{3,\text{area}}$$

Using the results of Sample Problem 9.6 and noting that the orientation of the axes corresponds to a 90° rotation, we have

$$(\bar{I}_{y'z'})_{3,\text{area}} = \frac{1}{72} b_3^2 h_3^2$$

Then

$$(I_{y'z'})_3 = \rho_{ST} t \left(\frac{1}{72} b_3^2 h_3^2\right) = \frac{1}{36} m_3 b_3 h_3$$

Also

$$\bar{y}_1 = \bar{x}_2 = 0$$
$$\bar{y}_2 = \frac{4 \times 0.195}{3\pi}\ \text{m} = 0.082761\ \text{m}$$

Finally,

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y})$$

$$= (0+0) + (0+0) + \left[0 + (0.45923\text{ kg})(0.35\text{ m})\left(-\frac{0.15}{3}\text{ m}\right)\right]$$

$$= -8.0365\times10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{xy} = -8.04\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\,\bar{z})$$

$$= (0+0) + [0 + (0.93775\text{ kg})(0.082761\text{ m})(0.195\text{ m})]$$

$$+\left[\frac{1}{36}(0.45923\text{ kg})(0.39\text{ m})(0.15\text{ m}) + (0.45923\text{ kg})\left(-\frac{0.15}{3}\text{ m}\right)\left(\frac{0.39}{3}\text{ m}\right)\right]$$

$$= [(15.1338) + (0.7462 - 2.9850)]\times10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (15.1338 - 2.2388)\times10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 12.8950\times10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{yz} = 12.90\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

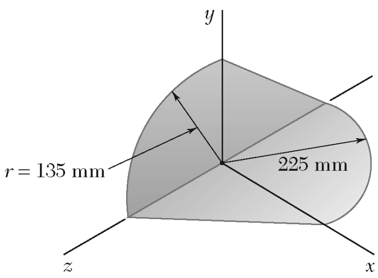
$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\,\bar{x})$$

$$= [0 + (2.14305\text{ kg})(0.175\text{ m})(0.195\text{ m})] + (0+0)$$

$$+\left[0 + (0.45923\text{ kg})\left(\frac{0.39}{3}\text{ m}\right)(0.35\text{ m})\right]$$

$$= (73.1316 + 20.8950)\times10^{-3}\text{ kg}\cdot\text{m}^2$$

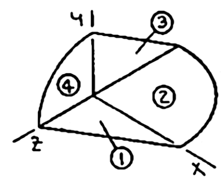
$$= 94.0266\times10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{zx} = 94.0\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM 9.156

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{ST}V = \rho_{ST}tA$$

Then

$$m_1 = m_3 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{1}{2}\times 0.225\times 0.135\right)\text{m}^2 = 0.23844 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{4}\times 0.225^2\right)\text{m}^2 = 0.62424 \text{ kg}$$

$$m_4 = (7850 \text{ kg/m}^3)(0.002 \text{ m})\left(\frac{\pi}{4}\times 0.135^2\right)\text{m}^2 = 0.22473 \text{ kg}$$

Now observe that the following centroidal products of inertia are zero because of symmetry.

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = 0 \qquad (\bar{I}_{x'y'})_2 = (\bar{I}_{y'z'})_2 = 0$$
$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0 \qquad (\bar{I}_{x'y'})_4 = (\bar{I}_{z'x'}) = 0$$

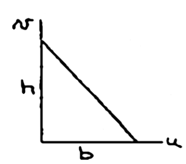
Also $\qquad \bar{y}_1 = \bar{y}_2 = 0 \qquad \bar{x}_3 = \bar{x}_4 = 0$

Now $\qquad I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y})$

so that $\qquad I_{xy} = 0$ ◀

Using the results of Sample Problem 9.6, we have

$$I_{uv,\text{area}} = \frac{1}{24}b^2h^2$$



Now

$$I_{uv,\text{mass}} = \rho_{ST}t\, I_{uv,\text{area}}$$
$$= \rho_{ST}t\left(\frac{1}{24}b^2h^2\right)$$
$$= \frac{1}{12}mbh$$

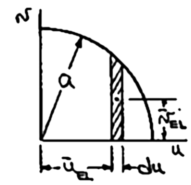
Thus, $\qquad (I_{zx})_1 = \frac{1}{12}m_1b_1h_1$

While $$(I_{yz})_3 = -\frac{1}{12} m_3 b_3 h_3$$

because of a 90° rotation of the coordinate axes.

To determine I_{uv} for a quarter circle, we have

$$dI_{uv} = d\cancelto{0}{\bar{I}_{u'v'}} + u_{\text{EL}} v_{\text{EL}}\, dm$$



Where $$\bar{u}_{\text{EL}} = u \quad \bar{v}_{\text{EL}} = \frac{1}{2} v = \frac{1}{2}\sqrt{a^2 - u^2}$$

$$dm = \rho_{\text{ST}} t\, dA = \rho_{\text{ST}} tv\, du = \rho_{\text{ST}} t \sqrt{a^2 - u^2}\, du$$

Then $$I_{uv} = \int dI_{uv} = \int_0^a (u)\left(\frac{1}{2}\sqrt{a^2 - u^2}\right)\left(\rho_{\text{ST}} t \sqrt{a^2 - u^2}\, du\right)$$

$$= \frac{1}{2}\rho_{\text{ST}} t \int_0^a u(a^2 - u^2)\, du$$

$$= \frac{1}{2}\rho_{\text{ST}} t \left[\frac{1}{2}a^2u^2 - \frac{1}{4}u^4\right]_0^a = \frac{1}{8}\rho_{\text{ST}} t\, a^4 = \frac{1}{2\pi} m a^4$$

Thus $$(I_{zx})_2 = -\frac{1}{2\pi} m_2 a_2^2$$

because of a 90° rotation of the coordinate axes. Also

$$(I_{yz})_4 = \frac{1}{2\pi} m_4 a_4^2$$

Finally,

$$I_{yz} = \Sigma(I_{yz}) = [(\cancelto{0}{\bar{I}_{y'z'}}) + m_1 \bar{y}_1 \bar{z}_1] + [(\cancelto{0}{\bar{I}_{y'z'}}) + m_2 \bar{y}_2 \bar{z}_2] + (I_{yz})_3 + (I_{yz})_4$$

$$= \left[-\frac{1}{12}(0.23844\text{ kg})(0.225\text{ m})(0.135\text{ m})\right] + \left[\frac{1}{2\pi}(0.22473\text{ kg})(0.135\text{ m})^2\right]$$

$$= (-0.60355 + 0.65185) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_{yz} = 48.3 \times 10^{-6}\text{ kg}\cdot\text{m}^2$ ◀

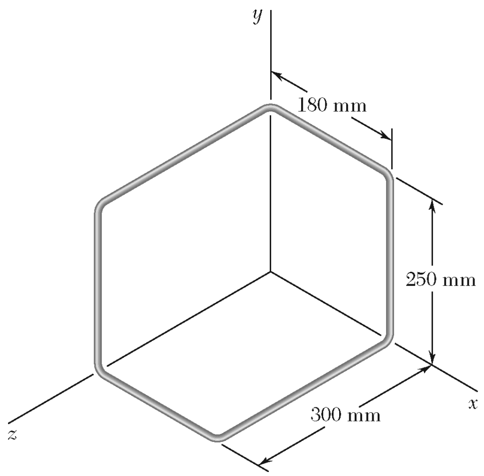
$$I_{zx} = \Sigma(I_{zx}) = (I_{zx})_1 + (I_{zx})_2 + [(\cancelto{0}{\bar{I}_{z'x'}})_3 + m_3 \bar{z}_3 \bar{x}_3] + [(\cancelto{0}{\bar{I}_{z'x'}})_4 + m_4 \bar{z}_4 \bar{x}_4]$$

$$= \left[\frac{1}{12}(0.23844\text{ kg})(0.225\text{ m})(0.135\text{ m})\right] + \left[-\frac{1}{2\pi}(0.62424\text{ kg})(0.225\text{ m})^2\right]$$

$$= (0.60355 - 5.02964) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

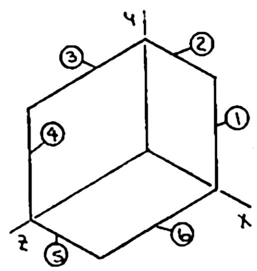
or $I_{zx} = -4.43 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

PROBLEM 9.157



The figure shown is formed of 1.5-mm-diameter aluminum wire. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \rho_{\text{AL}} AL$$

Then

$$m_1 = m_4 = (2800 \text{ kg/m}^3)\left[\frac{\pi}{4}(0.0015 \text{ m})^2\right](0.25 \text{ m})$$

$$= 1.23700 \times 10^{-3} \text{ kg}$$

$$m_2 = m_5 = (2800 \text{ kg/m}^3)\left[\frac{\pi}{4}(0.0015 \text{ m})^2\right](0.18 \text{ m})$$

$$= 0.89064 \times 10^{-3} \text{ kg}$$

$$m_3 = m_6 = (2800 \text{ kg/m}^3)\left[\frac{\pi}{4}(0.0015 \text{ m})^2\right](0.3 \text{ m})$$

$$= 1.48440 \times 10^{-3} \text{ kg}$$

PROBLEM 9.157 (Continued)

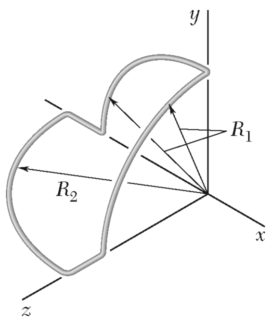
Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\,\bar{y}$, $\text{kg}\cdot\text{m}^2$	$m\bar{y}\,\bar{z}$, $\text{kg}\cdot\text{m}^2$	$m\bar{z}\,\bar{x}$, $\text{kg}\cdot\text{m}^2$
1	1.23700×10^{-3}	0.18	0.125	0	27.8325×10^{-6}	0	0
2	0.89064×10^{-3}	0.09	0.25	0	20.0394×10^{-6}	0	0
3	1.48440×10^{-3}	0	0.25	0.15	0	55.6650×10^{-6}	0
4	1.23700×10^{-3}	0	0.125	0.3	0	46.3875×10^{-6}	0
5	0.89064×10^{-3}	0.09	0	0.3	0	0	24.0473×10^{-6}
6	1.48440×10^{-3}	0.18	0	0.15	0	0	40.0788×10^{-6}
Σ					47.8719×10^{-6}	102.0525×10^{-6}	64.1261×10^{-6}

Then $$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) \qquad \text{or} \qquad I_{xy} = 47.9\times10^{-6}\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) \qquad \text{or} \qquad I_{yz} = 102.1\times10^{-6}\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

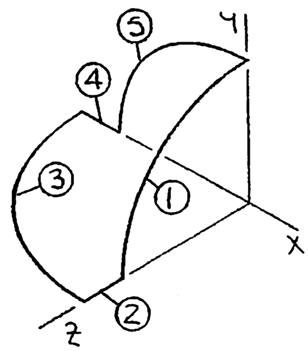
$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \qquad \text{or} \qquad I_{zx} = 64.1\times10^{-6}\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM 9.158

Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by m' the mass per unit length of the wire, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = m'L$$

Then
$$m_1 = m_5 = m'\left(\frac{\pi}{2}R_1\right) = \frac{\pi}{2}m'R_1$$

$$m_2 = m_4 = m'(R_2 - R_1)$$

$$m_3 = m'\left(\frac{\pi}{2}R_2\right) = \frac{\pi}{2}m'R_2$$

Now observe that because of symmetry the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of components 2 and 4 are zero and

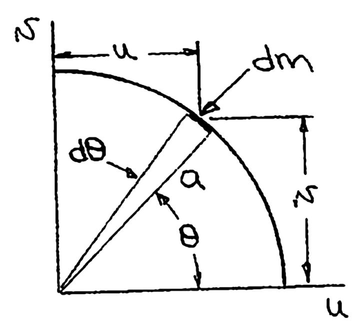
$$(\bar{I}_{x'y'})_1 = (\bar{I}_{z'x'})_1 = 0 \qquad (\bar{I}_{x'y'})_3 = (\bar{I}_{y'z'})_3 = 0$$

$$(\bar{I}_{y'z'})_5 = (\bar{I}_{z'x'})_5 = 0$$

Also
$$\bar{x}_1 = \bar{x}_2 = 0 \qquad \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = 0 \qquad \bar{z}_4 = \bar{z}_5 = 0$$

Using the parallel-axis theorem [Equations (9.47)], it follows that $I_{xy} = I_{yz} = I_{zx}$ for components 2 and 4.

To determine I_{uv} for one quarter of a circular arc, we have $dI_{uv} = uvdm$



where
$$u = a\cos\theta \qquad v = a\sin\theta$$

and
$$dm = \rho dV = \rho[A(ad\theta)]$$

where A is the cross-sectional area of the wire. Now

$$m = m'\left(\frac{\pi}{2}a\right) = \rho A\left(\frac{\pi}{2}a\right)$$

so that
$$dm = m'ad\theta$$

and
$$dI_{uv} = (a\cos\theta)(a\sin\theta)(m'ad\theta)$$
$$= m'a^3\sin\theta\cos\theta d\theta$$

Then $$I_{uv} = \int dI_{uv} = \int_0^{\pi/2} m'a^3 \sin\theta\cos\theta d\theta$$

$$= m'a^3\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \frac{1}{2}m'a^3$$

Thus, $$(I_{yz})_1 = \frac{1}{2}m'R_1^3$$

and $$(I_{zx})_3 = -\frac{1}{2}m'R_2^3 \quad (I_{xy})_5 = -\frac{1}{2}m'R_1^3$$

because of $90°$ rotations of the coordinate axes. Finally,

$$I_{xy} = \Sigma(I_{xy}) = [\cancelto{0}{(\bar{I}_{x'y'})_1} + m_1\bar{x}_1\bar{y}_1] + [\cancelto{0}{(\bar{I}_{x'y'})_3} + m_3\bar{x}_3\bar{y}_3] + (I_{xy})_5$$

or $I_{xy} = -\frac{1}{2}m'R_1^3$ ◀

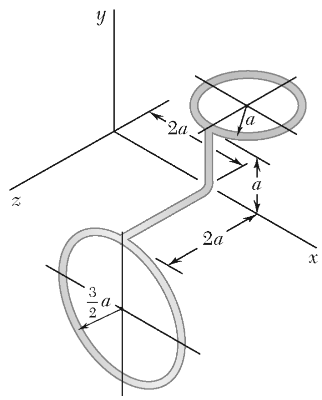
$$I_{yz} = \Sigma(I_{yz}) = (I_{yz})_1 + [\cancelto{0}{(\bar{I}_{y'z'})_3} + m_3\bar{y}_3\bar{z}_3] + [\cancelto{0}{(\bar{I}_{y'z'})_5} + m_5\bar{y}_5\bar{z}_5]$$

or $I_{yz} = \frac{1}{2}m'R_1^3$ ◀

$$I_{zx} = \Sigma(I_{zx}) = [\cancelto{0}{(\bar{I}_{z'x'})_1} + m_1\bar{z}_1\bar{x}_1] + (I_{zx})_3 + [\cancelto{0}{(\bar{I}_{z'x'})_5} + m_5\bar{z}_5\bar{x}_5]$$

or $I_{zx} = -\frac{1}{2}m'R_2^3$ ◀

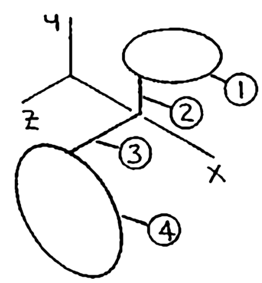
PROBLEM 9.159



Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have



$$m = \frac{W}{g} = \frac{1}{g} wL$$

Then

$$m_1 = \frac{W}{g}(2\pi \times a) = 2\pi \frac{w}{g} a$$

$$m_2 = \frac{w}{g}(a) = \frac{w}{g} a$$

$$m_3 = \frac{w}{g}(2a) = 2\frac{w}{g} a$$

$$m_4 = \frac{w}{g}\left(2\pi \times \frac{3}{2} a\right) = 3\pi \frac{w}{g} a$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'},$ and $\bar{I}_{z'x'},$ of each component are zero because of symmetry.

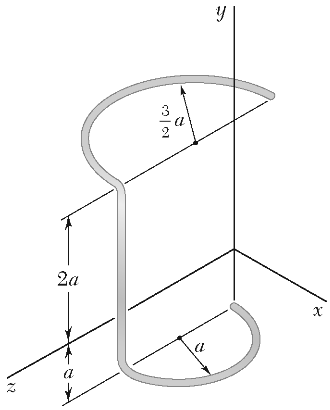
	m	$\bar{x}$	$\bar{y}$	$\bar{z}$	$m\bar{x}\,\bar{y}$	$m\bar{y}\,\bar{z}$	$m\bar{z}\,\bar{x}$
1	$2\pi \frac{w}{g} a$	$2a$	a	$-a$	$4\pi \frac{w}{g} a^3$	$-2\pi \frac{w}{g} a^3$	$-4\pi \frac{w}{g} a^3$
2	$\frac{w}{g} a$	$2a$	$\frac{1}{2} a$	0	$\frac{w}{g} a^3$	0	0
3	$2\frac{w}{g} a$	$2a$	0	a	0	0	$4\frac{w}{g} a^3$
4	$3\pi \frac{w}{g} a$	$2a$	$-\frac{3}{2} a$	$2a$	$-9\pi \frac{w}{g} a^3$	$-9\pi \frac{w}{g} a^3$	$12\pi \frac{w}{g} a^3$
Σ					$\frac{w}{g}(1-5\pi)a^3$	$-11\pi \frac{w}{g} a^3$	$4\frac{w}{g}(1+2\pi)a^3$

PROBLEM 9.159 (Continued)

Then $I_{xy} = \Sigma(\cancel{\bar{I}_{x'y'}}^{\,0} + m\bar{x}\,\bar{y})$ or $I_{xy} = \dfrac{w}{g}a^3(1-5\pi)$ ◀

$I_{yz} = \Sigma(\cancel{\bar{I}_{y'z'}}^{\,0} + m\bar{y}\,\bar{z})$ or $I_{yz} = -11\pi\dfrac{w}{g}a^3$ ◀

$I_{zx} = \Sigma(\cancel{\bar{I}_{z'x'}}^{\,0} + m\bar{z}\,\bar{x})$ or $I_{zx} = 4\dfrac{w}{g}a^3(1+2\pi)$ ◀

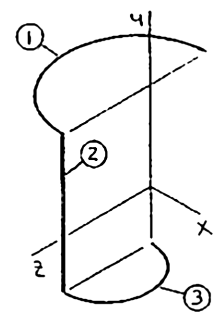


PROBLEM 9.160

Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have



$$m = \frac{W}{g} = \frac{1}{g} wL$$

Then

$$m_1 = \frac{W}{g}\left(\pi \times \frac{3}{2}a\right) = \frac{3}{2}\pi\frac{w}{g}a$$

$$m_2 = \frac{w}{g}(3a) = 3\frac{w}{g}a$$

$$m_3 = \frac{w}{g}(\pi \times a) = \pi\frac{w}{g}a$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m	$\bar{x}$	$\bar{y}$	$\bar{z}$	$m\bar{x}\,\bar{y}$	$m\bar{y}\,\bar{z}$	$m\bar{z}\,\bar{x}$
1	$\frac{3}{2}\pi\frac{w}{g}a$	$-\frac{2}{\pi}\left(\frac{3}{2}a\right)$	$2a$	$\frac{1}{2}a$	$-9\frac{w}{g}a^3$	$\frac{3}{2}\pi\frac{w}{g}a^3$	$-\frac{9}{4}\frac{w}{g}a^3$
2	$3\frac{w}{g}a$	0	$\frac{1}{2}a$	$2a$	0	$3\frac{w}{g}a^3$	0
3	$\pi\frac{w}{g}a$	$\frac{2}{\pi}(a)$	$-a$	a	$-2\frac{w}{g}a^3$	$-\pi\frac{w}{g}a^3$	$2\frac{w}{g}a^3$
Σ					$-11\frac{w}{g}a^3$	$\frac{w}{g}\left(\frac{\pi}{2}+3\right)a^3$	$-\frac{1}{4}\frac{w}{g}a^3$

PROBLEM 9.160 (Continued)

Then $$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y})$$ or $$I_{xy} = -11\frac{w}{g}a^3 \blacktriangleleft$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z})$$ or $$I_{yz} = \frac{1}{2}\frac{w}{g}a^3(\pi + 6) \blacktriangleleft$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x})$$ or $$I_{zx} = -\frac{1}{4}\frac{w}{g}a^3 \blacktriangleleft$$

PROBLEM 9.161

Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

SOLUTION

We have $$I_{xy} = \int xydm \quad I_{yz} = \int yzdm \quad I_{zx} = \int zxdm \qquad (9.45)$$

and $$x = x' + \bar{x} \quad y = y' + \bar{y} \quad z = z' + \bar{z} \qquad (9.31)$$

Consider $$I_{xy} = \int xydm$$

Substituting for x and for y

$$I_{xy} = \int (x' + \bar{x})(y' + \bar{y})\,dm$$
$$= \int x'y'dm + \bar{y}\int x'dm + \bar{x}\int y'dm + \bar{x}\,\bar{y}\int dm$$

By definition $$\bar{I}_{x'y'} = \int x'y'dm$$

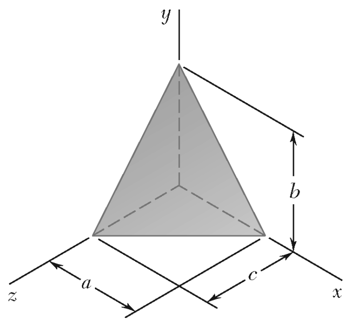
and $$\int x'dm = m\bar{x}'$$
$$\int y'dm = m\bar{y}'$$

However, the origin of the primed coordinate system coincides with the mass center G, so that

$$\bar{x}' = \bar{y}' = 0$$

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\,\bar{y} \quad \text{Q.E.D.} \blacktriangleleft$$

The expressions for I_{yz} and I_{zx} are obtained in a similar manner.

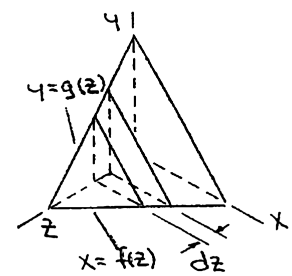


PROBLEM 9.162

For the homogeneous tetrahedron of mass m shown, (*a*) determine by direct integration the mass product of inertia I_{zx}, (*b*) deduce I_{yz} and I_{xy} from the result obtained in part *a*.

SOLUTION

(*a*) First divide the tetrahedron into a series of thin vertical slices of thickness dz as shown.



Now $$x = -\frac{a}{c}z + a = a\left(1-\frac{z}{c}\right)$$

and $$y = -\frac{b}{c}z + b = b\left(1-\frac{z}{c}\right)$$

The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xydz\right) = \frac{1}{2}\rho ab\left(1-\frac{z}{c}\right)^2 dz$$

Then $$m = \int dm = \int_0^c \frac{1}{2}\rho ab\left(1-\frac{z}{c}\right)^2 dz$$

$$= \frac{1}{2}\rho ab\left[\left(-\frac{c}{3}\right)\left(1-\frac{z}{c}\right)^3\right]_0^c = \frac{1}{6}\rho abc$$

Now $$dI_{zx} = d\bar{I}_{z'x'} + \bar{z}_{EL}\bar{x}_{EL}dm$$

where $$d\bar{I}_{z'x'} = 0 \quad \text{(symmetry)}$$

and $$\bar{z}_{EL} = z \quad \bar{x}_{EL} = \frac{1}{3}x = \frac{1}{3}a\left(1-\frac{z}{c}\right)$$

Then $$I_{zx} = \int dI_{zx} = \int_0^c z\left[\frac{1}{3}a\left(1-\frac{z}{c}\right)\right]\left[\frac{1}{2}\rho ab\left(1-\frac{z}{c}\right)^2 dz\right]$$

$$= \frac{1}{6}\rho a^2 b\int_0^c\left(z - 3\frac{z^2}{c} + 3\frac{z^3}{c^2} - \frac{z^4}{c^3}\right)dz$$

$$= \frac{m}{c}a\left[\frac{1}{2}z^2 - \frac{z^3}{c} + \frac{3}{4}\frac{z^4}{c^2} - \frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

or $$I_{zx} = \frac{1}{20}mac \blacktriangleleft$$

(*b*) Because of the symmetry of the body, I_{xy} and I_{yz} can be deduced by considering the circular permutation of (x, y, z) and (a, b, c).

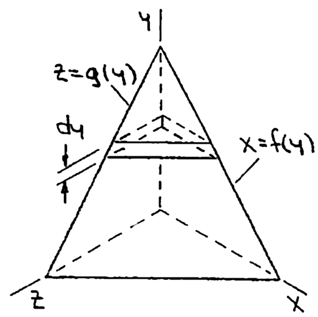
Thus,

$$I_{xy} = \frac{1}{20}mab \blacktriangleleft$$

$$I_{yz} = \frac{1}{20}mbc \blacktriangleleft$$

Alternative solution for part *a*:

First divide the tetrahedron into a series of thin horizontal slices of thickness dy as shown.



Now $$x = -\frac{a}{b}y + a = a\left(1 - \frac{y}{b}\right)$$

and $$z = -\frac{c}{b}y + c = c\left(1 - \frac{y}{b}\right)$$

The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xzdy\right) = \frac{1}{2}\rho ac\left(1 - \frac{y}{b}\right)^2 dy$$

Now $$dI_{zx} = \rho t dI_{zx,\text{area}}$$

where $$t = dy$$

and $dI_{zx,\text{area}} = \frac{1}{24}x^2z^2$ from the results of Sample Problem 9.6.

Then $$dIzx = \rho(dy)\left\{\frac{1}{24}\left[a\left(1 - \frac{y}{b}\right)^2\right]\left[c\left(1 - \frac{y}{b}\right)\right]^2\right\}$$

$$= \frac{1}{24}\rho a^2c^2\left(1 - \frac{y}{b}\right)^4 dy = \frac{1}{4}\frac{m}{b}ac\left(1 - \frac{y}{b}\right)^4 dy$$

Finally, $$I_{zx} = \int dI_{zx} = \int_0^b \frac{1}{4}\frac{m}{b}ac\left(1 - \frac{y}{b}\right)^4 dy$$

$$= \frac{1}{4}\frac{m}{b}ac\left[\left(-\frac{b}{5}\right)\left(1 - \frac{y}{b}\right)^5\right]_0^b$$

or $$I_{zx} = \frac{1}{20}mac \blacktriangleleft$$

Alternative solution for part *a*:

The equation of the included face of the tetrahedron is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

so that $$y=b\left(1-\frac{x}{a}-\frac{z}{c}\right)$$

For an infinitesimal element of sides $dx, dy,$ and dz:

$$dm=\rho dV=\rho dydxdz$$

From part *a* $$x=a\left(1-\frac{z}{c}\right)$$

Now

$$I_{zx}=\int zxdm=\int_0^c\int_0^{a(1-z/c)}\int_0^{b(1-x/a-z/c)} zx(\rho dydxdz)$$

$$=\rho\int_0^c\int_0^{a(1-z/c)} zx\left[b\left(1-\tfrac{x}{a}-\tfrac{z}{c}\right)\right]dxdz$$

$$=\rho b\int_0^c z\left[\frac{1}{2}x^2-\frac{1}{3}\frac{x^3}{a}-\frac{1}{2}\frac{z}{c}x^2\right]_0^{a(1-z/c)}dz$$

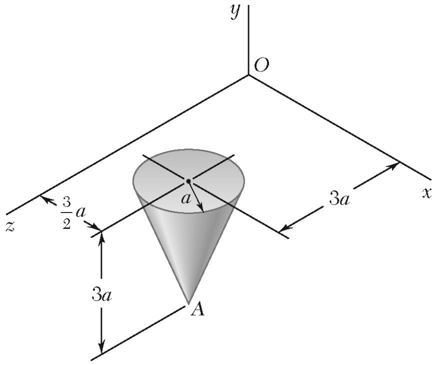
$$=\rho b\int_0^c z\left[\frac{1}{2}a^2\left(1-\frac{z}{c}\right)^2-\frac{1}{3a}a^3\left(1-\frac{z}{c}\right)^3-\frac{1}{2}\frac{z}{c}a^2\left(1-\frac{z}{c}\right)^2\right]dz$$

$$=\rho b\int_0^c \frac{1}{6}a^2z\left(1-\frac{z}{c}\right)^3dz$$

$$=\frac{1}{6}\rho a^2b\int_0^c\left(z-3\frac{z^2}{c}+3\frac{z^3}{c^2}-\frac{z^4}{c^3}\right)dz$$

$$=\frac{m}{c}a\left[\frac{1}{2}z^2-\frac{z^3}{c}+\frac{3}{4}\frac{z^4}{c^2}-\frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

or $I_{zx}=\dfrac{1}{20}mac$ ◀



PROBLEM 9.163

The homogeneous circular cone shown has a mass m. Determine the mass moment of inertia of the cone with respect to the line joining the origin O and Point A.

SOLUTION

First note that

$$d_{OA} = \sqrt{\left(\frac{3}{2}a\right)^2 + (-3a)^2 + (3a)^2} = \frac{9}{2}a$$

Then

$$\lambda_{OA} = \frac{1}{\frac{9}{2}a}\left(\frac{3}{2}a\mathbf{i} - 3a\mathbf{j} + 3a\mathbf{k}\right) = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

For a rectangular coordinate system with origin at Point A and axes aligned with the given x, y, z axes, we have (using Figure 9.28)

$$I_x = I_z = \frac{3}{5}m\left[\frac{1}{4}a^2 + (3a)^2\right] \qquad I_y = \frac{3}{10}ma^2$$

$$= \frac{111}{20}ma^2$$

Also, symmetry implies $\quad I_{xy} = I_{yz} = I_{zx} = 0$

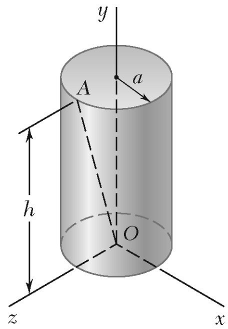
With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2$$

$$= \frac{111}{20}ma^2\left(\frac{1}{3}\right)^2 + \frac{3}{10}ma^2\left(-\frac{2}{3}\right)^2 + \frac{111}{20}ma^2\left(\frac{2}{3}\right)^2$$

$$= \frac{193}{60}ma^2$$

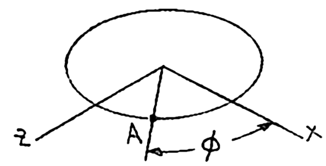
or $\quad I_{OA} = 3.22ma^2$ ◀



PROBLEM 9.164

The homogeneous circular cylinder shown has a mass m. Determine the mass moment of inertia of the cylinder with respect to the line joining the origin O and Point A that is located on the perimeter of the top surface of the cylinder.

SOLUTION



From Figure 9.28: $$I_y = \frac{1}{2}ma^2$$

and using the parallel-axis theorem

$$I_x = I_z = \frac{1}{12}m(3a^2 + h^2) + m\left(\frac{h}{2}\right)^2 = \frac{1}{12}m(3a^2 + 4h^2)$$

Symmetry implies $$I_{xy} = I_{yz} = I_{zx} = 0$$

For convenience, let Point A lie in the yz plane. Then

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(h\mathbf{j} + a\mathbf{k})$$

With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = \cancel{I_x \lambda_x^2}^{\,0} + I_y\lambda_y^2 + I_z\lambda_z^2$$

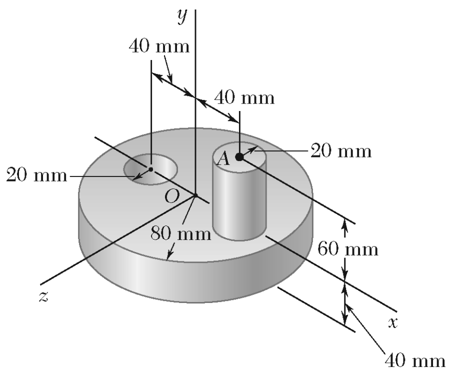
$$= \frac{1}{2}ma^2\left(\frac{h}{\sqrt{h^2 + a^2}}\right)^2 + \frac{1}{12}m(3a^2 + 4h^2)\left(\frac{a}{\sqrt{h^2 + a^2}}\right)^2$$

or $$I_{OA} = \frac{1}{12}ma^2\frac{10h^2 + 3a^2}{h^2 + a^2}$$ ◀

Note: For Point A located at an arbitrary point on the perimeter of the top surface, λ_{OA} is given by

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(a\cos\phi\mathbf{i} + h\mathbf{j} + a\sin\phi\mathbf{k})$$

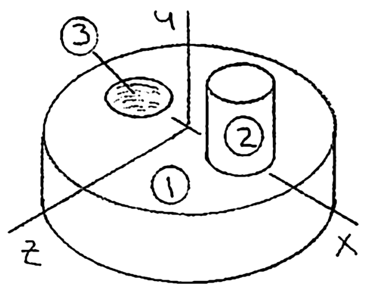
which results in the same expression for I_{OA}.



PROBLEM 9.165

Shown is the machine element of Problem 9.141. Determine its mass moment of inertia with respect to the line joining the origin O and Point A.

SOLUTION



First compute the mass of each component.

We have $$m = \rho_{ST} V = \frac{0.284 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} V = (0.008819 \text{ lb} \cdot \text{s}^2/\text{ft} \cdot \text{in}^3)V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)[\pi(0.08 \text{ m})^2(0.04 \text{ m})] = 6.31334 \text{ kg}$$
$$m_2 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.06 \text{ m})] = 0.59188 \text{ kg}$$
$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.04 \text{ m})] = 0.39458 \text{ kg}$$

Symmetry implies $$I_{yz} = I_{zx} = 0 \qquad (I_{xy})_1 = 0$$

and $$(\bar{I}_{x'y'})_2 = (\bar{I}_{x'y'})_3 = 0$$

Now
$$\begin{aligned} I_{xy} &= \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 - m_3\bar{x}_3\bar{y}_3 \\ &= [0.59188 \text{ kg } (0.04 \text{ m})(0.03 \text{ m})] - [0.39458 \text{ kg } (-0.04 \text{ m})(-0.02 \text{ m})] \\ &= (0.71026 - 0.31566)\times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ &= 0.39460 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

From the solution to Problem 9.141, we have

$$I_x = 13.98800 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$
$$I_y = 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$
$$I_z = 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

By observation $\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$

Substituting into Eq. (9.46)

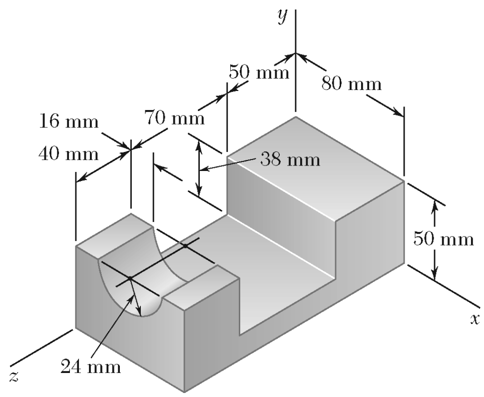
$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + \cancelto{0}{I_z\lambda_z^2} - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x$$

$$= \left[(13.98800)\left(\frac{2}{\sqrt{13}}\right)^2 + (20.55783)\left(\frac{3}{\sqrt{13}}\right)^2 - 2(0.39460)\left(\frac{2}{\sqrt{13}}\right)\left(\frac{2}{\sqrt{13}}\right)\right] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (4.30400 + 14.23234 - 0.36425) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_{OA} = 18.17 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.166



Determine the mass moment of inertia of the steel fixture of Problems 9.145 and 9.149 with respect to the axis through the origin that forms equal angles with the x, y, and z axes.

SOLUTION

From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145:
$$I_x = 26.4325 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_y = 31.1726 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_z = 8.5773 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Problem 9.149:
$$I_{xy} = 2.5002 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_{yz} = 4.0627 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_{zx} = 8.8062 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

From the problem statement it follows that

$$\lambda_x = \lambda_y = \lambda_z$$

Now $$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \Rightarrow 3\lambda_x^2 = 1$$

or $$\lambda_x = \lambda_y = \lambda_z = \frac{1}{\sqrt{3}}$$

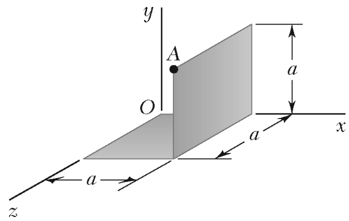
Substituting into Eq. (9.46)

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that $$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

We have $$I_{OL} = \frac{1}{3}[26.4325 + 31.1726 + 8.5773$$
$$-2(2.5002 + 4.0627 + 8.8062)] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

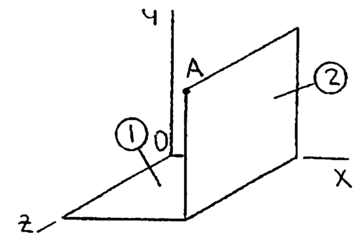
or $I_{OL} = 11.81 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.167

The thin bent plate shown is of uniform density and weight W. Determine its mass moment of inertia with respect to the line joining the origin O and Point A.

SOLUTION



First note that $$m_1 = m_2 = \frac{1}{2}\frac{W}{g}$$

and that $$\lambda_{OA} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$\begin{aligned}
I_x &= (I_x)_1 + (I_x)_2 \\
&= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right] \\
&\quad + \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2+a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12}+\frac{1}{4}\right)a^2 + \left(\frac{1}{6}+\frac{1}{2}\right)a^2\right] = \frac{1}{2}\frac{W}{g}a^2
\end{aligned}$$

$$\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 \\
&= \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2+a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&\quad + \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{6}+\frac{1}{2}\right)a^2 + \left(\frac{1}{12}+\frac{5}{4}\right)a^2\right] = \frac{W}{g}a^2
\end{aligned}$$

$$\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 \\
&= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right] \\
&\quad + \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12}+\frac{1}{4}\right)a^2 + \left(\frac{1}{12}+\frac{5}{4}\right)a^2\right] = \frac{5}{6}\frac{W}{g}a^2
\end{aligned}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also, $\bar{y}_1 = 0$

Then
$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 = \frac{1}{2}\frac{W}{g}(a)\left(\frac{a}{2}\right) = \frac{1}{4}\frac{W}{g}a^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_2\bar{y}_2\bar{z}_2 = \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{8}\frac{W}{g}a^2$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2$$

$$= \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)(a) = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Equation (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that

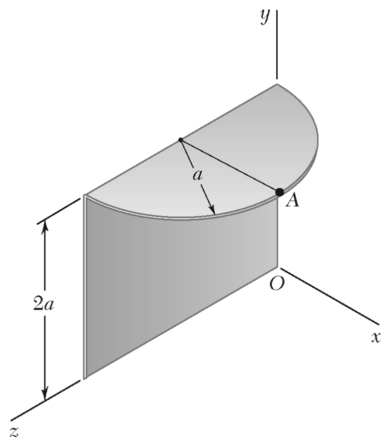
$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

We have

$$I_{OA} = \frac{1}{3}\left[\frac{1}{2}\frac{W}{g}a^2 + \frac{W}{g}a^2 + \frac{5}{6}\frac{W}{g}a^2 - 2\left(\frac{1}{4}\frac{W}{g}a^2 + \frac{1}{8}\frac{W}{g}a^2 + \frac{3}{8}\frac{W}{g}a^2\right)\right]$$

$$= \frac{1}{3}\left[\frac{14}{6} - 2\left(\frac{3}{4}\right)\right]\frac{W}{g}a^2$$

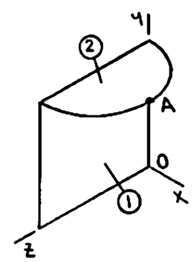
or $I_{OA} = \dfrac{5}{18}\dfrac{W}{g}a^2$ ◀



PROBLEM 9.168

A piece of sheet steel of thickness t and specific weight γ is cut and bent into the machine component shown. Determine the mass moment of inertia of the component with respect to the joining the origin O and Point A.

SOLUTION



First note that

$$\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

Next compute the mass of each component. We have

$$m = \rho V = \frac{\gamma}{g}(tA)$$

Then

$$m_1 = \frac{\gamma}{g}t(2a \times 2a) = 4\frac{\gamma t}{g}a^2$$

$$m_2 = \frac{\gamma}{g}t\left(\frac{\pi}{2} \times a^2\right) = \frac{\pi}{2}\frac{\gamma t}{g}a^2$$

Using Figure 9.28 for component 1 and the equations derived above (following the solution to Problem 9.134) for a semicircular plate for component 2, we have

$$\begin{aligned}
I_x &= (I_x)_1 + (I_x)_2 \\
&= \left\{\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)[(2a)^2 + (2a)^2] + 4\frac{\gamma t}{g}a^2(a^2 + a^2)\right\} \\
&\quad + \left\{\frac{1}{4}\left(\frac{\pi}{2}\frac{\gamma t}{g}a^2\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2[(2a)^2 + (a)^2]\right\} \\
&= 4\frac{\gamma t}{g}a^2\left(\frac{2}{3} + 2\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4} + 5\right)a^2 \\
&= 18.91335\frac{\gamma t}{g}a^4
\end{aligned}$$

PROBLEM 9.168 (Continued)

$$I_y = (I_y)_1 + (I_y)_2$$

$$= \left[\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)(2a)^2 + 4\frac{\gamma t}{g}a^2(a^2)\right]$$

$$+\left\{\frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left[\left(\frac{4a}{3\pi}\right)^2 + (a)^2\right]\right\}$$

$$= 4\frac{\gamma t}{g}a^2\left(\frac{1}{3}+1\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{2}-\frac{16}{9\pi^2}+\frac{16}{9\pi^2}+1\right)a^2$$

$$= 7.68953\frac{\gamma t}{g}a^4$$

$$I_z = (I_z)_1 + (I_z)_2$$

$$= \left[\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)(2a)^2 + 4\frac{\gamma t}{g}a^2(a^2)\right]$$

$$+\left\{\frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4}-\frac{16}{9\pi^2}\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left[\left(\frac{4a}{3\pi}\right)^2 + (2a)^2\right]\right\}$$

$$= 4\frac{\gamma t}{g}a^2\left(\frac{1}{3}+1\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4}-\frac{16}{9\pi^2}+\frac{16}{9\pi^2}+4\right)a^2$$

$$= 12.00922\frac{\gamma t}{g}a^4$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'},$ and $\bar{I}_{z'x'},$ of both components are zero because of symmetry. Also $\bar{x}_1 = 0.$

Then

$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 = \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{4a}{3\pi}\right)(2a)$$

$$= 1.33333\frac{\gamma t}{g}a^4$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_1\bar{y}_1\bar{z}_1 + m_2\bar{y}_2\bar{z}_2$$

$$= 4\frac{\gamma t}{g}a^2(a)(a) + \frac{\pi}{2}\frac{\gamma t}{g}a^2(2a)(a)$$

$$= 7.14159\frac{\gamma t}{g}a^4$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = m_2\bar{z}_2\bar{x}_2 = \frac{\pi}{2}\frac{\gamma t}{g}a^2(a)\left(\frac{4a}{3\pi}\right)$$

$$= 0.66667\frac{\gamma t}{g}a^4$$

Substituting into Eq. (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

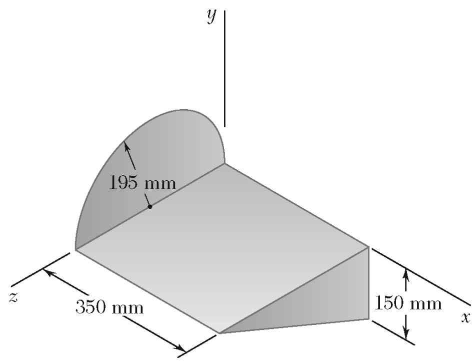
$$= 18.91335\frac{\gamma t}{g}a^4\left(\frac{1}{\sqrt{6}}\right)^2 + 7.68953\frac{\gamma t}{g}a^4\left(\frac{2}{\sqrt{6}}\right)^2$$

$$+12.00922\frac{\gamma t}{g}a^4\left(\frac{1}{\sqrt{6}}\right)^2 - 2\left(1.33333\frac{\gamma t}{g}a^4\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right)$$

$$-2\left(7.14159\frac{\gamma t}{g}a^4\right)\left(\frac{2}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right) - 2\left(0.66667\frac{\gamma t}{g}a^4\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)$$

$$= (3.15223 + 5.12635 + 2.00154 - 0.88889 - 4.76106 - 0.22222)\frac{\gamma t}{g}a^4$$

or $\quad I_{OA} = 4.41\dfrac{\gamma t}{g}a^4$ ◀



PROBLEM 9.169

Determine the mass moment of inertia of the machine component of Problems 9.136 and 9.155 with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})/9$.

SOLUTION

From the solutions to Problems 9.136 and 9.155. We have

Problem 9.136: $I_x = 175.503 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$

$$I_y = 308.629 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

$$I_z = 154.400 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

Problem 9.155: $I_{xy} = -8.0365 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$

$$I_{yz} = 12.8950 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

$$I_{zx} = 94.0266 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

Substituting into Eq. (9.46)

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

$$= \left[175.503\left(-\frac{4}{9}\right)^2 + 308.629\left(\frac{8}{9}\right)^2 + 154.400\left(\frac{1}{9}\right)^2 \right.$$

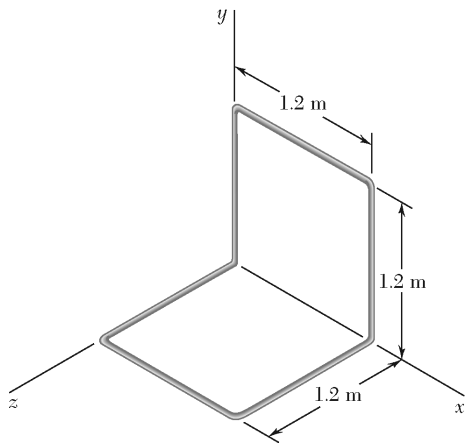
$$-2(-8.0365)\left(-\frac{4}{9}\right)\left(\frac{8}{9}\right) - 2(12.8950)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)$$

$$\left. -2(94.0266)\left(\frac{1}{9}\right)\left(-\frac{4}{9}\right)\right] \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

$$= (34.6673 + 243.855 + 1.906 - 6.350$$

$$-2.547 + 9.287) \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

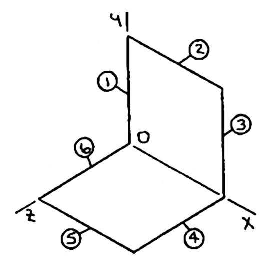
or $I_{OL} = 281 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀



PROBLEM 9.170

For the wire figure of Problem 9.148, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = 0.056 \text{ kg/m} \times 1.2 \text{ m}$$
$$= 0.0672 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also

$$\bar{x}_1 = \bar{x}_6 = 0 \qquad \bar{y}_4 = \bar{y}_5 = \bar{y}_6 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then

$$I_{xy} = \Sigma(\overset{0}{\cancel{\bar{I}_{x'y'}}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$
$$= (0.0672 \text{ kg})(0.6 \text{ m})(1.2 \text{ m}) + (0.0672 \text{ kg})(1.2 \text{ m})(0.6 \text{ m})$$
$$= 0.096768 \text{ kg} \cdot \text{m}^2$$

$$I_{yz} = \Sigma(\overset{0}{\cancel{\bar{I}_{y'z'}}} + m\bar{y}\,\bar{z}) = 0$$

$$I_{zx} = \Sigma(\overset{0}{\cancel{\bar{I}_{z'x'}}} + m\bar{z}\,\bar{x}) = m_4\bar{z}_4\bar{x}_4 + m_5\bar{z}_5\bar{x}_5$$
$$= (0.0672 \text{ kg})(0.6 \text{ m})(1.2 \text{ m}) + (0.0672 \text{ kg})(1.2 \text{ m})(0.6 \text{ m})$$
$$= 0.096768 \text{ kg} \cdot \text{m}^2$$

From the solution to Problem 9.148, we have

$$I_x = 0.32258 \text{ kg} \cdot \text{m}^2$$
$$I_y = I_z = 0.41933 \text{ kg} \cdot \text{m}^2$$

PROBLEM 9.170 (Continued)

Substituting into Eq. (9.46)

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancel{I_{yz}}^{\,0}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

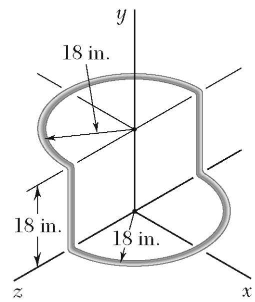
$$= \left[0.32258\left(-\frac{3}{7}\right)^2 + 0.41933\left(-\frac{6}{7}\right)^2 + 0.41933\left(\frac{2}{7}\right)^2\right.$$

$$\left. - 2(0.096768)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) - 2(0.096768)\left(\frac{2}{7}\right)\left(-\frac{3}{7}\right)\right] \text{kg}\cdot\text{m}^2$$

$$I_{OL} = (0.059249 + 0.30808 + 0.034231 - 0.071095 + 0.023698)\ \text{kg}\cdot\text{m}^2$$

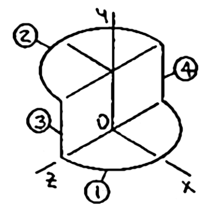
or $\quad I_{OL} = 0.354\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM 9.171



For the wire figure of Problem 9.147, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \frac{\gamma_{\text{ST}}}{g} AL$$

Then

$$m_1 = m_2 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8} \text{ in.}\right)^2\right] \times (\pi \times 18 \text{ in.}) \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= 6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = m_4 = \frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8} \text{ in.}\right)^2\right] \times 18 \text{ in.} \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= 1.9453 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also $\qquad \bar{x}_3 = \bar{x}_4 = 0 \qquad \bar{y}_1 = 0 \qquad \bar{z}_1 = \bar{z}_2 = 0$

Then $\qquad I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2 \bar{x}_2 \bar{y}_2$

$$= (6.1112 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})\left(-\frac{2 \times 18}{\pi} \text{ in.}\right)(18 \text{ in.})\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$= -8.75480 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_3 \bar{y}_3 \bar{z}_3 + m_4 \bar{y}_4 \bar{z}_4$$

Now $\qquad m_3 = m_4, \qquad \bar{y}_3 = \bar{y}_4, \qquad \bar{z}_4 = -\bar{z}_3 \qquad I_{yz} = 0$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \quad \text{or} \quad I_{zx} = 0$$

PROBLEM 9.171 (Continued)

From the solution to Problem 9.147, we have

$$I_x = 39.1721 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$
$$I_y = 36.2542 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$
$$I_z = 30.4184 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

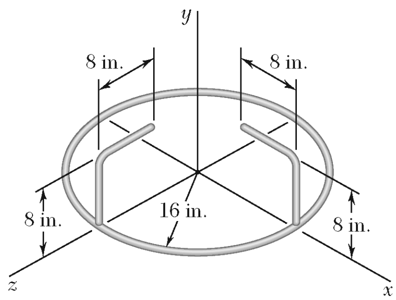
Substituting into Eq. (9.46)

$$I_{OL} = I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x$$

$$= \left[39.1721\left(-\frac{3}{7}\right)^2 + 36.2542\left(-\frac{6}{7}\right)^2 + 30.4184\left(\frac{2}{7}\right)^2 - 2(-8.75480)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right)\right] \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= (7.19488 + 26.6357 + 2.48313 + 6.43210) \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

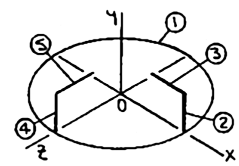
or $I_{OL} = 0.0427 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀



PROBLEM 9.172

For the wire figure of Problem 9.146, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g}(W/L)_{\text{AL}}\,L$$

Then

$$m_1 = \frac{1}{32.2\ \text{ft/s}^2}(0.033\ \text{lb/ft})(2\pi \times 16\ \text{in.}) \times \frac{1\ \text{ft}}{12\ \text{in.}}$$
$$= 8.5857 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = m_3 = m_4 = m_5 = \frac{1}{32.2\ \text{ft/s}^2}(0.033\ \text{lb/ft})(8\ \text{in.}) \times \frac{1\ \text{ft}}{12\ \text{in.}}$$
$$= 0.6832 \times 10^{-3}\ \text{lb} \cdot \text{ft/s}^2$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Also

$$\bar{x}_1 = \bar{x}_4 = \bar{x}_5 = 0 \qquad \bar{y}_1 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then

$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$
$$= 0.6832 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}\left(\frac{16}{12}\ \text{ft}\right)\left(\frac{4}{12}\ \text{ft}\right)$$
$$+ 0.6832 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}\left(\frac{12}{12}\ \text{ft}\right)\left(\frac{8}{12}\ \text{ft}\right)$$
$$= (0.30364 + 0.45547) \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$
$$= 0.75911 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

Symmetry implies

$$I_{yz} = I_{xy} \qquad I_{yz} = 0.75911 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = 0$$

From the solution to Problem 9.146, we have

$$I_x = I_z = 10.3642 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_y = 19.1097 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Substituting into Eq. (9.46)

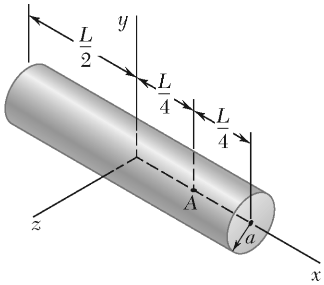
$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2\overset{0}{\cancel{I_{zx}}}\lambda_z\lambda_x$$

$$= \left[10.3642\left(-\frac{3}{7}\right)^2 + 19.1097\left(-\frac{6}{7}\right)^2 + 10.3642\left(\frac{2}{7}\right)^2 - 2(0.75911)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) - 2(0.75911)\left(-\frac{6}{7}\right)\left(\frac{2}{7}\right)\right]\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= (1.90663 + 14.03978 + 0.84606 - 0.55771 + 0.37181)\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

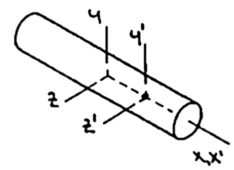
or $\quad I_{OL} = 16.61 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.173



For the homogeneous circular cylinder shown, of radius a and length L, determine the value of the ratio a/L for which the ellipsoid of inertia of the cylinder is a sphere when computed (a) at the centroid of the cylinder, (b) at Point A.

SOLUTION



(a) From Figure 9.28:

$$\bar{I}_x = \frac{1}{2}ma^2$$

$$\bar{I}_y = \bar{I}_z = \frac{1}{12}m(3a^2 + L^2)$$

Now observe that symmetry implies $\quad I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{1}{2}ma^2x^2 + \frac{1}{12}m(3a^2 + L^2)y^2 + \frac{1}{12}m(3a^2 + L^2) = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{2}ma^2 = \frac{1}{12}m(3a^2 + L^2) \qquad \text{or} \qquad \frac{a}{L} = \frac{1}{\sqrt{3}} \blacktriangleleft$$

(b) Using Figure 9.28 and the parallel-axis theorem, we have

$$I_{x'} = \frac{1}{2}ma^2$$

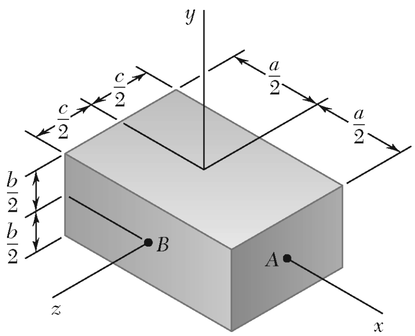
$$I_{y'} = I_{z'} = \frac{1}{12}m(3a^2 + L^2) + m\left(\frac{L}{4}\right)^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part a it then immediately follows that

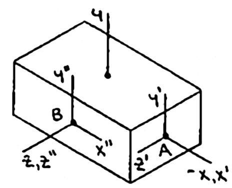
$$\frac{1}{2}ma^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right) \qquad \text{or} \qquad \frac{a}{L} = \sqrt{\frac{7}{12}} \blacktriangleleft$$



PROBLEM 9.174

For the rectangular prism shown, determine the values of the ratios b/a and c/a so that the ellipsoid of inertia of the prism is a sphere when computed (*a*) at Point *A*, (*b*) at Point *B*.

SOLUTION



(*a*) Using Figure 9.28 and the parallel-axis theorem, we have at Point *A*

$$I_{x'} = \frac{1}{12}m(b^2 + c^2)$$

$$I_{y'} = \frac{1}{12}m(a^2 + c^2) + m\left(\frac{a}{2}\right)^2$$

$$= \frac{1}{12}m(4a^2 + c^2)$$

$$I_{z'} = \frac{1}{12}m(a^2 + b^2) + m\left(\frac{a}{2}\right)^2 = \frac{1}{12}m(4a^2 + b^2)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_{x'}x^2 + I_{y'}y^2 + I_{z'}z^2 = 1: \quad \frac{1}{12}m(b^2 + c^2)x^2 + \frac{1}{12}m(4a^2 + c^2)y^2 + \frac{1}{12}m(4a^2 + b^2)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{12}m(b^2 + c^2) = \frac{1}{12}m(4a^2 + c^2)$$

$$= \frac{1}{12}m(4a^2 + b^2)$$

Then $\quad b^2 + c^2 = 4a^2 + c^2 \quad$ or $\quad \frac{b}{a} = 2$ ◀

and $\quad b^2 + c^2 = 4a^2 + b^2 \quad$ or $\quad \frac{c}{a} = 2$ ◀

PROBLEM 9.174 (Continued)

(*b*) Using Figure 9.28 and the parallel-axis theorem, we have at Point *B*

$$I_{x''} = \frac{1}{12}m(b^2 + c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(b^2 + 4c^2)$$

$$I_{y''} = \frac{1}{12}m(a^2 + c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(a^2 + 4c^2)$$

$$I_{z''} = \frac{1}{12}m(a^2 + b^2)$$

Now observe that symmetry implies

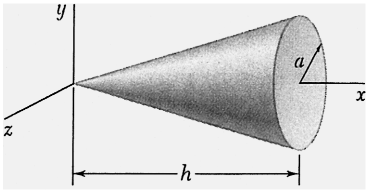
$$I_{x''y''} = I_{y''z''} = I_{z''x''} = 0$$

From part *a* it then immediately follows that

$$\frac{1}{12}m(b^2 + 4c^2) = \frac{1}{12}m(a^2 + 4c^2) = \frac{1}{12}m(a^2 + b^2)$$

Then $\quad b^2 + 4c^2 = a^2 + 4c^2 \quad$ or $\quad \frac{b}{a} = 1$ ◀

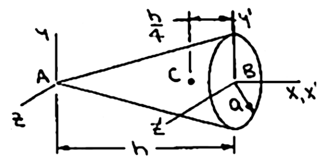
and $\quad b^2 + 4c^2 = a^2 + b^2 \quad$ or $\quad \frac{c}{a} = \frac{1}{2}$ ◀



PROBLEM 9.175

For the right circular cone of Sample Problem 9.11, determine the value of the ratio *a/h* for which the ellipsoid of inertia of the cone is a sphere when computed (*a*) at the apex of the cone, (*b*) at the center of the base of the cone.

SOLUTION



(*a*) From Sample Problem 9.11, we have at the apex *A*

$$I_x = \frac{3}{10}ma^2$$

$$I_y = I_z = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$$

Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{3}{10}ma^2x^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)y^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{3}{10}ma^2 = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right) \qquad \text{or} \qquad \frac{a}{h} = 2 \blacktriangleleft$$

(*b*) From Sample Problem 9.11, we have

$$I_{x'} = \frac{3}{10}ma^2$$

and at the centroid *C* $\quad I_{y''} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$

Then $\quad I_{y'} = I_{z'} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right) + m\left(\frac{h}{4}\right)^2 = \frac{1}{20}m(3a^2 + 2h^2)$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part *a* it then immediately follows that

$$\frac{3}{10}ma^2 = \frac{1}{20}m(3a^2 + 2h^2) \qquad \text{or} \qquad \frac{a}{h} = \sqrt{\frac{2}{3}} \blacktriangleleft$$

PROBLEM 9.176

Given an arbitrary body and three rectangular axes x, y, and z, prove that the mass moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the mass moments of inertia of the body with respect to the other two axes. That is, prove that the inequality $I_x \leq I_y + I_z$ and the two similar inequalities are satisfied. Further, prove that $I_y \geq \frac{1}{2} I_x$ if the body is a homogeneous solid of revolution, where x is the axis of revolution and y is a transverse axis.

SOLUTION

(i) To prove $I_y + I_z \geq I_x$

By definition

$$I_y = \int (z^2 + x^2)dm$$

$$I_z = \int (x^2 + y^2)dm$$

Then

$$I_y + I_z = \int (z^2 + x^2)dm + \int (x^2 + y^2)dm$$

$$= \int (y^2 + z^2)dm + 2\int x^2 dm$$

Now $\int (y^2 + z^2)dm = I_x$ and $\int x^2 dm \geq 0$

$$I_y + I_z \geq I_x \quad \text{Q.E.D.}$$

The proofs of the other two inequalities follow similar steps.

(ii) If the x axis is the axis of revolution, then

$$I_y = I_z$$

and from part (i) $I_y + I_z \geq I_x$

or $2I_y \geq I_x$

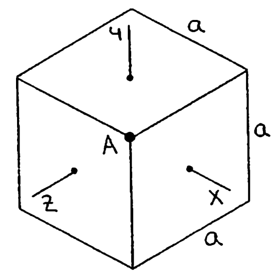
or

$$I_y \geq \frac{1}{2} I_x \quad \text{Q.E.D.}$$

PROBLEM 9.177

Consider a cube of mass m and side a. (a) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (b) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

SOLUTION



(a) At the center of the cube have (using Figure 9.28)

$$I_x = I_y = I_z = \frac{1}{12}m(a^2 + a^2) = \frac{1}{6}ma^2$$

Now observe that symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

Using Equation (9.48), the equation of the ellipsoid of inertia is

$$\left(\frac{1}{6}ma^2\right)x^2 + \left(\frac{1}{6}ma^2\right)y^2 + \left(\frac{1}{6}ma^2\right)z^2 = 1$$

or $$x^2 + y^2 + z^2 = \frac{6}{ma^2} \; (= R^2) \blacktriangleleft$$

which is the equation of a sphere.

Since the ellipsoid of inertia is a sphere, the moment of inertia with respect to any axis OL through the center O of the cube must always

be the same $\left(R = \dfrac{1}{\sqrt{I_{OL}}}\right)$.

$$I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

(b) The above sketch of the cube is the view seen if the line of sight is along the diagonal that passes through corner A. For a rectangular coordinate system at A and with one of the coordinate axes aligned with the diagonal, an ellipsoid of inertia at A could be constructed. If the cube is then rotated 120° about the diagonal, the mass distribution will remain unchanged. Thus, the ellipsoid will also remain unchanged after it is rotated. As noted at the end of Section 9.17, this is possible only if the ellipsoid is an ellipsoid of revolution, where the diagonal is both the axis of revolution and a principal axis.

It then follows that $$I_{x'} = I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

In addition, for an ellipsoid of revolution, the two transverse principal moments of inertia are equal and any axis perpendicular to the axis of revolution is a principal axis. Then, applying the parallel-axis theorem between the center of the cube and corner A for any perpendicular axis

$$I_{y'} = I_{z'} = \frac{1}{6}ma^2 + m\left(\frac{\sqrt{3}}{2}a\right)^2$$

or $$I_{y'} = I_{z'} = \frac{11}{12}ma^2 \blacktriangleleft$$

(*Note:* Part b can also be solved using the method of Section 9.18.)

PROBLEM 9.177 (Continued)

First note that at corner A

$$I_x = I_y = I_z = \frac{2}{3}ma^2$$

$$I_{xy} = I_{yz} = I_{zx} = \frac{1}{4}ma^2$$

Substituting into Equation (9.56) yields

$$k^3 - 2ma^2k^2 + \frac{55}{48}m^2a^6k - \frac{121}{864}m^3a^9 = 0$$

For which the roots are

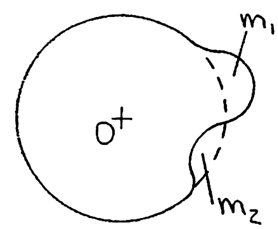
$$k_1 = \frac{1}{6}ma^2$$

$$k_2 = k_3 = \frac{11}{12}ma^2$$

PROBLEM 9.178

Given a homogeneous body of mass m and of arbitrary shape and three rectangular axes x, y, and z with origin at O, prove that the sum $I_x + I_y + I_z$ of the mass moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at O. Further, using the result of Problem 9.176, prove that if the body is a solid of revolution, where x is the axis of revolution, its mass moment of inertia I_y about a transverse axis y cannot be smaller than $3ma^2/10$, where a is the radius of the sphere of the same mass and the same material.

SOLUTION



(i) Using Equation (9.30), we have

$$I_x + I_y + I_z = \int (y^2 + z^2)dm + \int (z^2 + x^2)\,dm + \int (x^2 + y^2)dm$$

$$= 2\int (x^2 + y^2 + z^2)dm$$

$$= 2\int r^2 dm$$

where r is the distance from the origin O to the element of mass dm. Now assume that the given body can be formed by adding and subtracting appropriate volumes V_1 and V_2 from a sphere of mass m and radius a which is centered at O; it then follows that

$$m_1 = m_2 \ (m_{\text{body}} = m_{\text{sphere}} = m)$$

Then

$$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + (I_x + I_y + I_z)_{V_1}$$
$$-(I_x + I_y + I_z)_{V_2}$$

or $$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + 2\int_{m_1} r^2 dm - 2\int_{m_2} r^2 dm$$

Now, $m_1 = m_2$ and $r_1 \geq r_2$ for all elements of mass dm in volumes 1 and 2.

$$\int_{m_1} r^2 dm - \int_{m_2} r^2 dm \geq 0$$

so that $$(I_x + I_y + I_z)_{\text{body}} \geq (I_x + I_y + I_z)_{\text{sphere}}$$ Q.E.D.

(ii) First note from Figure 9.28 that for a sphere

$$I_x = I_y = I_z = \frac{2}{5}ma^2$$

Thus $$(I_x + I_y + I_z)_{\text{sphere}} = \frac{6}{5}ma^2$$

For a solid of revolution, where the x axis is the axis of revolution, we have

$$I_y = I_z$$

Then, using the results of part (i)

$$(I_x + 2I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

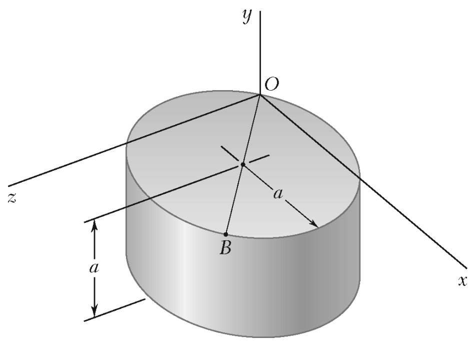
From Problem 9.178 we have $$I_y \geq \frac{1}{2}I_x$$

or $$(2I_y - I_x)_{\text{body}} \geq 0$$

Adding the last two inequalities yields

$$(4I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

or $$(I_y)_{\text{body}} \geq \frac{3}{10}ma^2 \qquad \text{Q.E.D.}$$

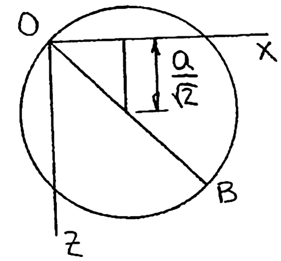


PROBLEM 9.179*

The homogeneous circular cylinder shown has a mass m, and the diameter OB of its top surface forms 45° angles with the x and z axes. (a) Determine the principal mass moments of inertia of the cylinder at the origin O. (b) Compute the angles that the principal axes of inertia at O form with the coordinate axes. (c) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

SOLUTION

(a) First compute the moments of inertia using Figure 9.28 and the parallel-axis theorem.



$$I_x = I_z = \frac{1}{12}m(3a^2 + a^2) + m\left[\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2\right] = \frac{13}{12}ma^2$$

$$I_y = \frac{1}{2}ma^2 + m(a)^2 = \frac{3}{2}ma^2$$

Next observe that the centroidal products of inertia are zero because of symmetry. Then

$$I_{xy} = \cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y} = m\left(\frac{a}{\sqrt{2}}\right)\left(-\frac{a}{2}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{yz} = \cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z} = m\left(-\frac{a}{2}\right)\left(\frac{a}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{zx} = \cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x} = m\left(\frac{a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = \frac{1}{2}ma^2$$

Substituting into Equation (9.56)

$$K^3 - \left(\frac{13}{12} + \frac{3}{2} + \frac{13}{12}\right)ma^2K^2$$

$$+\left[\left(\frac{13}{12}\times\frac{3}{2}\right) + \left(\frac{3}{2}\times\frac{13}{12}\right) + \left(\frac{13}{12}\times\frac{13}{12}\right) - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2\right](ma^2)^2K$$

$$-\left[\left(\frac{13}{12}\times\frac{3}{2}\times\frac{13}{12}\right) - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2\right.$$

$$\left.-\left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - 2\left(-\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)\right](ma^2)^3 = 0$$

Simplifying and letting $K = ma^2\zeta$ yields

$$\zeta^3 - \frac{11}{3}\zeta^2 + \frac{565}{144}\zeta - \frac{95}{96} = 0$$

Solving yields

$$\zeta_1 = 0.363383 \qquad \zeta_2 = \frac{19}{12} \qquad \zeta_3 = 1.71995$$

The principal moments of inertia are then

$$K_1 = 0.363ma^2 \blacktriangleleft$$

$$K_2 = 1.583ma^2 \blacktriangleleft$$

$$K_3 = 1.720ma^2 \blacktriangleleft$$

(*b*) To determine the direction cosines λ_x, λ_y, λ_z of each principal axis, we use two of the equations of Equations (9.54) and (9.57).

Thus,

$$(I_x - K)\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z = 0 \qquad (9.54a)$$

$$-I_{zx}\lambda_x - I_{yz}\lambda_y + (I_z - K)\lambda_z = 0 \qquad (9.54c)$$

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \qquad (9.57)$$

(*Note:* Since $I_{xy} = I_{yz}$, Equations (9.54a) and (9.54c) were chosen to simplify the "elimination" of λ_y during the solution process.)

Substituting for the moments and products of inertia in Equations (9.54a) and (9.54c)

$$\left(\frac{13}{12}ma^2 - K\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y - \left(\frac{1}{2}ma^2\right)\lambda_z = 0$$

$$-\left(\frac{1}{2}ma^2\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y + \left(\frac{13}{12}ma^2 - K\right)\lambda_z = 0$$

or

$$\left(\frac{13}{12} - \zeta\right)\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y - \frac{1}{2}\lambda_z = 0 \qquad \text{(i)}$$

and

$$-\frac{1}{2}\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y + \left(\frac{13}{12} - \zeta\right)\lambda_z = 0 \qquad \text{(ii)}$$

Observe that these equations will be identical, so that one will need to be replaced, if

$$\frac{13}{12} - \zeta = -\frac{1}{2} \qquad \text{or} \qquad \zeta = \frac{19}{12}$$

Thus, a third independent equation will be needed when the direction cosines associated with K_2 are determined. Then for K_1 and K_3

Eq. (i) through Eq. (ii): $$\left[\frac{13}{12}-\zeta-\left(-\frac{1}{2}\right)\right]\lambda_x+\left[-\frac{1}{2}-\left(\frac{13}{12}-\zeta\right)\right]\lambda_z=0$$

or $$\lambda_z=\lambda_x$$

Substituting into Eq. (i): $$\left(\frac{13}{12}-\zeta\right)\lambda_x+\frac{1}{2\sqrt{2}}\lambda_y-\frac{1}{2}\lambda_x=0$$

or $$\lambda_y=2\sqrt{2}\left(\zeta-\frac{7}{12}\right)\lambda_x$$

Substituting into Equation (9.57):

$$\lambda_x^2+\left[2\sqrt{2}\left(\zeta-\frac{7}{12}\right)\lambda_x\right]^2+(\lambda_x)^2=1$$

or $$\left[2+8\left(\zeta-\frac{7}{12}\right)^2\right]\lambda_x^2=1 \qquad \text{(iii)}$$

$\mathbf{K}_1$: Substituting the value of ζ_1 into Eq. (iii):

$$\left[2+8\left(0.363383-\frac{7}{12}\right)^2\right](\lambda_x)_1^2=1$$

or $$(\lambda_x)_1=(\lambda_z)_1=0.647249$$

and then $$(\lambda_y)_1=2\sqrt{2}\left(0.363383-\frac{7}{12}\right)(0.647249)$$

$$=-0.402662$$

$$(\theta_x)_1=(\theta_z)_1=49.7° \quad (\theta_y)_1=113.7° \blacktriangleleft$$

$\mathbf{K}_3$: Substituting the value of ζ_3 into Eq. (iii):

$$\left[2+8\left(1.71995-\frac{7}{12}\right)^2\right](\lambda_x)_3^2=1$$

or $$(\lambda_x)_3=(\lambda_z)_3=0.284726$$

and then $$(\lambda_y)_3=2\sqrt{2}\left(1.71995-\frac{7}{12}\right)(0.284726)$$

$$=0.915348$$

$$(\theta_x)_3=(\theta_z)_3=73.5° \quad (\theta_y)_3=23.7° \blacktriangleleft$$

PROBLEM 9.179* (Continued)

$\mathbf{K}_2$: For this case, the set of equations to be solved consists of Equations (9.54a), (9.54b), and (9.57). Now

$$-I_{xy}\lambda_x + (I_y - K)\lambda_y - I_{yz}\lambda_z = 0 \tag{9.54b}$$

Substituting for the moments and products of inertia.

$$-\left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_x + \left(\frac{3}{2}ma^2 - K\right)\lambda_y - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_z = 0$$

or
$$\frac{1}{2\sqrt{2}}\lambda_x + \left(\frac{3}{2} - \xi\right)\lambda_y + \frac{1}{2\sqrt{2}}\lambda_z = 0 \tag{iv}$$

Substituting the value of ξ_2 into Eqs. (i) and (iv):

$$\left(\frac{13}{12} - \frac{19}{12}\right)(\lambda_x)_2 + \frac{1}{2\sqrt{2}}(\lambda_y)_2 - \frac{1}{2}(\lambda_z)_2 = 0$$

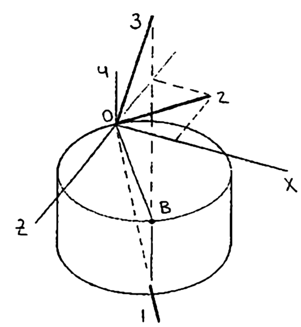
$$\frac{1}{2\sqrt{2}}(\lambda_x)_2 + \left(\frac{3}{2} - \frac{19}{12}\right)(\lambda_y)_2 + \frac{1}{2\sqrt{2}}(\lambda_z)_2 = 0$$

or
$$-(\lambda_x)_2 + \frac{1}{\sqrt{2}}(\lambda_y)_2 - (\lambda_z)_2 = 0$$

and
$$(\lambda_x)_2 - \frac{\sqrt{2}}{6}(\lambda_y)_2 + (\lambda_z)_2 = 0$$

Adding yields
$$(\lambda_y)_2 = 0$$

and then
$$(\lambda_y)_2 = -(\lambda_x)_2$$



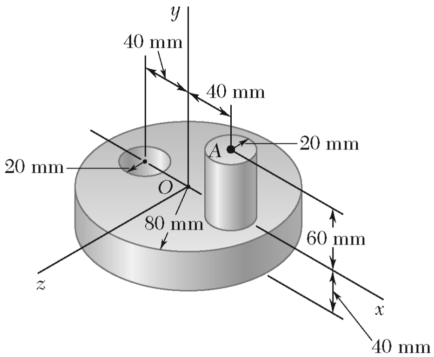
Substituting into Equation (9.57)

$$(\lambda_x)_2^2 + \cancel{(\lambda_y)_2^2}^{\,0} + (-\lambda_x)_2^2 = 1$$

or
$$(\lambda_x)_2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad (\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$

$$(\theta_x)_2 = 45.0° \quad (\theta_y)_2 = 90.0° \quad (\theta_z)_2 = 135.0° \blacktriangleleft$$

(*c*) Principal axes 1 and 3 lie in the vertical plane of symmetry passing through Points *O* and *B*. Principal axis 2 lies in the *xz* plane.



PROBLEM 9.180

For the component described in Problem 9.165, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.141 and 9.165 we have

Problem 9.141:

$$I_x = 13.98800 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 20.55783 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Problem 9.165:

$$I_{yz} = I_{zx} = 0$$

$$I_{xy} = 0.39460 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Eq. (9.55) then becomes

$$\begin{vmatrix} I_x - K & -I_{xy} & 0 \\ -I_{xy} & I_y - K & 0 \\ 0 & 0 & I_z - K \end{vmatrix} = 0 \quad \text{or} \quad (I_x - K)(I_y - K)(I_z - K) - (I_z - K)I_{xy}^2 = 0$$

Thus $\qquad I_z - K = 0 \qquad I_x I_y - (I_x + I_y)K + K^2 - I_{xy}^2 = 0$

Substituting: $\qquad K_1 = 14.30368 \times 10^{-3} \text{ kg} \cdot \text{m}^2$

or $\quad K_1 = 14.30 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

and $\quad (13.98800 \times 10^{-3})(20.55783 \times 10^{-3}) - (13.98800 + 20.55783)(10^{-3})K + K^2 - (0.39460 \times 10^{-3})^2 = 0$

or $\qquad K^2 - (34.54583 \times 10^{-3})K + 287.4072 \times 10^{-6} = 0$

Solving yields $\qquad K_2 = 13.96438 \times 10^{-3} \text{ kg} \cdot \text{m}^2$

or $\quad K_2 = 13.96 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

and $\qquad K_3 = 20.58145 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \qquad$ or $\quad K_3 = 20.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.180 (Continued)

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*) with $I_{yz} = I_{zx} = 0.$

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 = 0$$

Substituting:

$$[(13.98800 - 14.30368)\times 10^{-3}](\lambda_x)_1 - (0.39460\times 10^{-3})(\lambda_y)_1 = 0$$
$$-(0.39460\times 10^{-3})(\lambda_x)_1 + [(20.55783 - 14.30368)\times 10^{-3}](\lambda_y)_1 = 0$$

Adding yields $\qquad (\lambda_x)_1 = (\lambda_y)_1 = 0$

Then using Eq. (9.57) $\qquad \cancelto{0}{(\lambda_x)_1^2} + \cancelto{0}{(\lambda_y)_1^2} + (\lambda_z)_1^2 = 1$

or $\qquad (\lambda_z)_1 = 1$

$$(\theta_x)_1 = 90.0° \quad (\theta_y)_1 = 90.0° \quad (\theta_z)_1 = 0° \;◀$$

K_2: Begin with Eqs. (9.54*b*) and (9.54*c*) with $I_{yz} = I_{zx} = 0.$

$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 = 0 \qquad \text{(i)}$$
$$(I_z - K_2)(\lambda_z)_2 = 0$$

Now $\qquad I_z \neq K_2 \Rightarrow (\lambda_z)_2 = 0$

Substituting into Eq. (i):

$$-(0.39460\times 10^{-3})(\lambda_x)_z + [(20.55783 - 13.96438)\times 10^{-3}](\lambda_y)_2 = 0$$

or $\qquad (\lambda_y)_2 = 0.059847(\lambda_x)_2$

Using Eq. (9.57): $\qquad (\lambda_x)_2^2 + [0.05984](\lambda_x)_2]^2 + \cancelto{0}{(\lambda_z)_2^2} = 1$

or $\qquad (\lambda_x)_2 = 0.998214$

and $\qquad (\lambda_y)_2 = 0.059740$

$$(\theta_x)_2 = 3.4° \qquad (\theta_y)_2 = 86.6° \qquad (\theta_z)_2 = 90.0° \;◀$$

K_3: Begin with Eqs. (9.54*b*) and (9.54*c*) with $I_{yz} = I_{zx} = 0.$

$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 = 0 \qquad \text{(ii)}$$
$$(I_z - K_3)(\lambda_z)_3 = 0$$

PROBLEM 9.180 (Continued)

Now $$I_z \neq K_3 \Rightarrow (\lambda_z)_3 = 0$$

Substituting into Eq. (ii):

$$-(0.39460\times10^{-3})(\lambda_x)_3 + [(20.55783-20.58145)\times10^{-3}](\lambda_y)_3 = 0$$

or $$(\lambda_y)_3 = -16.70618(\lambda_x)_3$$

Using Eq. (9.57): $$(\lambda_x)_3^2 + [-16.70618(\lambda_x)_3]^2 + \overset{0}{\cancel{(\lambda_z)_3^2}} = 1$$

or $$(\lambda_x)_3 = -0.059751 \quad \text{(axes right-handed set } \Rightarrow \text{"\_")}$$

and $$(\lambda_y)_3 = 0.998211$$

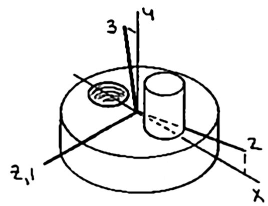
$$(\theta_x)_3 = 93.4° \qquad (\theta_y)_3 = 3.43° \qquad (\theta_z)_3 = 90.0° \blacktriangleleft$$

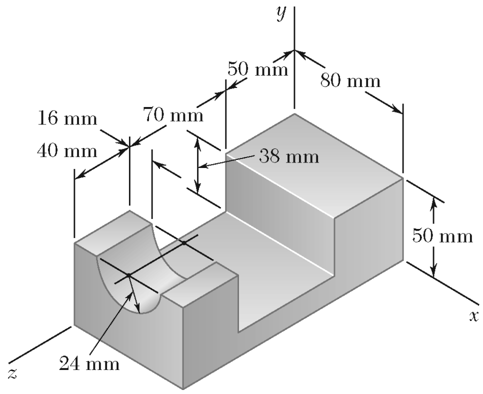
Note: Since the principal axes are orthogonal and $(\theta_z)_2 = (\theta_z)_3 = 90°$, it follows that

$$|(\lambda_x)_2| = |(\lambda_y)_3| \qquad |(\lambda_y)_2| = |(\lambda_z)_3|$$

The differences in the above values are due to round-off errors.

(*c*) Principal axis 1 coincides with the *z* axis, while principal axes 2 and 3 lie in the *xy* plane.





PROBLEM 9.181*

For the component described in Problems 9.145 and 9.149, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145: $I_x = 26.4325\times10^{-3}\text{ kg}\cdot\text{m}^2$ Problem 9.149: $I_{xy} = 2.5002\times10^{-3}\text{ kg}\cdot\text{m}^2$

$I_y = 31.1726\times10^{-3}\text{ kg}\cdot\text{m}^2$ $I_{yz} = 4.0627\times10^{-3}\text{ kg}\cdot\text{m}^2$

$I_z = 8.5773\times10^{-3}\text{ kg}\cdot\text{m}^2$ $I_{zx} = 8.8062\times10^{-3}\text{ kg}\cdot\text{m}^2$

Substituting into Eq. (9.56):

$$\begin{aligned}
&K^3 - [(26.4325+31.1726+8.5773)(10^{-3})]K^2\\
&\quad + [(26.4325)(31.1726)+(31.1726)(8.5773)+(8.5773)(26.4325)\\
&\quad - (2.5002)^2 - (4.0627)^2 - (8.8062)^2](10^{-6})K\\
&\quad - [(26.4325)(31.1726)(8.5773) - (26.4325)(4.0627)^2\\
&\quad - (31.1726)(8.8062)^2 - (8.5773)(2.5002)^2\\
&\quad - 2(2.5002)(4.0627)(8.8062)](10^{-9}) = 0
\end{aligned}$$

or $$K^3 - (66.1824\times10^{-3})K^2 + (1217.76\times10^{-6})K - (3981.23\times10^{-9}) = 0$$

Solving yields

$K_1 = 4.1443\times10^{-3}\text{ kg}\cdot\text{m}^2$ or $K_1 = 4.14\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$K_2 = 29.7840\times10^{-3}\text{ kg}\cdot\text{m}^2$ or $K_2 = 29.8\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$K_3 = 32.2541\times10^{-3}\text{ kg}\cdot\text{m}^2$ or $K_3 = 32.3\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

(*b*) To determine the direction cosines λ_x, λ_y, λ_z of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting:

$$[(26.4325 - 4.1443)(10^{-3})](\lambda_x)_1 - (2.5002\times10^{-3})(\lambda_y)_1 - (8.8062\times10^{-3})(\lambda_z)_1 = 0$$

$$-(2.5002\times10^{-3})(\lambda_x)_1 + [(31.1726 - 4.1443)(10^{-3})](\lambda_y)_1 - (4.0627\times10^{-3})(\lambda_z)_1 = 0$$

Simplifying

$$8.9146(\lambda_x)_1 - (\lambda_y)_1 - 3.5222(\lambda_z)_1 = 0$$
$$-0.0925(\lambda_x)_1 + (\lambda_y)_1 - 0.1503(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 2.4022(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [8.9146 - 3.5222(2.4022)](\lambda_x)_1$$
$$= 0.45357(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.45357(\lambda_x)_1]^2 + [2.4022(\lambda_x)_1]^2 = 1$$

or $\quad (\lambda_x)_1 = 0.37861$

and $\quad (\lambda_y)_1 = 0.17173 \quad (\lambda_z)_1 = 0.90950$

$$(\theta_x)_1 = 67.8° \quad (\theta_y)_1 = 80.1° \quad (\theta_z)_1 = 24.6° \blacktriangleleft$$

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting:

$$[(26.4325 - 29.7840)(10^{-3})](\lambda_x)_2 - (2.5002\times10^{-3})(\lambda_y)_2 - (8.8062\times10^{-3})(\lambda_z)_2 = 0$$

$$-(2.5002\times10^{-3})(\lambda_x)_2 + [(31.1726 - 29.7840)(10^{-3})](\lambda_y)_2 - (4.0627\times10^{-3})(\lambda_z)_2 = 0$$

Simplifying

$$-1.3405(\lambda_x)_2 - (\lambda_y)_2 - 3.5222(\lambda_z)_2 = 0$$
$$-1.8005(\lambda_x)_2 + (\lambda_y)_2 - 2.9258(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$:

$$(\lambda_z)_2 = -0.48713(\lambda_x)_2$$

and then

$$(\lambda_y)_2 = [-1.3405 - 3.5222(-0.48713)](\lambda_x)_2$$
$$= 0.37527(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [0.37527(\lambda_x)_2]^2 + [-0.48713(\lambda_x)_2]^2 = 1$$

or $$(\lambda_x)_2 = 0.85184$$

and $$(\lambda_y)_2 = 0.31967 \qquad (\lambda_z)_2 = -0.41496$$

$$(\theta_x)_1 = 31.6° \quad (\theta_y)_2 = 71.4° \quad (\theta_z)_2 = 114.5° \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$
$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_z)_3 = 0$$

Substituting:

$$[(26.4325 - 32.2541)(10^{-3})](\lambda_x)_3 - (2.5002 \times 10^{-3})(\lambda_y)_3 - (8.8062 \times 10^{-3})(\lambda_z)_3 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_3 + [(31.1726 - 32.2541)(10^{-3})](\lambda_y)_3 - (4.0627 \times 10^{-3})(\lambda_z)_3 = 0$$

Simplifying

$$-2.3285(\lambda_x)_3 - (\lambda_y)_3 - 3.5222(\lambda_z)_3 = 0$$
$$2.3118(\lambda_x)_3 + (\lambda_y)_3 + 3.7565(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$:

$$(\lambda_z)_3 = 0.071276(\lambda_x)_3$$

and then

$$(\lambda_y)_3 = [-2.3285 - 3.5222(0.071276)](\lambda_x)_3$$
$$= -2.5795(\lambda_x)_3$$

Now substitute into Eq. (9.57):

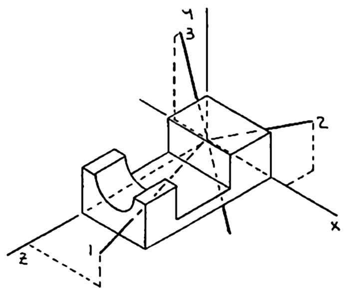
$$(\lambda_x)_3^2 + [-2.5795(\lambda_x)_3]^2 + [0.071276(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or $$(\lambda_x)_3 = 0.36134$$

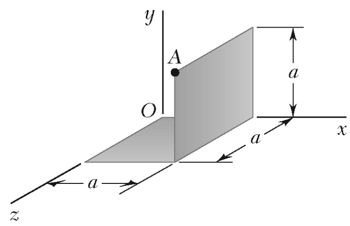
and $$(\lambda_y)_3 = 0.93208 \qquad (\lambda_z)_3 = 0.025755$$

$$(\theta_x)_3 = 68.8° \quad (\theta_y)_3 = 158.8° \quad (\theta_z)_3 = 88.5° \blacktriangleleft$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.36134$.



Then $$(\theta_x)_3 = 111.2° \quad (\theta_y)_3 = 21.2° \quad (\theta_z)_3 = 91.5° \blacktriangleleft$$



PROBLEM 9.182*

For the component described in Problem 9.167, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution of Problem 9.167, we have

$$I_x = \frac{1}{2}\frac{W}{g}a^2 \qquad I_{xy} = \frac{1}{4}\frac{W}{g}a^2$$

$$I_y = \frac{W}{g}a^2 \qquad I_{yz} = \frac{1}{8}\frac{W}{g}a^2$$

$$I_z = \frac{5}{6}\frac{W}{g}a^2 \qquad I_{zx} = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Eq. (9.56):

$$\begin{aligned}K^3 &- \left[\left(\frac{1}{2}+1+\frac{5}{6}\right)\left(\frac{W}{g}a^2\right)\right]K^2 \\ &+ \left[\left(\frac{1}{2}\right)(1)+(1)\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)-\left(\frac{1}{4}\right)^2-\left(\frac{1}{8}\right)^2-\left(\frac{3}{8}\right)^2\right]\left(\frac{W}{g}a^2\right)^2 K \\ &- \left[\left(\frac{1}{2}\right)(1)\left(\frac{5}{6}\right)-\left(\frac{1}{2}\right)\left(\frac{1}{8}\right)^2-(1)\left(\frac{3}{8}\right)^2-\left(\frac{5}{6}\right)\left(\frac{1}{4}\right)^2-2\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\left(\frac{3}{8}\right)\right]\left(\frac{W}{g}a^2\right)^3 = 0\end{aligned}$$

Simplifying and letting $K = \frac{W}{g}a^2 K$ yields

$$K^3 - 2.33333K^2 + 1.53125K - 0.192708 = 0$$

Solving yields

$$K_1 = 0.163917 \quad K_2 = 1.05402 \quad K_3 = 1.11539$$

The principal moments of inertia are then

$$K_1 = 0.1639\frac{W}{g}a^2 \blacktriangleleft$$

$$K_2 = 1.054\frac{W}{g}a^2 \blacktriangleleft$$

$$K_3 = 1.115\frac{W}{g}a^2 \blacktriangleleft$$

(*b*) To determine the direction cosines λ_x, λ_y, λ_z of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[\left(\frac{1}{2} - 0.163917\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_1 - \left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_1 - \left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_1 + \left[(1 - 0.163917)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_1 - \left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

Simplifying

$$1.34433(\lambda_x)_1 - (\lambda_y)_1 - 1.5(\lambda_z)_1 = 0$$

$$-0.299013(\lambda_x)_1 + (\lambda_y)_1 - 0.149507(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 0.633715(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [1.34433 - 1.5(0.633715)](\lambda_x)_1$$
$$= 0.393758(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.393758(\lambda_x)_1]^2 + (0.633715(\lambda_x)_1]^2 = 1$$

or $(\lambda_x)_1 = 0.801504$

and $(\lambda_y)_1 = 0.315599$ $\quad$ $(\lambda_z)_1 = 0.507925$

$$(\theta_x)_1 = 36.7° \quad (\theta_y)_1 = 71.6° \quad (\theta_z)_1 = 59.5° \blacktriangleleft$$

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - k_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - k_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting

$$\left[\left(\frac{1}{2}-1.05402\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_2-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_2-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_2+\left[(1-1.05402)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_2-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

Simplifying

$$-2.21608(\lambda_x)_2-(\lambda_y)_2-1.5(\lambda_z)_2=0$$

$$4.62792(\lambda_x)_2+(\lambda_y)_2+2.31396(\lambda_z)_2=0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2=-2.96309(\lambda_x)_2$$

and then

$$(\lambda_y)_2=[-2.21608-1.5(-2.96309)](\lambda_x)_2$$
$$=2.22856(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2+[2.22856(\lambda_x)_2]^2+[-2.96309(\lambda_x)_2]^2=1$$

or $(\lambda_x)_2=0.260410$

and $(\lambda_y)_2=0.580339$ $\quad (\lambda_z)_2=-0.771618$

$(\theta_x)_2=74.9°\quad (\theta_y)_2=54.5°\quad (\theta_z)_2=140.5°$ ◀

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x-K_3)(\lambda_x)_3-I_{xy}(\lambda_y)_3-I_{zx}(\lambda_z)_3=0$$
$$-I_{xy}(\lambda_x)_3+(I_y-K_3)(\lambda_y)_3-I_{yz}(\lambda_z)_3=0$$

Substituting

$$\left[\left(\frac{1}{2}-1.11539\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_3-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_3-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_3+\left[(1-1.11539)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_3-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

Simplifying

$$-2.46156(\lambda_x)_3-(\lambda_y)_3-1.5(\lambda_z)_3=0$$
$$2.16657(\lambda_x)_3+(\lambda_y)_3+1.08328(\lambda_z)_3=0$$

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = -0.707885(\lambda_x)_3$$

and then
$$(\lambda_y)_3 = [-2.46156 - 1.5(-0.707885)](\lambda_x)_3$$
$$= -1.39973(\lambda_x)_3$$

Now substitute into Eq. (9.57):

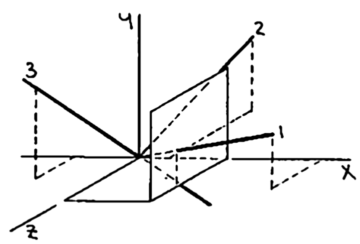
$$(\lambda_x)_3^2 + [-1.39973(\lambda_x)_3]^2 + [-0.707885(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or
$$(\lambda_x)_3 = 0.537577$$

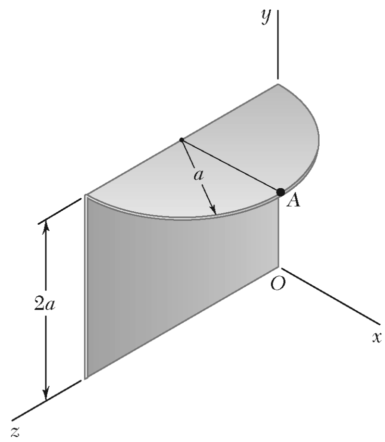
and
$$(\lambda_y)_3 = -0.752463 \qquad (\lambda_z)_3 = -0.380543$$

$$(\theta_x)_3 = 57.5° \quad (\theta_y)_3 = 138.8° \quad (\theta_z)_3 = 112.4° \blacktriangleleft$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.537577$.



Then
$$(\theta_x)_3 = 122.5° \quad (\theta_y)_3 = 41.2° \quad (\theta_z)_3 = 67.6° \blacktriangleleft$$



PROBLEM 9.183*

For the component described in Problem 9.168, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution to Problem 9.168, we have

$$I_x = 18.91335\frac{\gamma t}{g}a^4 \qquad I_{xy} = 1.33333\frac{\gamma t}{g}a^4$$

$$I_y = 7.68953\frac{\gamma t}{g}a^4 \qquad I_{yz} = 7.14159\frac{\gamma t}{g}a^4$$

$$I_z = 12.00922\frac{\gamma t}{g}a^4 \qquad I_{zx} = 0.66667\frac{\gamma t}{g}a^4$$

Substituting into Eq. (9.56):

$$K^3 - \left[(18.91335 + 7.68953 + 12.00922)\left(\frac{\gamma t}{g}a^4\right)\right]K^2$$
$$+ [(18.91335)(7.68953) + (7.68953)(12.00922) + (12.00922)(18.91335)$$
$$- (1.33333)^2 - (7.14159)^2 + (0.66667)^2]\left(\frac{\gamma t}{g}a^4\right)^2 K$$
$$- [(18.91335)(7.68953)(12.00922) - (18.91335)(7.14159)^2$$
$$- (7.68953)(0.66667)^2 - (12.00922)(1.33333)^2$$
$$- 2(1.33333)(7.14159)(0.66667)]\left(\frac{\gamma t}{g}a^4\right)^3 = 0$$

Simplifying and letting

$$K = \frac{\gamma t}{g}a^4 K \quad \text{yields}$$

$$K^3 - 38.61210K^2 + 411.69009K - 744.47027 = 0$$

Solving yields

$$K_1 = 2.25890 \quad K_2 = 17.27274 \quad K_3 = 19.08046$$

The principal moments of inertia are then $K_1 = 2.26\dfrac{\gamma t}{g}a^4$ ◀

$$K_2 = 17.27\frac{\gamma t}{g}a^4 \;◀$$

$$K_3 = 19.08\frac{\gamma t}{g}a^4 \;◀$$

(*b*) To determine the direction cosines λ_x, λ_y, λ_z of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[(18.91335 - 2.25890)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_x)_1 - \left(1.33333\frac{\gamma t}{a}a^4\right)(\lambda_y)_1 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_1 = 0$$

$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_1 + \left[(7.68953 - 2.25890)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_1 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_1 = 0$$

Simplifying

$$12.49087(\lambda_x)_1 - (\lambda_y)_1 - 0.5(\lambda_z)_1 = 0$$
$$-0.24552(\lambda_x)_1 + (\lambda_y)_1 - 1.31506(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$

$$(\lambda_z)_1 = 6.74653(\lambda_x)_1$$

and then $(\lambda_y)_1 = [12.49087 - (0.5)(6.74653)](\lambda_x)_1$

$$= 9.11761(\lambda_x)_1$$

Now substitute into Eq. (9.57): $(\lambda_x)_1^2 + [9.11761(\lambda_x)_1]^2 + [6.74653(\lambda_x)_1]^2 = 1$

or $(\lambda_x)_1 = 0.087825$

and $(\lambda_y)_1 = 0.80075 \qquad (\lambda_z)_1 = 0.59251$

$$(\theta_x)_1 = 85.0° \quad (\theta_y)_1 = 36.8° \quad (\theta_z)_1 = 53.7° \;◀$$

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting

$$[(18.91335 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_2 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$
$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 + \left[(7.68953 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_2 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$

Simplifying

$$1.23046(\lambda_x)_2 - (\lambda_y)_2 - 0.5(\lambda_z)_2 = 0$$
$$0.13913(\lambda_x)_2 + (\lambda_y)_2 + 0.74522(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2 = -5.58515(\lambda_x)_2$$

and then
$$(\lambda_y)_2 = [1.23046 - (0.5)(-5.58515)](\lambda_x)_2 = 4.02304\,(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [4.02304(\lambda_x)_2]^2 + [-5.58515(\lambda_x)_2]^2 = 1$$

or
$$(\lambda_x)_2 = 0.14377$$

and
$$(\lambda_y)_2 = 0.57839 \qquad (\lambda_z)_2 = -0.80298$$

$$(\theta_x)_2 = 81.7° \quad (\theta_y)_2 = 54.7° \quad (\theta_z)_2 = 143.4° \;◀$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$
$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_y)_3 = 0$$

Substituting

$$[(18.91335 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_3 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$
$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 + \left[(7.68953 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_3 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$

Simplifying

$$-0.12533(\lambda_x)_3 - (\lambda_y)_3 - 0.5(\lambda_z)_3 = 0$$
$$0.11705(\lambda_x)_3 + (\lambda_y)_3 + 0.62695(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = 0.06522(\lambda_x)_3$$

and then

$$(\lambda_y)_3 = [-0.12533 - (0.5)(0.06522)](\lambda_x)_3$$
$$= -0.15794(\lambda_x)_3$$

Now substitute into Eq. (9.57):

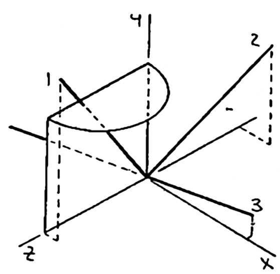
$$(\lambda_x)_3^2 + [-0.15794(\lambda_x)_3]^2 + [0.06522(\lambda_x]_3]^2 = 1 \qquad \text{(i)}$$

or $(\lambda_x)_3 = 0.98571$

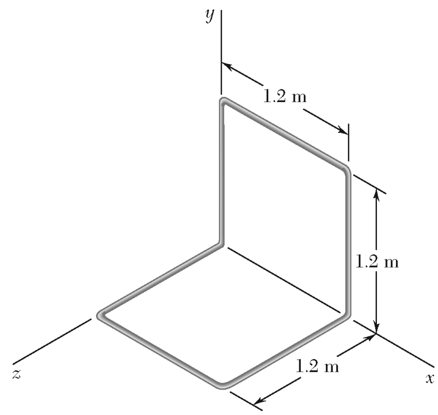
and $(\lambda_y)_3 = -0.15568 \qquad (\lambda_z)_3 = 0.06429$

$(\theta_x)_3 = 9.7° \quad (\theta_y)_3 = 99.0° \quad (\theta_z)_3 = 86.3°$ ◀

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.98571$.



Then $(\theta_x)_3 = 170.3° \quad (\theta_y)_3 = 81.0° \quad (\theta_z)_3 = 93.7°$ ◀



PROBLEM 9.184*

For the component described in Problems 9.148 and 9.170, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

SOLUTION

(*a*) From the solutions to Problems 9.148 and 9.170. We have

$$I_x = 0.32258 \text{ kg} \cdot \text{m}^2 \qquad I_y = I_z = 0.41933 \text{ kg} \cdot \text{m}^2$$

$$I_{xy} = I_{zx} = 0.096768 \text{ kg} \cdot \text{m}^2 \qquad I_{yz} = 0$$

Substituting into Eq. (9.56) and using

$$I_y = I_z \qquad I_{xy} = I_{zx} \qquad I_{yz} = 0$$

$$\begin{aligned} &K^3 - [0.32258 + 2(0.41933)]K^2 \\ &\quad + [2(0.32258)(0.41933) + (0.41933)^2 - 2(0.096768)^2]K \\ &\quad - [(0.32258)(0.41933)^2 - 2(0.41933)(0.096768)^2] = 0 \end{aligned}$$

Simplifying

$$K^3 - 1.16124K^2 + 0.42764K - 0.048869 = 0$$

Solving yields

$$K_1 = 0.22583 \text{ kg} \cdot \text{m}^2 \qquad \text{or} \qquad K_1 = 0.226 \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$K_2 = 0.41920 \text{ kg} \cdot \text{m}^2 \qquad \text{or} \qquad K_2 = 0.419 \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$K_3 = 0.51621 \text{ kg} \cdot \text{m}^2 \qquad \text{or} \qquad K_3 = 0.516 \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

(*b*) To determine the direction cosines λ_x, λ_y, λ_z of each principal axis, use two of the equations of Eqs. (9.54) and (9.57). Then

K_1: Begin with Eqs. (9.54*b*) and (9.54*c*):

$$-I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - \cancelto{0}{I_{yz}}(\lambda_z)_3 = 0$$

$$-I_{zx}(\lambda_x)_1 - \cancelto{0}{I_{yz}}(\lambda_y)_1 + (I_z - K_1)(\lambda_z)_3 = 0$$

PROBLEM 9.184* (Continued)

Substituting

$$-(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_y)_1 = 0$$
$$-(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_z)_1 = 0$$

Simplifying yields

$$(\lambda_y)_1 = (\lambda_z)_1 = 0.50009(\lambda_x)_1$$

Now substitute into Eq. (9.54):

$$(\lambda_x)^2 + 2[0.50009(\lambda_x)_1]^2 = 1$$

or $$(\lambda_x)_1 = 0.81645$$

and $$(\lambda_y)_1 = (\lambda_z)_1 = 0.40830$$

$$(\theta_x)_1 = 35.3^\circ \quad (\theta_y)_1 = (\theta_z)_1 = 65.9^\circ \blacktriangleleft$$

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*)

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - \cancelto{0}{I_{yz}}(\lambda_z)_2 = 0$$

Substituting

$$(0.32258 - 0.41920)(\lambda_x)_2 - (0.096768)(\lambda_y)_2 - (0.096768)(\lambda_z)_2 = 0 \qquad \text{(i)}$$
$$-(0.96768)(\lambda_x)_2 + (0.41933 - 0.41920)(\lambda_y)_2 = 0 \qquad \text{(ii)}$$

Eq. (ii) $\Rightarrow$ $(\lambda_x)_2 = 0$

and then Eq. (i) $\Rightarrow$ $(\lambda_z)_2 = -(\lambda_y)_2$

Now substitute into Eq. (9.57):

$$\cancelto{0}{(\lambda_x)_2^2} + (\lambda_y)_2^2 + [-(\lambda_y)_2]^2 = 1$$

or $$(\lambda_y)_2 = \frac{1}{\sqrt{2}}$$

and $$(\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$

$$(\theta_x)_2 = 90.0^\circ \quad (\theta_y)_2 = 45.0^\circ \quad (\theta_z)_2 = 135.0^\circ \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*b*) and (9.54*c*):

$$I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 + \cancelto{0}{I_{yz}}(\lambda_z)_3 = 0$$

$$-I_{zx}(\lambda_x)_3 - \cancelto{0}{I_{yz}}(\lambda_y)_3 + (I_z - K_3)(\lambda_z)_3 = 0$$

Substituting

$$-(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_y)_3 = 0$$

$$-(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_z)_3 = 0$$

Simplifying yields

$$(\lambda_y)_3 = (\lambda_z)_3 = -(\lambda_x)_3$$

Now substitute into Eq. (9.57):

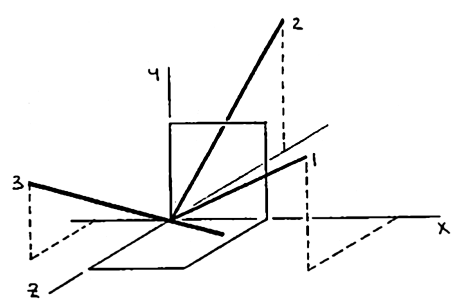
$$(\lambda_x)_3^2 + 2[-(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or $$(\lambda_x)_3 = \frac{1}{\sqrt{3}}$$

and $$(\lambda_y)_3 = (\lambda_z) = -\frac{1}{\sqrt{3}}$$

$(\theta_x)_3 = 54.7° \quad (\theta_y)_3 = (\theta_z)_3 = 125.3°$ ◀

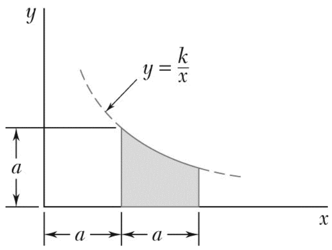
(*c*)



Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen;

That is, $$(\lambda_x)_3 = -\frac{1}{\sqrt{3}}$$

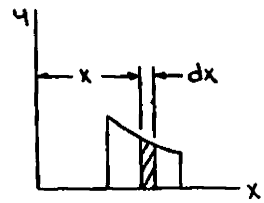
Then $(\theta_x)_3 = 125.3° \qquad (\theta_y)_3 = (\theta_z)_3 = 54.7°$ ◀



PROBLEM 9.185

Determine by direct integration the moments of inertia of the shaded area with respect to the x and y axes.

SOLUTION



At $x = a, \quad y = a$: $\quad a = \frac{k}{a} \quad$ or $\quad k = a^2$

Then $\quad y = \frac{a^2}{x}$

Now
$$dI_x = \frac{1}{3}y^3 dx = \frac{1}{3}\left(\frac{a^2}{x}\right)^3 dx$$
$$= \frac{1}{3}\frac{a^6}{x^3}dx$$

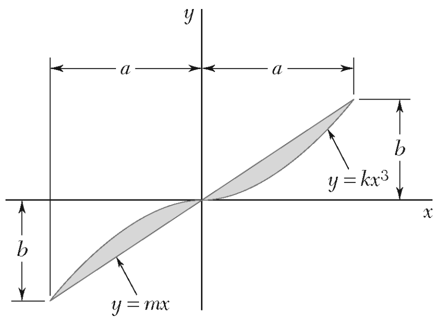
Then
$$I_x = \int dI_x = \int_a^{2a} \frac{1}{3}\frac{a^6}{x^3}dx = \frac{1}{3}a^6\left[-\frac{1}{2}\frac{1}{x^2}\right]_a^{2a}$$
$$= -\frac{1}{6}a^6\left[\frac{1}{(2a)^2} - \frac{1}{(a)^2}\right]$$

or $\quad I_x = \frac{1}{8}a^4$ ◀

Now
$$dI_y = x^2 \quad dA = x^2(y\,dx)$$
$$= x^2\left(\frac{a^2}{x}dx\right) = a^2 x\,dx$$

Then
$$I_y = \int dI_y = \int_a^{2a} a^2 x\,dx = a^2\left[\frac{1}{2}x^2\right]_a^{2a} = \frac{a^2}{2}[(2a)^2 - (a)^2]$$

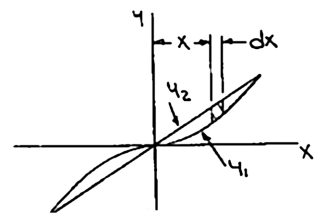
or $\quad I_y = \frac{3}{2}a^4$ ◀



PROBLEM 9.186

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION



At $x = a, \quad y_1 = y_2 = b$: $\quad y_1: \; b = ka^3 \;$ or $\; k = \dfrac{b}{a^3}$

$$y_2: \; b = ma \quad \text{or} \quad m = \frac{b}{a}$$

Then
$$y_1 = \frac{b}{a^3}x^3$$
$$y_2 = \frac{b}{a}x$$

Now
$$dA = (y_2 - y_1)dx = \left(\frac{b}{a}x - \frac{b}{a^3}x^3\right)dx$$

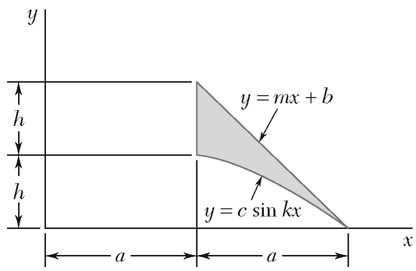
Then
$$A = \int dA = 2\int_0^a \frac{b}{a}\left(x - \frac{1}{a^2}x^3\right)dx$$
$$= 2\frac{b}{a}\left[\frac{1}{2}x^2 - \frac{1}{4a^2}x^4\right]_0^a = \frac{1}{2}ab$$

Now
$$dI_y = x^2 dA = x^2\left[\left(\frac{b}{a}x - \frac{b}{a^3}x^3\right)dx\right]$$

Then
$$I_y = \int dI_y = 2\int_0^a \frac{b}{a}x^2\left(x - \frac{1}{a^2}x^3\right)dx$$
$$= 2\frac{b}{a}\left[\frac{1}{4}x^4 - \frac{1}{6}\frac{1}{a^2}x^6\right]_0^a \qquad \text{or} \quad I_y = \frac{1}{6}a^3b \;\blacktriangleleft$$

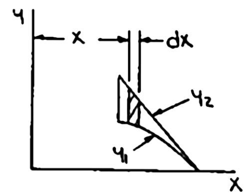
and
$$k_y^2 = \frac{I_y}{A} = \frac{\frac{1}{6}a^3b}{\frac{1}{2}ab} = \frac{1}{3}a^2 \qquad \text{or} \quad k_y = \frac{a}{\sqrt{3}} \;\blacktriangleleft$$

PROBLEM 9.187



Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION



y_1: At $x = 2a, \quad y = 0$: $\quad 0 = c\sin k(2a)$

$$2ak = \pi \quad \text{or} \quad k = \frac{\pi}{2a}$$

At $x = a, \quad y = h$: $\quad h = c\sin\frac{\pi}{2a}(a) \quad \text{or} \quad c = h$

y_2: At $x = a, \quad y = 2h$: $\quad 2h = ma - b$

At $x = 2a, \quad y = 0$: $\quad 0 = m(2a) + b$

Solving yields $\quad m = -\frac{2h}{a}, \quad b = 4h$

Then $\quad y_1 = h\sin\frac{\pi}{2a}x \qquad y_2 = -\frac{2h}{a}x + 4h$

$$= \frac{2h}{a}(-x + 2a)$$

Now $\quad dA = (y_2 - y_1)dx = \left[\frac{2h}{a}(-x + 2a) - h\sin\frac{\pi}{2a}x\right]dx$

Then $\quad A = \int dA = \int_a^{2a} h\left[\frac{2}{a}(-x + 2a) - \sin\frac{\pi}{2a}x\right]dx$

$$= h\left[-\frac{1}{a}(-x + 2a)^2 + \frac{2a}{\pi}\cos\frac{\pi}{2a}x\right]_a^{2a}$$

$$= h\left[\left(-\frac{2a}{\pi}\right) + \frac{1}{a}(-a + 2a)^2\right] = ah\left(1 - \frac{2}{\pi}\right)$$

$$= 0.36338ah$$

Find: I_x and k_x

We have
$$dI_x = \left(\frac{1}{3}y_2 - \frac{1}{3}y_1\right)dx = \frac{1}{3}\left\{\left[\frac{2h}{a}(-x+2a)\right]^3 - \left(h\sin\frac{\pi}{2a}x\right)^3\right\}dx$$
$$= \frac{h^3}{3}\left[\frac{8}{a^3}(-x+2a)^3 - \sin^3\frac{\pi}{2a}x\right]$$

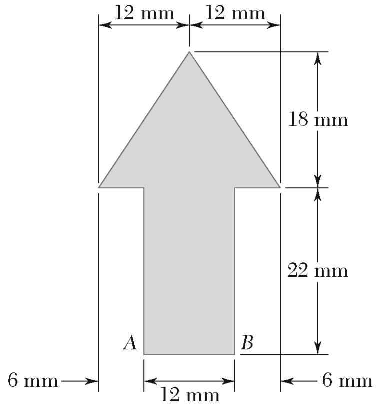
Then
$$I_x = \int dI_x = \int_a^{2a}\frac{h^3}{3}\left[\frac{8}{a^3}(-x+2a)^3 - \sin^3\frac{\pi}{2a}x\right]dx$$

Now
$$\sin^3\theta = \sin\theta(1-\cos^2\theta) = \sin\theta - \sin\theta\cos^2\theta$$

Then
$$I_x = \frac{h^3}{3}\int_a^{2a}\left[\frac{8}{a^3}(-x+2a)^3 - \left(\sin\frac{\pi}{2a}x - \sin\frac{\pi}{2a}x\cos^2\frac{\pi}{2a}x\right)\right]dx$$
$$= \frac{h^3}{3}\left[-\frac{2}{a^3}(-x+2a)^4 + \frac{2a}{\pi}\cos\frac{\pi}{2a}x - \frac{2a}{3\pi}\cos^3\frac{\pi}{2a}x\right]_a^{2a}$$
$$= \frac{h^3}{3}\left[\left(-\frac{2a}{\pi}+\frac{2a}{3\pi}\right) + \frac{2}{a^3}(-a+2a)^4\right]$$
$$= \frac{2}{3}ah^3\left(1-\frac{2}{3\pi}\right)$$

$$I_x = 0.52520ah^3 \qquad \text{or} \qquad I_x = 0.525ah^3 \blacktriangleleft$$

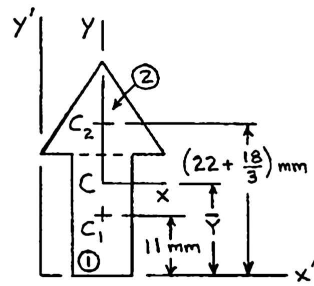
and
$$k_x^2 = \frac{I_x}{A} = \frac{0.52520ah^3}{0.36338ah} \qquad \text{or} \qquad k_x = 1.202h \blacktriangleleft$$



PROBLEM 9.188

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate C of the area:

Symmetry implies $\bar{X} = 12$ mm.

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$12 \times 22 = 264$	11	2904
2	$\frac{1}{2}(24)(18) = 216$	28	6048
Σ	480		8952

Then $\quad \bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(480 \text{ mm}^2) = 8952 \text{ mm}^3$

$$\bar{Y} = 18.65 \text{ mm}$$

Now $\quad \bar{I}_x = (I_x)_1 + (I_x)_2$

where $\quad (I_x)_1 = \frac{1}{12}(12 \text{ mm})(22 \text{ mm})^3 + (264 \text{ mm}^2)[(18.65 - 11) \text{ mm}]^2$

$$= 26{,}098 \text{ mm}^4$$

$$(I_x)_2 = \frac{1}{36}(24 \text{ mm})(18 \text{ mm})^3 + (216 \text{ mm}^2)[(28 - 18.65) \text{ mm}]^2$$

$$= 22{,}771 \text{ mm}^4$$

Then $\quad \bar{I}_x = (26.098 + 22.771) \times 10^3 \text{ mm}^4$

or $\quad \bar{I}_x = 48.9 \times 10^3 \text{ mm}^4$ ◀

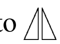
PROBLEM 9.188 (Continued)

Also $$\bar{I}_y = (I_y)_1 + (I_y)_2$$

where $$(I_y)_1 = \frac{1}{12}(22\text{ mm})(12\text{ mm})^3 = 3168\text{ mm}^4$$

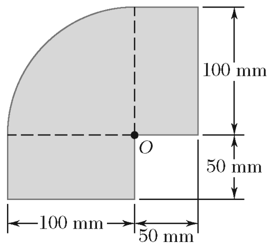
$$(I_y)_2 = 2\left[\frac{1}{36}(18\text{ mm})(12\text{ mm})^3 + \left(\frac{1}{2}\times 18\text{ mm}\times 12\text{ mm}\right)(4\text{ mm})^2\right]$$

$$= 5184\text{ mm}^4$$

[$(I_y)_2$ is obtained by dividing A_2 into ]

Then $$\bar{I}_y = (3168 + 5184)\text{ mm}^4$$

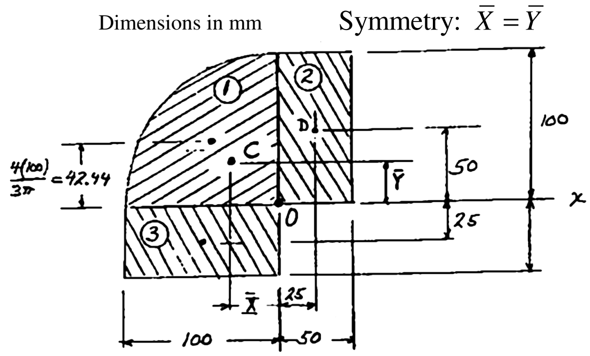
or $\bar{I}_y = 8.35\times 10^3\text{ mm}^4$ ◀



PROBLEM 9.189

Determine the polar moment of inertia of the area shown with respect to (*a*) Point O, (*b*) the centroid of the area.

SOLUTION



Determination of centroid C of entire section.

Section	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{\pi}{4}(100)^2 = 7.854\times10^3$	42.44	333.3×10^3
2	$(50)(100) = 5\times10^3$	50	250×10^3
3	$(100)(50) = 5\times10^3$	–25	-125×10^3
Σ	17.854×10^3		458.3×10^3

$$\bar{Y}\Sigma A = \Sigma\bar{y}A: \quad \bar{Y}(17.854\times10^3 \text{ mm}^2) = 458.3\times10^3 \text{ mm}^3$$

$$\bar{Y} = 25.67 \text{ mm} \quad \bar{X} = \bar{Y} = 25.67 \text{ mm}$$

Distance O to C: $\overline{OC} = \sqrt{2}\bar{Y} = \sqrt{2}(25.67) = 36.30 \text{ mm}$

(*a*) Section 1: $J_O = \frac{\pi}{8}(100)^4 = 39.27\times10^6 \text{ mm}^4$

Section 2: $J_O = J + A(\overline{OD})^2 = \frac{1}{12}(50)(100)[50^2 + 100^2] + (50)(100)\left[\left(\frac{50}{2}\right)^2 + \left(\frac{100}{2}\right)^2\right]$

$$J_O = 5.208\times10^6 + 15.625\times10^6 = 20.83\times10^6 \text{ mm}^4$$

PROBLEM 9.189 (Continued)

Section 3: Same as Section 2; $J_O = 20.83 \times 10^6 \text{ mm}^4$

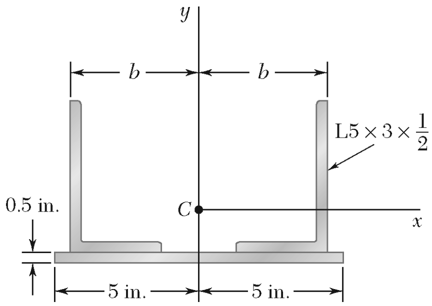
Entire section:

$$J_O = 39.27 \times 10^6 + 2(20.83 \times 10^6)$$
$$= 80.94 \times 10^6 \qquad J_O = 80.9 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

(*b*) Recall that, $\overline{OC} = 36.30 \text{ mm}$ and $A = 17.854 \times 10^3 \text{ mm}^2$

$$J_O = \bar{J}_C + A(\overline{OC})^2$$
$$80.94 \times 10^6 \text{ mm}^4 = \bar{J}_C + (17.854 \times 10^3 \text{ mm}^2)(36.30 \text{ mm})^2$$
$$\bar{J}_C = 57.41 \times 10^6 \text{ mm}^4 \qquad \bar{J}_C = 57.4 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

PROBLEM 9.190



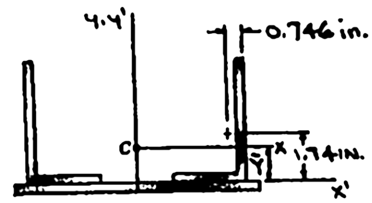
Two L5 × 3 × $\frac{1}{2}$-in. angles are welded to a $\frac{1}{2}$-in. steel plate. Determine the distance b and the centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the combined section, knowing that $\bar{I}_y = 4\bar{I}_x$.

SOLUTION

Angle: $A = 3.75\,\text{in}^2$

$\bar{I}_x = 9.43\,\text{in}^4$

$\bar{I}_y = 2.55\,\text{in}^4$



First locate centroid C of the section.

	Area, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
Angle	$2(3.75) = 7.50$	1.74	13.05
Plate	$(10)(0.5) = 5$	–0.25	−1.25
Σ	12.50		11.80

Then $\quad \bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(12.50\,\text{in}^2) = 11.80\,\text{in}^3$

or $\quad \bar{Y} = 0.944\,\text{in.}$

Now $\quad \bar{I}_x = 2(I_x)_{\text{angle}} + (I_x)_{\text{plate}}$

where $\quad (I_x)_{\text{angle}} = \bar{I}_x + Ad^2 = 9.43\,\text{in}^4 + (3.75\,\text{in}^2)[(1.74 - 0.944)\,\text{in.}]^2$

$$= 11.8061\,\text{in}^4$$

$$(I_x)_{\text{plate}} = \bar{I}_x + Ad^2 = \frac{1}{12}(10\,\text{in.})(0.5\,\text{in.})^3 + (5\,\text{in}^2)[(0.25 + 0.944)\,\text{in.}]^2$$

$$= 7.2323\,\text{in}^4$$

Then $\quad \bar{I}_x = [2(11.8061) + 7.2323]\,\text{in}^4 = 30.8445\,\text{in}^4$

or $\quad \bar{I}_x = 30.8\,\text{in}^4$ ◀

PROBLEM 9.190 (Continued)

We have $\bar{I}_y = 4\bar{I}_x = 4(30.8445 \text{ in}^4) = 123.378 \text{ in}^4$

or $\bar{I}_y = 123.4 \text{ in}^4$ ◀

Now $\bar{I}_y = 2(I_y)_{\text{angle}} + (\bar{I}_y)_{\text{plate}}$

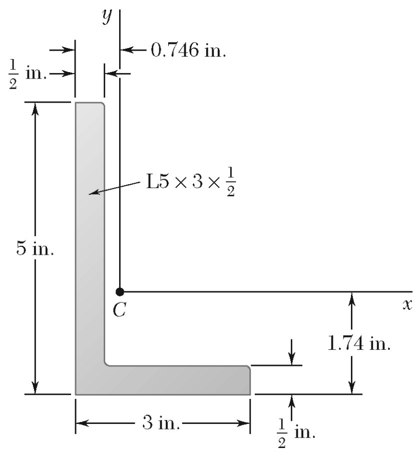
where $(\bar{I}_y)_{\text{angle}} = \bar{I}_y + Ad^2 = 2.55 \text{ in}^4 + (3.75 \text{ in}^2)[(b - 0.746) \text{ in.}]^2$

$$(\bar{I}_y)_{\text{plate}} = \frac{1}{12}(0.5 \text{ in.})(10 \text{ in.})^3 = 41.6667 \text{ in}^4$$

Then $123.378 \text{ in}^4 = 2[2.55 + 3.75(b - 0.746)^2] \text{ in}^4 + 41.6667 \text{ in}^4$

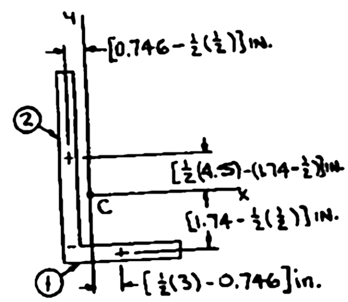
or $b = 3.94 \text{ in.}$ ◀

PROBLEM 9.191



Using the parallel-axis theorem, determine the product of inertia of the L5 × 3 × $\frac{1}{2}$-in. angle cross section shown with respect to the centroidal x and y axes.

SOLUTION



We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each rectangle:

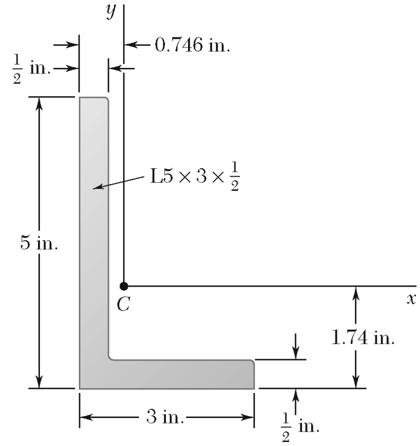
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

and $$\bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

Thus $$I_{xy} = \Sigma \bar{x}\,\bar{y}A$$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
1	$3 \times \frac{1}{2} = 1.5$	0.754	−1.49	−1.68519
2	$4.5 \times \frac{1}{2} = 2.25$	−0.496	1.01	−1.12716
Σ				−2.81235

$$\bar{I}_{xy} = -2.81 \text{ in}^4 \blacktriangleleft$$

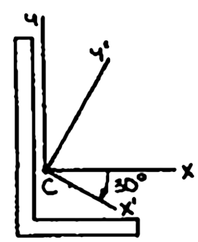


PROBLEM 9.192

For the L5 × 3 × $\frac{1}{2}$-in. angle cross section shown, use Mohr's circle to determine (*a*) the moments of inertia and the product of inertia with respect to new centroidal axes obtained by rotating the *x* and *y* axes 30° clockwise, (*b*) the orientation of the principal axes through the centroid and the corresponding values of the moments of inertia.

SOLUTION

From Figure 9.13*a*:



$$\bar{I}_x = 9.43 \text{ in}^4 \qquad \bar{I}_y = 2.55 \text{ in}^4$$

From the solution to Problem 9.191

$$\bar{I}_{xy} = -2.81235 \text{ in}^4$$

The Mohr's circle is defined by the diameter *XY*, where

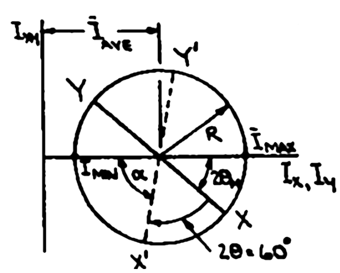
$$X(9.43 - 2.81235), \qquad Y(2.55, 2.81235)$$

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(9.43 + 2.55) = 5.99 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} = \sqrt{\left[\frac{1}{2}(9.43 - 2.55)\right]^2 + (-2.81235)^2}$$

$$= 4.4433 \text{ in}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-2.81235)}{9.43 - 2.55}$$

$$= 0.81754$$

or $$2\theta_m = 39.267°$$

and $$\theta_m = 19.6335°$$

Then $$\alpha = 180° - (39.267° + 60°)$$

$$= 80.733°$$

PROBLEM 9.192 (Continued)

(*a*) We have $\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = 5.99 - 4.4433\cos 80.733°$

or $\bar{I}_{x'} = 5.27 \text{ in}^4$ ◀

$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = 5.99 + 4.4433\cos 80.733°$

or $\bar{I}_{y'} = 6.71 \text{ in}^4$ ◀

$\bar{I}_{x'y'} = -R\sin\alpha = -4.4433\sin 80.733°$

or $\bar{I}_{x'y'} = -4.39 \text{ in}^4$ ◀

(*b*) First observe that the principal axes are obtained by rotating the *xy* axes through

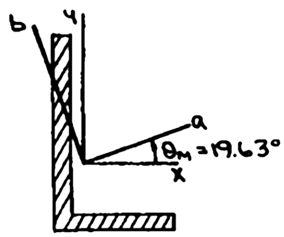
19.63° counterclockwise ◀

about *C*.

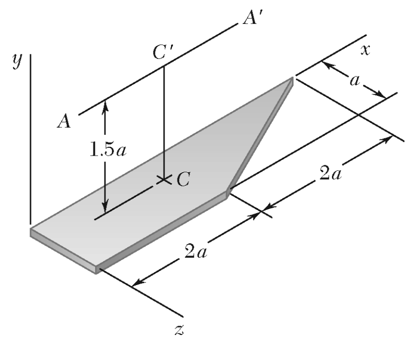
Now $\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = 5.99 \pm 4.4433$

or $\bar{I}_{\text{max}} = 10.43 \text{ in}^4$ ◀

$\bar{I}_{\text{min}} = 1.547 \text{ in}^4$ ◀



From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\text{max}}$ and the *b* axis corresponds to $\bar{I}_{\text{min}}$.



PROBLEM 9.193

A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the x axis, (*b*) the y axis.

SOLUTION

First note

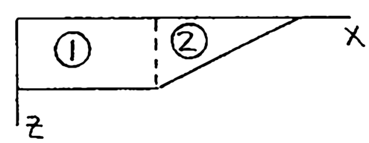
$$\text{mass} = m = \rho V = \rho t A$$

$$= \rho t\left[(2a)(a) + \frac{1}{2}(2a)(a)\right] = 3\rho t a^2$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{3a^2} I_{\text{area}}$$

(*a*) Now



$$I_{x,\text{area}} = (I_x)_{1,\text{area}} + (I_x)_{2,\text{area}}$$

$$= \frac{1}{3}(2a)(a)^3 + \frac{1}{12}(2a)(a)^3$$

$$= \frac{5}{6}a^4$$

Then

$$I_{x,\text{mass}} = \frac{m}{3a^2} \times \frac{5}{6}a^4 \qquad \text{or} \quad I_{x,\text{ mass}} = \frac{5}{18}ma^2 \blacktriangleleft$$

(*b*) We have

$$I_{z,\text{area}} = (I_z)_{1,\text{area}} + (I_z)_{2,\text{ area}}$$

$$= \left[\frac{1}{3}(a)(2a)^3\right] + \left[\frac{1}{36}(a)(2a)^3 + \frac{1}{2}(2a)(a)\left(2a + \frac{1}{3} \times 2a\right)^2\right] = 10a^4$$

Then

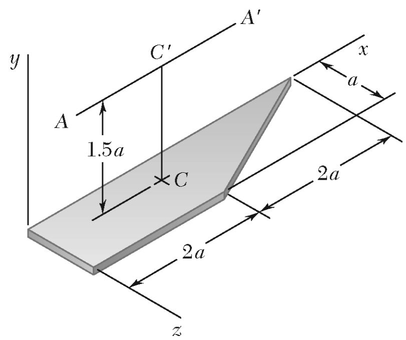
$$I_{x,\text{mass}} = \frac{m}{3a^2} \times 10a^4 = \frac{10}{3}ma^2$$

Finally,

$$I_{y,\text{mass}} = I_{x,\text{mass}} + I_{z,\text{mass}}$$

$$= \frac{5}{18}ma^2 + \frac{10}{3}ma^2$$

$$= \frac{65}{18}ma^2 \qquad \text{or} \quad I_{y,\text{mass}} = 3.61ma^2 \blacktriangleleft$$



PROBLEM 9.194

A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axis CC' that is perpendicular to the plate, (*b*) the axis AA' that is parallel to the x axis and is located at a distance $1.5a$ from the plate.

SOLUTION

First locate the centroid C.

$$\bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(2a^2 + a^2) = a(2a^2) + \left(2a + \frac{1}{3}\times 2a\right)(a^2)$$

or

$$\bar{X} = \frac{14}{9}a$$

$$\bar{Z}\Sigma A = \Sigma \bar{z}A: \quad \bar{Z}(2a^2 + a^2) = \left(\frac{1}{2}a\right)(2a^2) + \left(\frac{1}{3}a\right)(a^2)$$

or

$$\bar{Z} = \frac{4}{9}a$$

(*a*) We have $\quad I_{y,\,\text{mass}} = \bar{I}_{CC',\,\text{mass}} + m(\bar{X}^2 + \bar{Z}^2)$

From the solution to Problem 9.117:

$$I_{y,\text{mass}} = \frac{65}{18}ma^2$$

Then $\quad \bar{I}_{cc',\text{mass}} = \frac{65}{18}ma^2 - m\left[\left(\frac{14}{9}a\right)^2 + \left(\frac{4}{9}a\right)^2\right]$

or $\quad \bar{I}_{cc'} = 0.994ma^2$ ◀

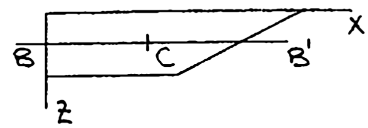
(*b*) We have $\quad I_{x,\text{mass}} = \bar{I}_{BB',\text{mass}} + m(\bar{Z})^2$

and $\quad I_{AA',\,\text{mass}} = \bar{I}_{BB',\,\text{mass}} + m(1.5a)^2$

Then $\quad I_{AA',\,\text{mass}} = I_{x,\,\text{mass}} + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$

PROBLEM 9.194 (Continued)

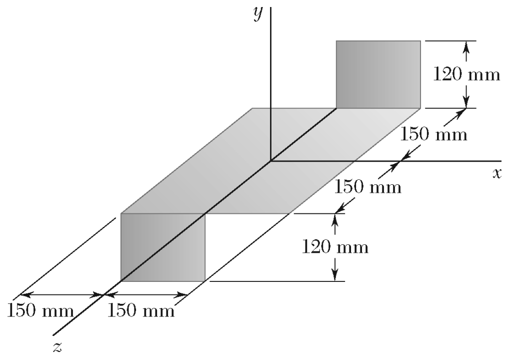
From the solution to Problem 9.193:



$$I_{x,\text{mass}} = \frac{5}{18}ma^2$$

Then $$I_{AA',\text{mass}} = \frac{5}{18}ma^2 + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$$

or $I_{AA'} = 2.33ma^2$ ◀

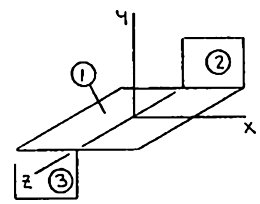


PROBLEM 9.195

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have



$$m = \rho_{ST} V = \rho_{ST} t A$$

Then

$$m_1 = (7850 \text{ kg/m}^3)(0.002 \text{ m})(0.3 \text{ m})^2$$
$$= 1.413 \text{ kg}$$

$$m_2 = m_3 = (7850 \text{ kg/m}^3)(0.002 \text{ m}) \times (0.15 \times 0.12) \text{ m}^2$$
$$= 0.2826 \text{ kg}$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + 2(I_x)_2$$
$$= \left[\frac{1}{12}(1.413 \text{ kg})(0.3 \text{ m})^2\right]$$
$$+ 2\left[\frac{1}{12}(0.2826 \text{ kg})(0.12 \text{ m})^2 + (0.2828 \text{ kg})(0.15^2 + 0.06^2) \text{ m}^2\right]$$
$$= [(0.0105975) + 2(0.0003391 + 0.0073759)] \text{ kg} \cdot \text{m}^2$$
$$= [(0.0105975) + 2(0.0077150)] \text{ kg} \cdot \text{m}^2$$

or $I_x = 26.0 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

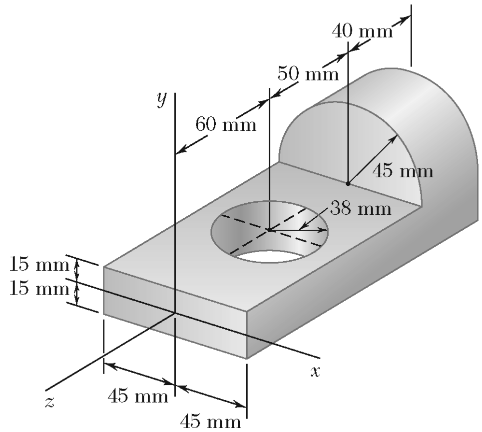
$$I_y = (I_y)_1 + 2(I_y)_2$$
$$= \left[\frac{1}{12}(1.413 \text{ kg})(0.3^2 + 0.3^2) \text{ m}^2\right]$$
$$+ 2\left[\frac{1}{12}(0.2826 \text{ kg})(0.15 \text{ m})^2 + (0.2826 \text{ kg})(0.075^2 + 0.15^2) \text{ m}^2\right]$$
$$= [(0.0211950) + 2(0.0005299 + 0.0079481)] \text{ kg} \cdot \text{m}^2$$
$$= [(0.0211950) + 2(0.0084780)] \text{ kg} \cdot \text{m}^2$$

or $I_y = 38.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.195 (Continued)

$$
\begin{aligned}
I_z &= (I_z)_1 + 2(I_z)_2 \\
&= \left[\frac{1}{12}(1.413\text{ kg})(0.3\text{ m})^2\right] \\
&\quad + 2\left[\frac{1}{12}(0.2826\text{ kg})(0.15^2 + 0.12^2)\text{ m}^2 + (0.2826\text{ kg})(0.075^2 + 0.06^2)\text{ m}^2\right] \\
&= [(0.0105975) + 2(0.0008690 + 0.0026070)]\text{ kg}\cdot\text{m}^2 \\
&= [(0.0105975) + 2(0.0034760)]\text{ kg}\cdot\text{m}^2
\end{aligned}
$$

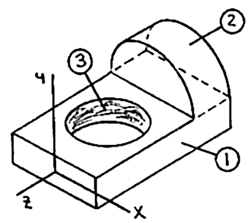
or $\quad I_z = 17.55\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM 9.196

Determine the mass moment of inertia and the radius of gyration of the steel machine element shown with respect to the x axis. (The density of steel is $7850\ \text{kg/m}^3$.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(0.09 \times 0.03 \times 0.15)\ \text{m}^3$$
$$= 3.17925\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)\left[\frac{\pi}{2}(0.045)^2 \times 0.04\right]\ \text{m}^3$$
$$= 0.998791\ \text{kg}$$

$$m_3 = (7850\ \text{kg/m}^3)[\pi(0.038)^2 \times 0.03]\ \text{m}^3 = 1.06834\ \text{kg}$$

Using Figure 9.28 for components 1 and 3 and the equation derived above (before the solution to Problem 9.142) for a semicylinder, we have

$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$
$$= \left[\frac{1}{12}(3.17925\ \text{kg})(0.03^2 + 0.15^2)\ \text{m}^2 + (3.17925\ \text{kg})(0.075\ \text{m})^2\right]$$
$$+ \left\{(0.998791\ \text{kg})\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)(0.045\ \text{m})^2 + \frac{1}{12}(0.04\ \text{m})^2\right]\right.$$
$$\left. + (0.998791\ \text{kg})\left[(0.13)^2 + \left(\frac{4 \times 0.045}{3\pi} + 0.015\right)^2\right]\ \text{m}^2\right\}$$
$$- \left\{\frac{1}{12}(1.06834\ \text{kg})[3(0.038\ \text{m})^2 + (0.03\ \text{m})^2] + (1.06834\ \text{kg})(0.06\ \text{m})^2\right\}$$

PROBLEM 9.196 (Continued)

$$= [(0.0061995 + 0.0178833) + (0.0002745 + 0.0180409) - (0.0004658 + 0.0038460)] \text{ kg} \cdot \text{m}^2$$

$$= (0.0240828 + 0.0183154 - 0.0043118) \text{ kg} \cdot \text{m}^2$$

$$= 0.0380864 \text{ kg} \cdot \text{m}^2$$

or $I_x = 38.1 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

Now

$$m = m_1 + m_2 - m_3 = (3.17925 + 0.998791 - 1.06834) \text{ kg}$$
$$= 3.10970 \text{ kg}$$

and

$$k_x^2 = \frac{I_x}{m} = \frac{0.0380864 \text{ kg} \cdot \text{m}^2}{3.10970 \text{ kg}}$$

or $k_x = 110.7 \text{ mm}$ ◀