14.4 Chain Rule.

Recall : (Cal. 1)

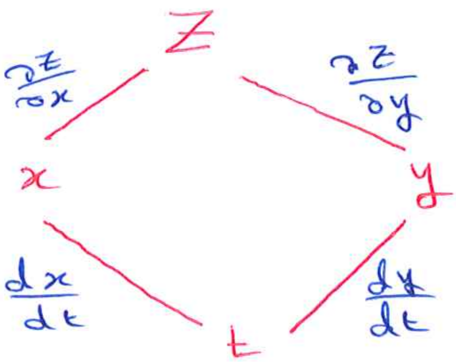
If $y = f(t)$ and $t = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Now : (Cal. 3):

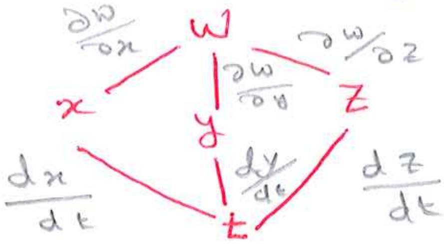
• If $z = f(x,y)$ and $x = x(t)$ & $y = y(t)$

then $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$



• If $w = f(x,y,z)$ with $x = x(t)$, $y = y(t)$ & $z = z(t)$

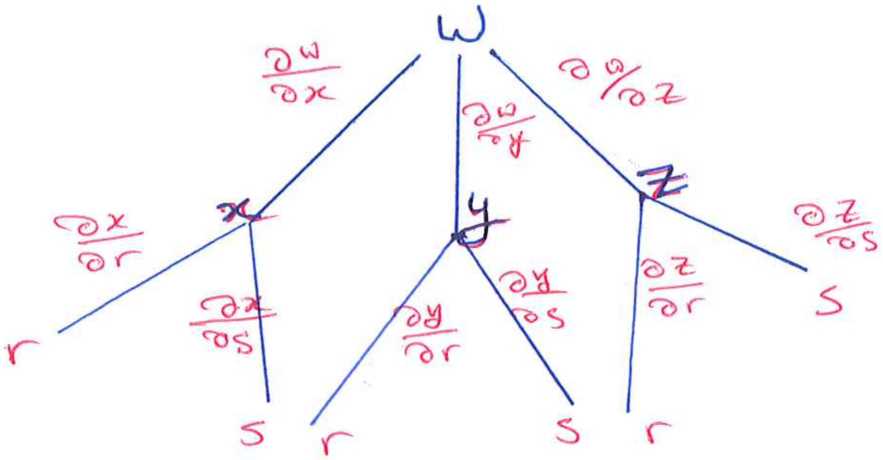
then $$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



• If $w = f(x, y, z)$ and

$x = x(r, s)$ and $y = y(r, s)$ & $z = z(r, s)$

then.



$$\Rightarrow \quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

and
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}.$$

Example: Let $w = xy$, $x = \cos t$, $y = \sin t$

Find $\left.\dfrac{dw}{dt}\right|_{t = \frac{\pi}{2}}$

At $= \frac{\pi}{2}$
$x = 0$
$y = 1$

sol: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial w}{\partial y} \cdot \dfrac{dy}{dt}$

$= -y \sin t + x \cos t$

$$\Rightarrow \left.\frac{dw}{dt}\right|_{t=0,\, x=0,\, y=1} = -1 \sin \frac{\pi}{2} + 0 \cos \frac{\pi}{2} = \boxed{-1}$$

Example: $\omega = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$

Find $\frac{dw}{dt}$ at $t = 0$

sol: At $t = 0 \Rightarrow x = 1,\ y = 0,\ z = 0.$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= -y\sin t + x\cos t + 1\cdot 1$$

$$\Rightarrow \left.\frac{dw}{dt}\right|_{t=0,\ x=1,\ y=0,\ z=0} = 0 + 1\cos 0 + 1 = \boxed{2}$$

Example: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $\omega = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$

$z = 2r$.

sol: $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= 1\cdot\left(\frac{1}{s}\right) + 2(2r) + 2z\cdot(2)$$

$$= \frac{1}{s} + 4r + 4(2r).$$

$$= \frac{1}{s} + 12r.$$

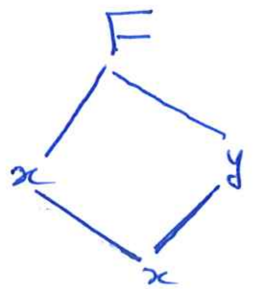
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= 1\cdot\left(-\frac{r}{s^2}\right) + 2\left(\frac{1}{s}\right) + 2z\cdot(0)$$

$$= \frac{-r}{s^2} + \frac{2}{s} = \frac{2s-r}{s^2}.$$

Implicit Differentiation Revisited.

If $F(x,y)=0$. Find $\frac{dy}{dx}$.



$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}, \quad F_y \neq 0$$

Theorem ⑧: A Formula for Implicit Differentiation.

Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y)=0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, $\frac{dy}{dx} = \frac{-F_x}{F_y}$.

Example : Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin(xy) = 0$.

Sol: (Cal. 1) : $2y \frac{dy}{dx} - 2x - \cos(xy)\left[x \frac{dy}{dx} + y\right] = 0$

$$\Rightarrow \underbrace{2y \frac{dy}{dx}} - 2x - \underbrace{x \cos(xy) \frac{dy}{dx}} - y \cos(xy) = 0$$

$$\Rightarrow \left[2y - x \cos(xy)\right] \frac{dy}{dx} = y \cos(xy) + 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y \cos(xy)}{2y - x \cos(xy)}$$

Calc. 3 : Using Thm 8 :

$$y^2 - x^2 - \sin(xy) = 0$$

$$F(x,y) = y^2 - x^2 - \sin(xy) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$= \frac{2x + y \cos(xy)}{2y - x \cos(xy)}.$$

• If $F(x,y,z)=0$, $z=z(x,y)$, then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}.$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if

$$\underbrace{x^3+z^2+ye^{xz}+z\cos y}_{F(x,y,z)}=0.$$

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)} = \left.\frac{-F_x}{F_z}\right|_{(0,0,0)} = \left.\frac{-(3x^2+yze^{xz})}{2z+xye^{xz}+\cos y}\right|_{(0,0,0)}$$

$$= \frac{-(0+0)}{0+0+1} = \boxed{0}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)} = \left.\frac{-F_y}{F_z}\right|_{(0,0,0)} = \left.-\frac{(e^{xz}-z\sin y)}{(2z+xye^{xz}+\cos y)}\right|_{(0,0,0)}$$

$$= \frac{-1+0}{0+0+1} = \boxed{-1}.$$

Example: If $z^3 - xz - y = 0$, where

z defines a function of x and y.

Find $\frac{\partial^2 z}{\partial x \partial y}$

Sol: $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(-1)}{(3z^2 - x)} = \frac{1}{(3z^2 - x)}$

Now, $\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1)\left(6z\frac{\partial z}{\partial x} - 1\right)}{(3z^2 - x)}$.

Q37). If $z = 5\tan^{-1} x$, $x = e^u + \ln v$

Find $\frac{\partial z}{\partial u}$ at $(u, v) = (\ln 2, 1)$

$\frac{\partial z}{\partial u} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial u} = \frac{1(5)}{1 + x^2} \cdot (e^u)$

$= \frac{5e^u}{1 + (e^u + \ln v)^2}$

$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{(\ln 2, 1)} = \frac{5e^{\ln 2}}{1 + (e^{\ln 2} + \ln 1)^2} = \frac{5(2)}{1 + (2)^2} = \frac{10}{5} = \boxed{2}$