Exercises of chapter 13 :

Q4: list all zero divisor in $\underset{\sim}{Z_{20}}^{\{0,\dots,19\}}$. can you see a relation between the zero divisors of Z_{20} and the units of Z_{20} .

the zero divisors in Z_{20} : $\{2, 10, 4, 5, 6, 10, 8, 12, 16, 14, 18\}$.

units of Z_{20} = $U(20) = \{a \in Z$; a is invertable $\Leftrightarrow$ a is a unit $\Leftrightarrow$ $g.c.d(a,n) = 1\}$

$= \{1, 3, 7, 9, 11, 13, 17, 19\}$.

the relation : Z_{20} = zero divisors $\overset{\text{union}}{\underset{\sim}{U}}$ units .

claim : If $g.c.d(a,n) > 1$, then a is a zero divisor .

Q5: Show that every nonzero elements of Z_n is a unit or a zero divisor .

let $K \in Z$ s.t $K \neq 0$

case 1 : If $g.c.d(k,n) = 1$

case 2 : If $g.c.d(K,n) = d$, $d > 1$

$\Rightarrow \exists\ s, t$ s.t $Ks + nt = 1 \Rightarrow Ks = 1 \Rightarrow K$ is a unit .

$\Rightarrow \exists\ m \in Z^{+}$ s.t $K = md$

$$K\left(\frac{n}{d}\right) = md\left(\frac{n}{d}\right) = m.n = 0 \Rightarrow k \text{ is zero divisor .}$$

Q6: Find a nonzero element in a Ring that is neither a zero divisor nor a unit .

In $(Z, +, \cdot)$, $2 \in Z$

$\rightarrow$ 2 not a unit $\rightsquigarrow 2K = 1 \rightsquigarrow K = \frac{1}{2} \notin Z$.

$\rightarrow$ 2 not a zero divisor $\rightsquigarrow 2.K = 0 \Leftrightarrow K = 0$. since Z is integral domain .

Q7: let R be a finite commutative ring with unity. Prove that every nonzero element of R is either a zero divisor or a unit. what happens if we drop the «finite» condition on R?

let $0 \neq a \in R$. since R is finite the list $1 = a^0, a, a^2, \ldots$ must contain a repetition.

Thus, $a^L = a^n$ for some $0 \leq L < n$

we may assume that n is the smallest such integer.

Note that $n-1 \geq 0$

suppose $L = 0$ then $1 = a^0 = a^n = a a^{n-1} = a^{n-1} a$

which means a^{n-1} is an inverse of a.

suppose $L > 0$ then $0 \leq L-1 < n-1$ and we deduce

$0 = a(a^{n-1} - a^{L-1})$ from $a^L = a^n$

But $a^{n-1} \neq a^{L-1}$ By the minimality of n.

thus, $a^{n-1} - a^{L-1} \neq 0$

We have shown that a is a zero divisor. ∎

Q8: let $a \neq 0$ belong to a commutative Ring. prove that a is a zero divisor iff $a^2 b = 0$ for some $b \neq 0$.

Q15: let a belong to a Ring R with unity and suppose that $a^n = 0$ for some positive integer n (such an element is called nilpotent). Prove that $1-a$ has a multiplicative inverse in R.

$(1-a)(1+a+a^2+\cdots+a^{n-1}) = 1+\cancel{a}+\cancel{a^2}+\cdots+a^{n-1}-\cancel{a}-\cancel{a^2}-\cdots-a^n$

$= 1-a^n$

But $a^n=0 \Rightarrow \quad = 1-0$

$= 1$

Q16: Show that the nilpotent element of a commutative Ring form a subring.

let N be the set of all nilpotent elements of R, show N subring

assume $a,b \in N$, and we know $a^n=0$ and $b^n=0$

$\rightarrow a^n+b^n = 0+0 = 0 \in N$.

$\rightarrow a^n \times a^n = 0.0 = 0 \in N$

$\rightarrow 0_R \in N$

$\rightarrow a+x=0 \Rightarrow x=-a \in N$

So N is a subring of R

Q17: show that 0 is the only nilpotent element in an integral domain.

suppose that a is nilpotent element in integral domain

$0 = a^n = \cancel{a}.0 = \cancel{a}.a^{n-1}$

$\Rightarrow a^{n-1} = 0$

$\Rightarrow a = 0$

So 0 is the only ...

Q18: a Ring element a is called an idempotent if $a^2 = a$. prove that the only idempotent in an integral domain are 0 and 1.

assume R is an integral domain $\Rightarrow$ R has a unity 1

let a be an idempotent in R, then

$$a^2 = a \Rightarrow a^2 - a = 0$$

$$\Rightarrow a(a-1) = 0$$

$$\Rightarrow a = 0 \text{ or } a-1 = 0$$

$$\Rightarrow a = 0 \text{ or } a = 1$$

Q20: show that Zn has a nonzero nilpotent element iff n is divisibly by the square of some prime.

exercises of chapter 13 :

Q1 + Q2 Done on Notes .

Q3 : show that a commutative Ring with the cancellation property (under multiplication) has No zero divisor .

let $ab = 0$ and $a \neq 0$ then $ab = a.0$ so $b = 0$.

Q4 : Find elements a, b and c in the Ring $Z \oplus Z \oplus Z$ s.t ab, ac and bc are zero divisors but abc is Not a zero divisor .

$a = (1, 1, 0)$, $b = (1, 0, 1)$, $c = (0, 1, 1)$.

Q6 : Describe all zero divisor and units of $Z \oplus Q \oplus Z$.

the Zero divisor are elements with one or two entries that are zero

Formally , {(a, b, c) : exactly one or two of the units are elements composes of units so they have ±1 in the first and last coordinate , and the middle coordinate is Not zero .

so $\{(\pm 1, b, \pm 1) : b \neq 0\}$

Q11 : let d be an integer. Prove that $Z[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in Z\}$ is an integral domain .

→

→ $a, b \in Z[\sqrt{d}] \Rightarrow (a_1 + b_1\sqrt{d}) - (a_2 + b_2\sqrt{d}) = (a_1 - a_2) + (b_1 - b_2)\sqrt{d}$

→ $a \in Z[\sqrt{d}]$, $r \in R \Rightarrow (a_1 + b_1\sqrt{d})(a_2 + b_2\sqrt{d}) = (a_1 a_2 + b_1 b_2 d) + (a_1 b_2 + a_2 b_1)\sqrt{d}$

→ Thus , its Ring .

→ since $Z[\sqrt{d}]$ is a subring of the Ring of complex numbers , it has No zero divisor .

Q12: In $\mathbb{Z}_7$, give a reasonable interpretation for the expressions $\frac{1}{2}$, $-\frac{2}{3}$, $\sqrt{-3}$ and $-\frac{1}{6}$.

since $2\times4 = 8 = 1$, $4 = \frac{1}{2}$

since $3\times5 = 15 = 1$, 5 acts like $\frac{1}{3}$ so

$$-\frac{2}{3} = -1 + \frac{1}{3} = 6+5 = 11 = 4$$

[Alternately, $\frac{2}{3} = 2\times5 = 3$ so $-\frac{2}{3} = -3 = 4$]

Now, $-3 = 4$ so $\sqrt{-3} = 2$

Finally, $6\times6 = 36 = 1$ so $\frac{1}{6} = 6$ and $-\frac{1}{6} = -6 = 1$.

Q13: give an example of a comm. Ring without zero divisors that is Not an integral domain.

The even integers.

Q14: Find two elements a and b in a Ring such that both a and b are zero divisors, $a+b \neq 0$ and $a+b$ is Not a zero divisor.