4.8: distributions of $\bar{x}$, $\frac{nS^2}{\sigma^2}$

Recall:

- $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

- $S^2 = S_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$

- $S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

proposition.

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2.$$

proof:

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X} + \bar{X} - \mu)^2$$

$$= \sum_{i=1}^{n} (X_1 - \bar{X})^2 + 2 \sum_{i=1}^{n} (X_i - \bar{X})(\bar{X} - \mu) + \sum_{i=1}^{n} (\bar{X} - \mu)^2.$$

$$= \sum_{i=1}^{n} (X_i - \bar{X})^2 + 2(\bar{X} - \mu) \underbrace{\sum_{i=1}^{n} (X_i - \bar{X})}_{=0} + n(\bar{X} - \mu)^2.$$

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

→ since $\sum_{i=1}^{n} (X_i - \bar{X}) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \bar{X}$

$= n\bar{X} - n\bar{X}$

$= 0$

Thm: let $x_1, \ldots, x_n$ be a Random sample of size n from $N(\mu, \sigma^2)$, then

1. $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$
2. $\bar{X}$ and S_n^2 independent.
3. $\frac{n S_n^2}{\sigma^2} \sim \chi^2(n-1)$ OR $\frac{(n-1) S_{n-1}^2}{\sigma^2} \sim \chi^2(n-1)$.

proof 3:

$$\frac{n S_n^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2}$$

// multiply the prop. by $\frac{1}{\sigma^2}$ //: $\underbrace{\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2}_{A} = \underbrace{\sum \left(\frac{x_i - \bar{x}}{\sigma} \right)}_{B} + \underbrace{\left(\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \right)^2}_{C}$

$M_A(t) = (1-2t)^{-\frac{n}{2}}$, $t < \frac{1}{2}$

$M_C(t) = (1-2t)^{-\frac{1}{2}}$, $t < \frac{1}{2}$

$\rightarrow M_A(t) = E(e^{t(A)}) = E(e^{tB} \cdot e^{tC})$

$\underset{B, C \text{ indep.}}{=} E(e^{tB}) \cdot E(e^{tC})$

$\Rightarrow M_A(t) = M_B(t) \cdot M_c(t)$.

$(1-2t)^{-\frac{n}{2}} = M_B(t) \cdot (1-2t)^{-\frac{1}{2}}$, $t < \frac{1}{2}$

$\Rightarrow M_B(t) = (1-2t)^{-\frac{n-1}{2}}$, $t < \frac{1}{2}$

$\Rightarrow B = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma} \right) = \frac{n S_n^2}{\sigma^2} \sim \chi^2(n-1)$. ■

Note: $B = \frac{n S_n^2}{\sigma^2}$, $C = \left(\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \right)^2$

$S_n^2, \bar{X}$ indep. $\Rightarrow$ B, C indep.