1 State transition Matrix

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$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$ ——— (1)

Homogeneous Solution (Zero-input solution)

IC = x(0) exist ($u=0$)

$\Rightarrow$ $\dot{x}(t) = Ax(t)$, $y(t) = Cx$ ← IC ✓ ($u=0$)

take laplace transformation

$$sX(s) - x(0) = AX(s)$$

$$[sI - A]X(s) = x(0) \Rightarrow X(s) = (sI - A)^{-1} x(0)$$

s-domain

based on that, lets take laplace inverse

$$x(t) = \{\mathcal{L}^{-1}(sI - A)^{-1}\} x(0)$$ ——— (1)

→ State trans. matrix

$$\mathcal{L}^{-1}(sI - A)^{-1} = \Phi(t) \triangleq e^{At}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\omega t)$$ $F(t) =$

Forced w.r.t b + damped

Complementary + steady state solution (Particular)

$$\ddot{x}_c + \frac{c}{m}\dot{x}_c + \frac{k}{m}x = 0$$ ← Free + damped sys

IC x(t)

$z_1 = x$

$z_2 = \dot{x}$

$\dot{z}_1 = z_2$ — (1)

$\dot{z}_2 = -\frac{c}{m}z_2 - \frac{k}{m}z_1$ — (2)

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Ex: Compute $x(t)$ for the homogenous solution of the Sys shown below

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \quad \text{if } x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad t=0$$

$$x(t) = \mathcal{L}^{-1}(sI-A)^{-1} x(0)$$

$$[sI-A] = s \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -2 & s-3 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s-3 & 1 \\ 2 & s \end{bmatrix} \frac{1}{s^2-3s-2}$$

$$\phi(s) = \mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{s-3}{s^2-3s-2} & \frac{1}{s^2-3s-2} \\ \frac{1}{s^2-3s-2} & \frac{s}{s^2-3s-2} \end{bmatrix}$$

$$\phi(t) = \text{ilaplace}(\phi(s))$$

$$x(t) = [\phi(t)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{when } t=0$$

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Non-Homogeneous Solution

$$\dot{x}(t) = Ax + Bu$$
$$y(t) = Cx + Du$$

solve the problem under the effect of input force, IC = 0

take laplace

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$(sI - A)^{-1} X(s) = BU(s)$$

Particular Solution

$$X(s) = (sI - A)^{-1} BU(s)$$

take the inverse of laplace

$$x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] BU(s)$$

Total Solution IC is given $\neq 0$ also $u \neq 0$

$$\dot{x}(t) = Ax(t) + BU(t)$$

take laplace

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$(sI - A)X(s) = x(0) + BU(s)$$
$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

to find the solution in time domain lets take laplace inverse

$$x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] x(0) + \mathcal{L}^{-1}\left[(sI - A)^{-1} BU(s)\right]$$

$$x(t) = \Phi(t)\, x(0) + \int_0^t \Phi(t - \tau)\, BU(\tau)\, d\tau$$

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if you start from $t=0$

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,B\,u(\tau)\,d\tau \quad (*)$$

$$x(t) = \phi(t-t_0)\,x(t_0) + \int_{t_0}^{t} \phi(t-\tau)\,B\,u(\tau)\,d\tau \quad (**)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

} if you start from $t=t_0$ SSR

lets sub Eq (*) in to the second Eq of SSR

$$y(t) = C\left[\phi(t-t_0)\,x(t_0) + \int_{t_0}^{t} \phi(t-t_0)\,B\,u(\tau)\,d\tau\right] + Du(t)$$

$$= C\,\phi(t-t_0)\,x(t_0) + \int_{t_0}^{t} C\,\phi(t-t_0)\,B\,u(t)\,d\tau + Du(t)$$

Zero input response
= "free vibrations"
the motion under the effect of IC ($u = 0$)

Zero state response
= "forced vib."
= "Particular solution"
the motion under the effect of input force $u(t)$
IC = 0

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Example find the zero input response forth following sys

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{if } x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

for the zero-input respone $u(t) = 0$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}(sI - A)^{-1} x(0)$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{s^2 + 3s + 2}$$

$$\mathcal{L}^{-1}(sI - A)^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = R$$

$$y(t) = C x(t)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} * R \leftarrow \text{zero input respone}$$

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Consider the system which is shown below :-

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

find the responce of the system $x(t)$ if the input force is unit step $(u(t) = 1)$.

$$x(t) = \underbrace{\phi(t)\, x(0)}_{\text{Free responce}} + \underbrace{\int_0^t \phi(t-\tau)\, B\, u(\tau)\, d\tau}_{\text{Forced responce}}$$

IC

$u(t)$

$$\phi(t) = \mathcal{L}^{-1}(sI-A)^{-1} = e^{(At)} \text{ : transition matrix}$$

$$(sI-A) = \begin{bmatrix} s & 0 \\ 6 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}$$

$$\phi(s) = (sI-A)^{-1} = \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix} \frac{1}{s^2+6s+8}$$

$$\phi(t) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{s+6}{s^2+6s+8} & \frac{1}{s^2+6s+8} \\ \frac{-8}{s^2+6s+8} & \frac{s}{s^2+6s+8} \end{bmatrix} \right)$$

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syms t
Phi = expm(A*t)        " time domain"
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$$\phi(t) = \begin{bmatrix} 2e^{-2t} - e^{-4t} & \frac{e^{-2t}}{2} - \frac{e^{-4t}}{2} \\ 4e^{-4t} - 4e^{-2t} & 2e^{-4t} - e^{-2t} \end{bmatrix}$$

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$$\phi(t)\,x(0) = \begin{bmatrix} 2e^{-2t} - e^{-4t} & \frac{e^{-2t}}{2} - \frac{e^{-4t}}{2} \\ 4e^{-4t} - 4e^{-2t} & 2e^{-4t} - e^{-2t} \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}$$

$$\phi(t)\,x(0) = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ 4e^{-4t} - 4e^{-2t} \end{bmatrix}$$

$$\phi(t-\tau) \begin{bmatrix} 2e^{-2(t-\tau)} - e^{-4(t-\tau)} & \frac{e^{-2(t-\tau)}}{2} - \frac{e^{-4(t-\tau)}}{2} \\ -4e^{-2(t-\tau)} & 2e^{-4(t-\tau)} - e^{-2(t-\tau)} \end{bmatrix}$$

$$\phi(t-\tau)\,B = \begin{bmatrix} 2e^{-2(t-\tau)} - e^{-4(t-\tau)} & \frac{e^{-2(t-\tau)}}{2} - \frac{e^{-4(t-\tau)}}{2} \\ 4e^{-4(t-\tau)} - 4e^{-2(t-\tau)} & 2e^{-4(t-\tau)} - e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\phi(t-\tau)\,B = \begin{bmatrix} \frac{1}{2}e^{-2(t-\tau)} - \frac{1}{2}e^{-4(t-\tau)} \\ -e^{-2(t-\tau)} + 2e^{-4(t-\tau)} \end{bmatrix}$$

$$\int_0^t \phi(t-\tau)\,B\,u(\tau)\,d\tau = \begin{bmatrix} \frac{1}{2}e^{-2t}\int_0^t e^{2\tau}\,d\tau - \frac{1}{2}e^{-4t}\int_0^t e^{4\tau}\,d\tau \\ -e^{-2t}\int_0^t e^{2\tau}\,d\tau + 2e^{-4t}\int_0^t e^{4\tau}\,d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

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$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,B\,u(\tau)\,d\tau$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ 4e^{-4t} - 4e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{8}e^{-4t} \end{bmatrix}$$

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