Chapter 3

Z transform

Fourier transform $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$

F. T Converges if $\sum_{n=-\infty}^{\infty} |x(n)|$ Converges (i.e finite)

Or $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ (absolutly summable)

$\Rightarrow$ the system is stable

* If sequence $x[n]$ is absolutly summable, then F.T of $x[n]$ converges (i.e exist)

If the system is stable $\Rightarrow$ frequency response (Fourier) exist

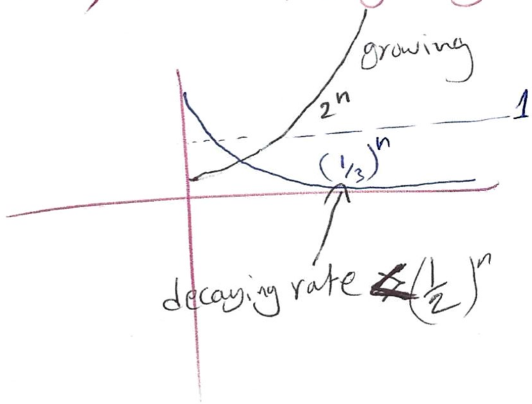
Examples: ① $x[n] = (\frac{1}{2})^n u(n)$

$\sum_{n=-\infty}^{\infty} |x[n]| = 2 \Rightarrow$ F.T. Converges (exist)

② $x[n] = 2^n u[n]$

$\sum_{n=-\infty}^{\infty} |x[n]| = \infty \Rightarrow$ F.T. does not exist

So to make the Signal Summable, a multiplying by a decaying exponantial is applied.



the product has a F.T

$$\left(\frac{1}{r}\right)^n = r^{-n} \longleftarrow \text{decaying exponantial}$$

$$X_r[n] = X[n]\, r^{-n}$$

$$\mathcal{F}(X_r[n]) = \sum_{n=-\infty}^{\infty} X_r[n]\, e^{-j\omega n}$$

$$X_r(e^{j\omega}) = \sum_{n=-\infty}^{\infty} [X[n]\, r^{-n}]\, e^{-j\omega n}$$

r^{-n}: decaying exponantial

r should be choosed so that $(X[n]\, r^{-n})$ is absolutly Summable

$$X_r(e^{j\omega}) = \sum_{n=-\infty}^{\infty} X[n]\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} X[n]\, (r e^{j\omega})^{-n}$$

↳ new Complex Varable called z

$$z = r e^{j\omega}$$

$$|z| = r \qquad \angle z = \omega$$

$$X(z) = \sum_{-\infty}^{\infty} X[n]\, z^{-n} \qquad \text{Z-transform}$$

$X(e^{jw}) = X(z)\Big|_{z=e^{jw}}$ when magnitude of $z = 1$

$|z| = 1$

This is true when z transform converges

for some values of r, z transform converges

$$\sum_{n=-\infty}^{\infty} |x[n]\, r^{-n}| < \infty$$

for other values of r, z transform diverges

Examples- $x[n] = (\frac{1}{2})^n u[n]$ (called Right-sided exponantial)

$$X(z) = \sum_{n=-\infty}^{\infty} (\tfrac{1}{2})^n u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{2})^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{2} z^{-1})^n$$

$\alpha = \frac{1}{2} z^{-1}$ if $|\alpha| < 1$ → Converges

$|\frac{1}{2} z^{-1}| < 1$ multiplying by z

$\frac{1}{2} < |z|$

So. $X(z)$ converges when $|z|$ or $r > \frac{1}{2}$

all values of $|z|$ or r which makes $X(z)$ converges are called Region of Convergence (ROC)

$$\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

$\Downarrow$

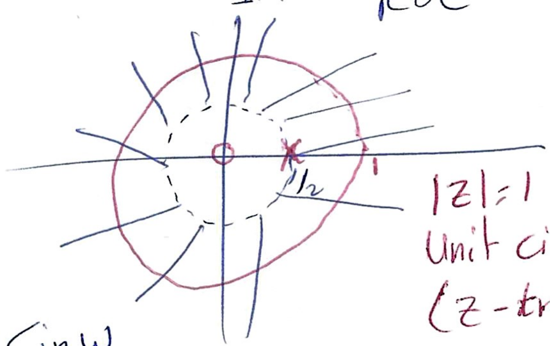
easier to find (zeros and poles) roots

$$\text{FT of } x[n] = \frac{1}{1-\frac{1}{2}e^{-jw}}$$

$|z| = 1$

$z > \frac{1}{2}$ (ROC)

Z-plane



$z = r e^{jw}$

$z = r\cos w + jr\sin w$

$|z| = 1$
Unit circle
(z-transform) = (Fourier)

Right-sided exponential sequence

$$X(z) = \frac{\text{Polynomial}}{\text{Polynomial}}$$

roots of numerator polynomial $\Rightarrow$ zeros of transform 0

roots of denominator polynomial $\Rightarrow$ poles of transform X

$$X(z) = \frac{z}{z-\frac{1}{2}}$$

roots of numerator $= 0$ zeros

roots of denominator $= \frac{1}{2}$ poles

Poles are borders of RoC   Poles $<$ RoC

Example: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$ left sided exponential

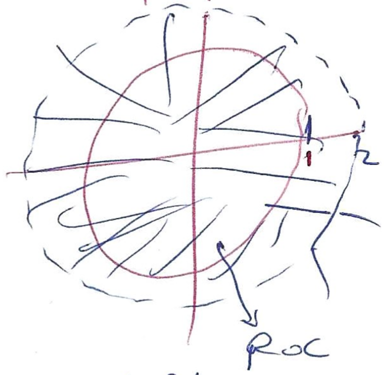
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

e.g. $X(z) = \frac{1}{2-z}$

$|2-z| > 0$

$|z| < 2$

$$X(z) = \sum_{n=-\infty}^{\infty} -\left(\tfrac{1}{2}\right)^n u[-n-1]\, z^{-n}$$



left

$$= -\sum_{n=-\infty}^{-1} \left(\tfrac{1}{2}\right)^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^{-n} z^{+n}$$

$$= -\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{-n} z^{n} - (-1)$$

$$= 1 - \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} \Big(\underbrace{\left(\tfrac{1}{2}\right)^{-1} z}_{a}\Big)^{n}$$

$X(z)$ converges if $\left|\left(\tfrac{1}{2}\right)^{-1} z\right| < 1$

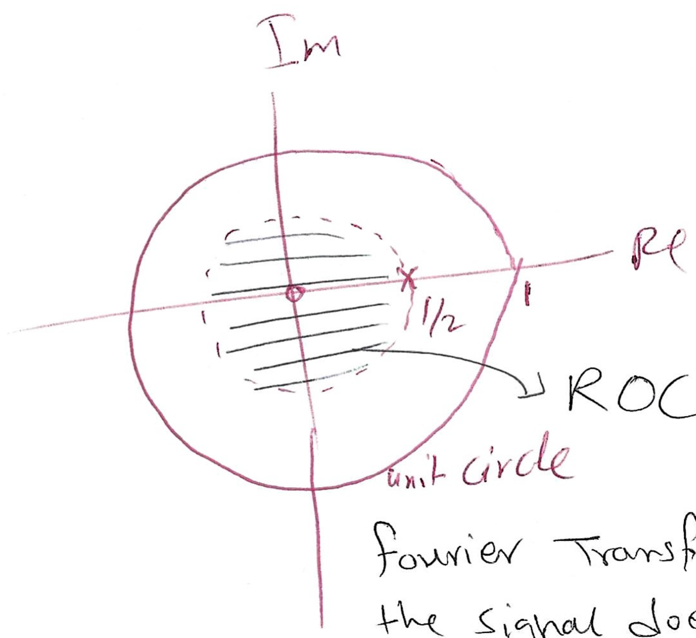
$$|z| < \frac{1}{2}$$

$$X(z) = 1 - \frac{1}{1-\left(\frac{1}{2}\right)^{-1} z} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

zero's at $z=0$

poles at $z=\frac{1}{2}$

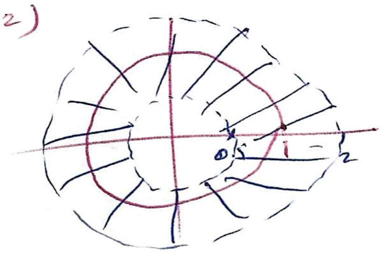
$0 < \text{RoC} < \text{poles}$



fourier Transform of the signal does not exist

eg. $X(z) = \frac{1}{(z-0.5)(z-2)}$

$0.5 < |z| < 2$



Example:- $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u(-n-1)$

two-sided sequence

$$\left(-\frac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3} \text{ RoC}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2} \text{ RoC}$$

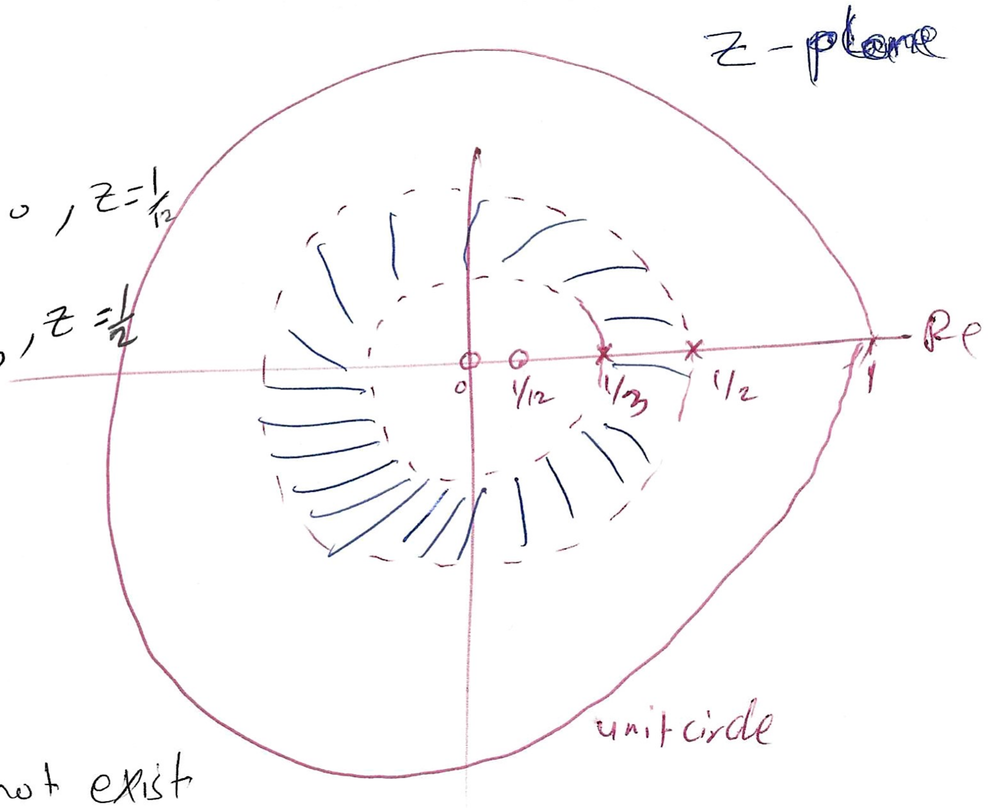
$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z|, \ |z| < \frac{1}{2}$$

$$X(z) = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

ROC

Zero's :- at $Z=0$, $Z=\frac{1}{12}$

Poles :- at $Z=-\frac{1}{3}$, $Z=\frac{1}{2}$

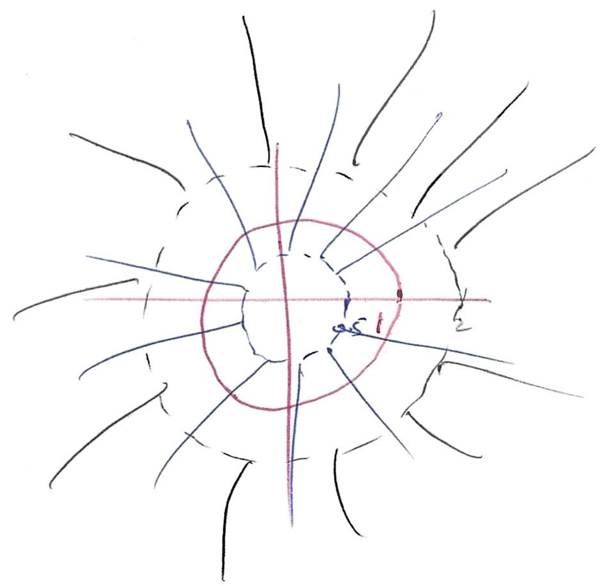


Twosided

F.T does not exist

example

$$X(Z)=\frac{1}{(Z-0.5)(Z-2)}$$

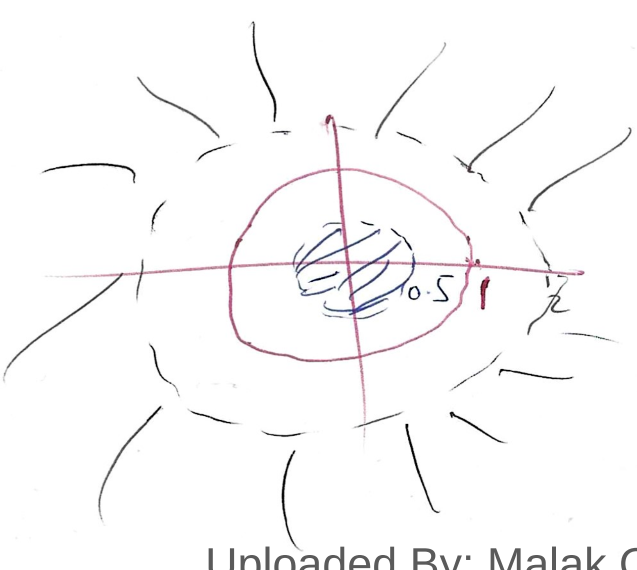


$|Z|>0.5$

$|Z|>2$

ROC $|Z|>2$

example $X(Z)=\frac{1}{(0.5-Z)(Z-2)}$



Example:- consider a signal that the sum of two real exponential

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]$$

Evaluate $X(z)$, plot ROC, specify zeros and poles

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\tfrac{1}{3}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\tfrac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\tfrac{1}{3} z^{-1}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}}$$

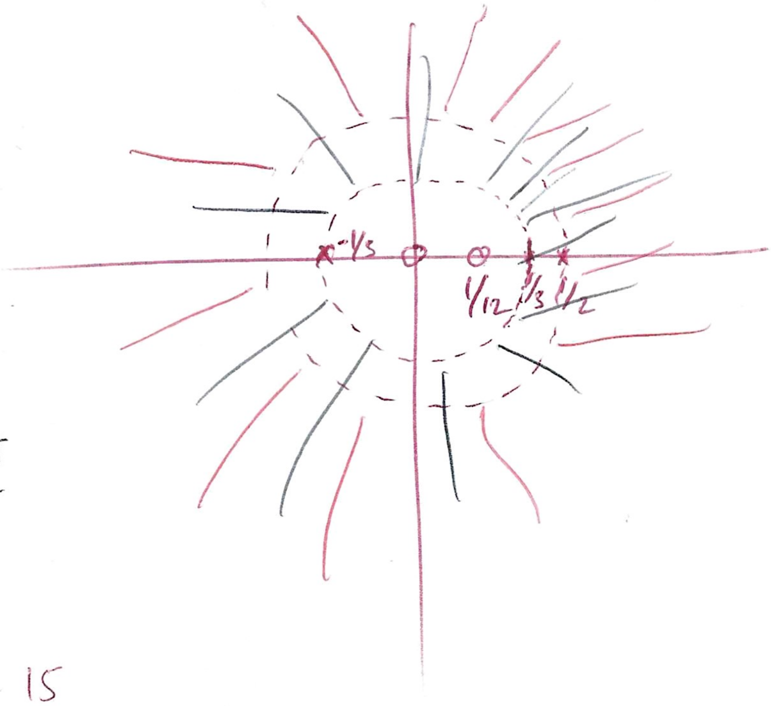
$$= \frac{z\left(z+\frac{1}{3}\right) + z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

Right sided

ROC $z > 1/2$

outer most



If the sequence is left sided
ROC $\Rightarrow$ inner most

* Finite sequence

Example: $x[n] = \delta[n] + \delta[n-5]$

$$X[z] = 1 + z^{-5}$$

$$X[e^{jw}] = 1 + 1e^{-j5w}$$

If the signal is finite then the Fourier transform and the z-transform both exist

Example:

$$x[n] = \begin{cases} a^n & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases}$$

evaluate $X(z)$

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

$N_2 \geq N_1$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{(az^{-1})^0 - (az^{-1})^{N-1+1}}{1 - az^{-1}}$$

$$= \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{z^N - a^N}{z^N - az^{N-1}}$$

$$= \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

another example solution If the previous example defined up to N

$$x[n] = \begin{cases} a^n & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N} a^n z^{-n} = \sum_{n=0}^{N} (a z^{-1})^n$$

$$= \frac{(a z^{-1})^0 - (a z^{-1})^{N+1}}{1 - a z^{-1}}$$

$$= \frac{1 - (a z^{-1})^{N+1}}{1 - a z^{-1}}$$

$$\frac{1 - a^{N+1} z^{-N-1}}{1 - a z^{-1}} * \frac{z^{N+1}}{z^{N+1}}$$

$$= \frac{z^{N+1} - a^{N+1}}{z^{N+1} - a z^N}$$

No of zeros are N+1

$$= \frac{1}{z^N} \frac{z^{N+1} - a^{N+1}}{z - a}$$

The finite signal has only poles at $z=0$

or It may have No poles

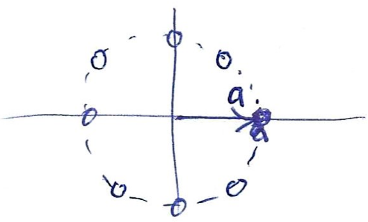
but It has $(N-)$ zeros

The region of convergence for a finite signal is the all z-plane, in some cases exept the origin, the infinity or both

The zeros are $z^N = a^N$

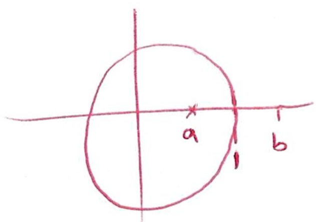
The zeros are complex $= |a| e^{j(\frac{2\pi}{k})}$

$= a e^{j(\frac{2\pi}{k})} \quad k = 0, 1, 2, 3, \dots N-1$

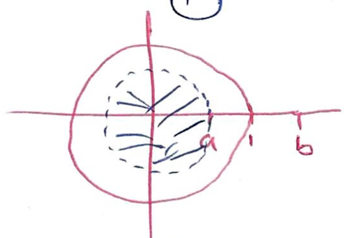


$N=8$ as an example

Example: Determine RoC, which sided, FT converge?



①

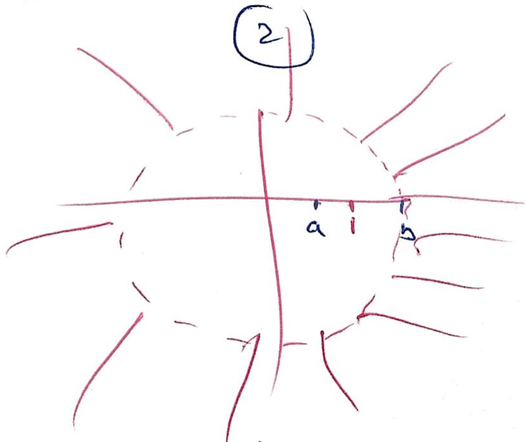


$z < a$

left sided

F.T. does not exist

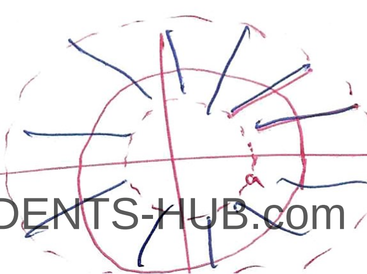
②



$z > b$

right sided

F.T. does not exist



$a < z < b$

two-sided

F T exist

Examples:- Find the z-transform and ROC of the sequence

$x[n] = \{ \underset{\uparrow}{1}, 0, 3, -1, 2 \}$

Right-sided
Finite sequence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{4} x[n] z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$$

$$= 1 + 3z^{-2} - z^{-3} + 2z^{-4}$$

ROC :- all the values of z except 0

$x[n] = \{ 3, -2, -1, 0, \underset{\uparrow}{1} \}$

left-sided

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-4}^{0} x[n] z^{-n} = Z\{x[n]\}$$

$$= 1 - z^{2} - 2z^{3} + 3z^{4}$$

ROC :- all the values of z except ∞

$x[n] = \{ 2, -1, 3, 2, \underset{\uparrow}{1}, 0, 2, 3, -1 \}$

two-sided

$$X(z) = 2z^{4} - z^{3} + 3z^{2} + 2z + 1 + 2z^{-2} + 3z^{-3} - z^{4}$$

ROC :- all the values of z except 0 and ∞

Causality and stability:

Example: Consider the following difference equation of the system

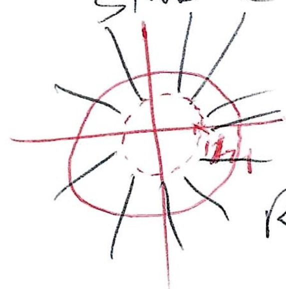
$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

check the stability and the causality of the system

-

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

Causal and stable



Right sided

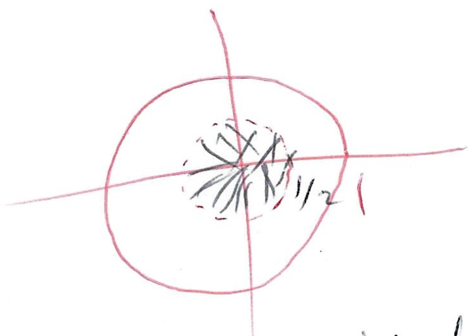
$$Y(z) - \frac{1}{2}Y(z)z^{-1} = X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$\rightarrow |z| > \frac{1}{2}, \ h[n] = a^n u[n]$

$\rightarrow |z| < \frac{1}{2}, \ h[n] = -a^n u[-n-1]$



left sided

non-Causal

non-stable

Inverse Z-transform

1- Inspection Method

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$-a^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad |z|<|a|$$

example :- Evaluate the inverse Z-transform of the

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z|> 1/2$$

$$h[n] = (1/2)^n u[n]$$

2- partial fraction expansion

Example:- consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z|> 1/2$$

Evaluate $x[n]$

$$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$1 = A_1(1 - \tfrac{1}{2}z^{-1}) + A_2(1 - \tfrac{1}{4}z^{-1})$$

either $(1-\frac{1}{2}z^{-1}) = 0$ or $(1 - \frac{1}{4}z^{-1}) = 0$

$1 - \frac{1}{2}z^{-1} = 0$ $\qquad$ $1 - \frac{1}{4}z^{-1} = 0$

$\frac{1}{2}z^{-1} = 1$ $\qquad$ $\frac{1}{4}z^{-1} = 1$

$z^{-1} = 2$ $\qquad$ $z^{-1} = 4$

when $z^{-1} = 2$

$$1 = A_1(0) + A_2(\tfrac{1}{2}) \Rightarrow A_2 = 2$$

when $z^{-1} = 4$

$$1 = A_1(-1) + A_2(0)$$

$$A_1 = -1$$

$$X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

$$x[n] = -1(\tfrac{1}{4})^n u[n] + 2(\tfrac{1}{2})^n u[n]$$

Example:

$N \geq M$

degree of numerator $\geq$ degree of denominator

same power

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \quad |z|>1$$

$$\begin{array}{r|l} & 2 \\ \hline \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & -\left(z^{-2}-3z^{-1}+2\right) \\ \hline & 5z^{-1}-1 \end{array}$$

$$X(z) = 2 + \frac{5z^{-1}-1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$= 2 + \frac{5z^{-1}-1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

$$= B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

$$= 2\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right) + A_1\left(1-z^{-1}\right) + A_2\left(1-\frac{1}{2}z^{-1}\right)$$

If $z^{-1}=2 \quad \therefore \quad A_1 = 9$

If $z^{-1}=1 \quad \therefore \quad A_2 = 8$

$$= 2 + \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$