CH. Two : Discrete-Time Signals and Systems

* The term signal is applied to something that conveys information

* Mathematically : signals are represented as functions of one or more independent variables

examples:

Signal	independent variables
$s_1(t) = 5t$	t
$s_2(t) = 20t^2$	t
$s_3(x,y) = 3x+2xy+10y^2$	x, y

speech signal	$x(t)$	1 indep. variable
image =	$z(x,y)$	2 indep. =
video	$g(x,y,t)$	3 = =

* Continuous-time signals : are represented by a continuous independent variable.

* Discrete-time signals : are defined at discrete times hence, the independent variable (t) has discrete values.

That is, discrete time signals are represented as a sequence of numbers.

* Continuous-time signals are often referred to as Analog Signals

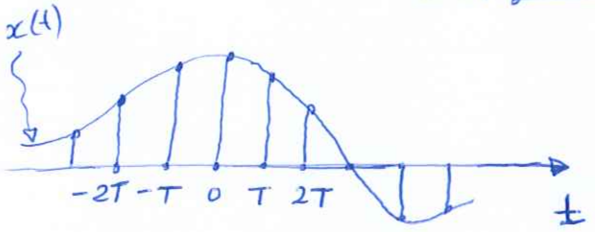
* Digital signals are those for which both time and amplitude are discrete.

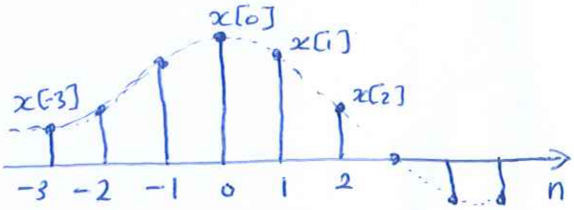
Discrete - Time signals:

* Discrete - time signals are represented as a sequence of numbers.

* it can be obtained by sampling an analog (i.e., cont. - time) signal $x(t)$. such that

$$x(t) \xrightarrow{t = nT} x(nT) = x[n]$$





T : sampling period (sec)

f : " frequency (Hz) or (Cycle/sec)

$$T = \frac{1}{f}$$

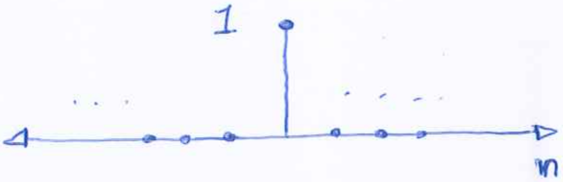
$$\Rightarrow \quad x[\underset{\text{integer}}{n}] \equiv x(nT) \qquad -\infty < n < \infty$$

Basic Sequences (Discrete-time Signals)

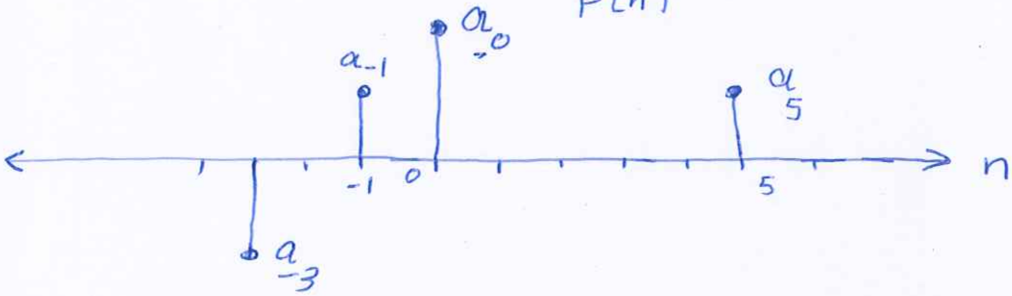
$\boxed{1}$ Unit Sample Sequence (impulse)

* it is defined as the sequence

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 1 \end{cases}$$



* it is used to represent any arbitrary sequence as a sum of scaled, delayed impulses



$$p[n] = a_{-3}\,\delta[n+3] + a_{-1}\,\delta[n+1] + a_0\,\delta[n] + a_5\,\delta[n-5]$$

OR $p[n] = \{a_{-3}, 0, a_{-1}, \underset{\uparrow n=0}{a_0}, 0, 0, 0, 0, a_5\}$

In general, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

* Some Properties :-

$\delta[n]$ is an even function (i.e., $\delta[-n] = \delta[n]$)

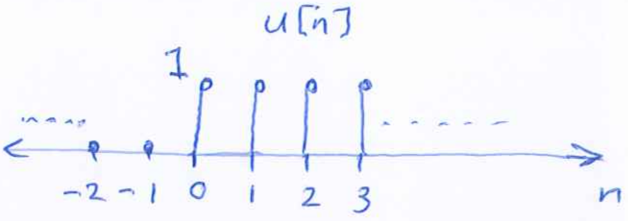
$\delta[kn] = \delta[n]$

$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$ (sampling)

$x[n] * \delta[n-n_0] = x[n-n_0]$ (convolution)

$\boxed{2}$ Unit step Signal /sequence (u[n])

* $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



* u[n] can be expressed in terms of δ[n] in two ways

① Sum of delayed impulses

from $x[n] = \sum_{k=0}^{\infty} x[k]\,\delta[n-k]$

$$u[n] = \sum \overset{1}{\cancel{u[k]}}\,\delta[n-k]$$

$$\Rightarrow \boxed{u[n] = \sum_{k=0}^{\infty} \delta[n-k]} = \delta[n] + \delta[n-1] + \cdots$$

② The value of u[n] at time index (n) is equal to the accomulated sum of the value at index (n) and all previous values of δ[n]

$$u[-1] = \cdots + \delta[-2] + \delta[-1] = 0$$

$$u[0] = \cdots + \delta[-1] + \delta[0] = 1$$

$$u[1] = \cdots + \overset{0}{\cancel{\delta[-1]}} + \delta[0] + \overset{0}{\cancel{\delta[1]}} = 1$$

$$\Rightarrow \boxed{u[n] = \sum_{k=-\infty}^{n} \delta[k]}$$

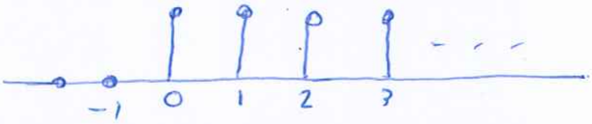
also this representation can be obtained from the previous one by changing of variable $m = n - k$

* δ[n] can be expressed as the first backward difference of u[n]

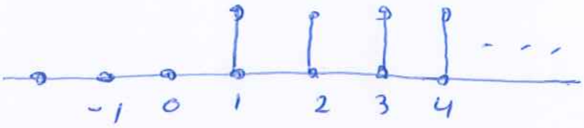
$$\boxed{\delta[n] = u[n] - u[n-1]}$$

← plot !!

$u[n]$



$u[n-1]$



$\delta[n]$

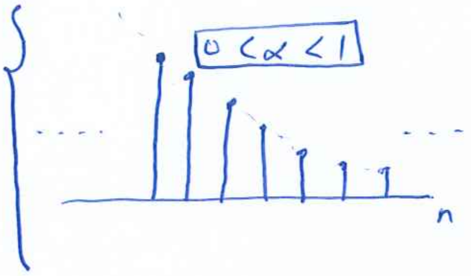
$\boxed{3}$ Exponential seqnence

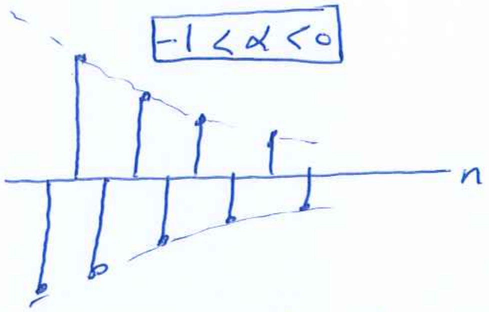
$$x[n] = A\alpha^n$$

Case 1: if A, α are both real, then $x[n]$ is real.

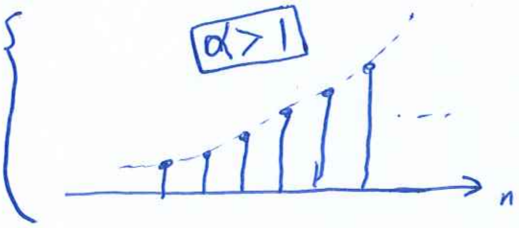
- * if A : +ve , $0 < \alpha < 1$, $x[n]$ is exp. decreasing as n↑
- * if A : +ve , $-1 < \alpha < 0$, $x[n]$ alternate in sign but again decrease as n↑

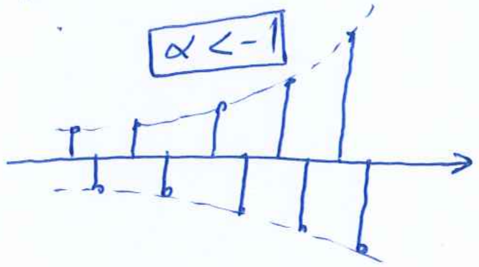
$|\alpha| < 1$



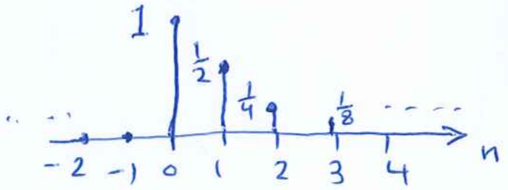


$|\alpha| > 1$

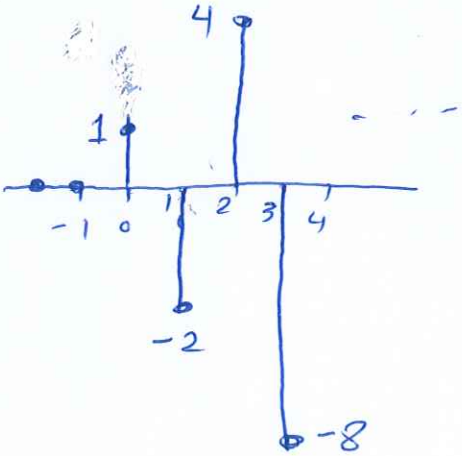




example plot $x[n] = (\frac{1}{2})^n u[n]$



plot $x[n] = (-2)^n u[n]$



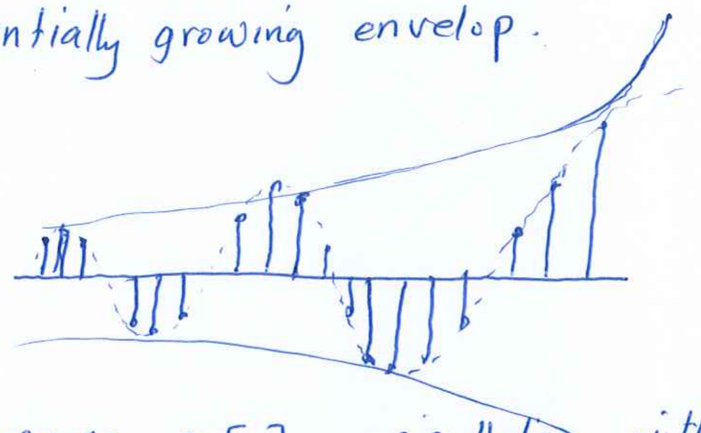
CASE II: if A, α are complex, ~~then~~ such that

$$A = |A| e^{j\phi} \qquad \alpha = |\alpha| e^{jw_0}, \text{ then}$$

$$x[n] = A\alpha^n = |A||\alpha|^n e^{j(w_0 n + \phi)}$$

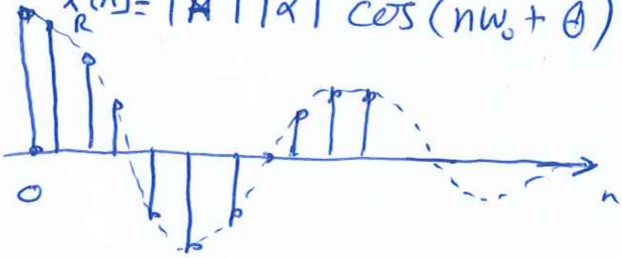
$$= |A||\alpha|^n \left[\cos(w_0 n + \phi) + j \sin(w_0 n + \phi)\right]$$

* if $|\alpha| > 1$, the sequence $x[n]$ oscillates with an exponentially growing envelop.

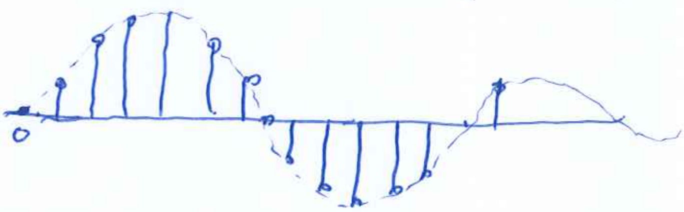


* if $|\alpha| < 1$, the sequence $x[n]$ oscillates with an exponentially decreasing envelop.

$x_R[n] = |A||\alpha|^n \cos(nw_0 + \theta)$



$x_I[n] = |A||\alpha|^n \sin(nw_0 + \theta)$



The concept of Frequency

⊠ Continuous - time sinusoidal signals

$$x(t) = A\cos(\Omega t + \Theta)$$
$$= A\cos(2\pi F t + \Theta) \qquad -\infty < t < \infty$$

* For every fixed value of the frequency F, $x(t)$ is periodic

$$x(t+T) = A\cos(2\pi F t + 2\pi F T + \Theta)$$
$$= A\cos(2\pi F t + \Theta)$$

if $2\pi F T = 2\pi$

$$\Rightarrow FT = 1 \Rightarrow F = \frac{1}{T}$$

T : signal period (fundamental)

F : frequency (HZ or cycle/sec)

$\Omega = 2\pi F$ (radian /sec) radian frequency.

notes :-

* Continuous - time sinusoidal signals with distinict (different) frequencies are themselves distinict.

* Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in a given time interval.

* The relationships we have described for a sinusoidal signals carry over to the class of complex exponential signals

$$x(t) = A\, e^{j(\Omega t + \Theta)}$$

⊛ Discrete-time sinusoidal signals

$$x[n] = \underset{\text{Amplitude}}{A} \cos(\omega n + \theta) \quad , -\infty < n < \infty$$

θ : phase (radian)

n : integer (sample number)

ω : frequency (radian or radian/sample)

If instead of ω we use the frequency variable f defined by

$$\omega \equiv 2\pi f$$

then $x[n] = A\cos(2\pi f n + \theta)$, $-\infty < n < \infty$

f ↳ (cycles/sample)

Properties :

P.1 one can argue that ω has no physical meaning from $x(t) = A\cos(\Omega t + \theta)$

by sampling $t = nT_s \Rightarrow x[nT_s] = A\cos(\Omega T_s n + \theta)$

$$= A\cos(\omega n + \theta)$$

$$\Rightarrow \boxed{\omega = \Omega T_s} = 2\pi F T_s = 2\pi \frac{T_s}{T}$$

$\frac{\text{radian}}{\cancel{\text{sec}}} \cdot \cancel{\text{sec}} = \text{radian}$ (No meaning)

$\boxed{P.2}$ A discrete-time sinusoid is periodic only if its frequency f is a rational number (ratio of two integers)

* By definition, $x[n]$ is periodic with period N ($N>0$) if and only if

$$x[n+N] = x[n] \qquad \text{for all } n.$$

* the smallest value of N is called the fundamental period.

* the proof :-

$$x[n+N] = A\cos(2\pi f_0(N+n)+\Theta)$$
$$= A\cos(2\pi f_0 n + 2\pi f_0 N + \Theta)$$
$$= A\cos(2\pi f_0 n + \Theta) = x[n]$$

this relation is true iff there exist an integer k such that

$$2\pi f_0 N = 2\pi k \qquad \omega_0 N = 2\pi k$$

$$\boxed{f_0 = \frac{k}{N}} \qquad \boxed{N = \frac{2\pi k}{\omega_0}}$$

k and N are integers

f_0 should be simplified rational number.

Examples:

① $x_1[n] = 5\cos(2\pi \frac{31}{60} n)$ $\qquad \omega_0 = \frac{61}{30}\pi$

$$f_0 = \frac{31}{60} = \frac{k}{N} \Rightarrow k = 31 \text{ (integer)}$$
$$N = 60 \text{ samples}$$

② $x_2[n] = 5\cos(2\pi \frac{30}{60} n) \Rightarrow f_0 = \frac{30}{60} = \frac{1}{2} \Rightarrow \begin{matrix} k=1 \\ N=2 \end{matrix}$

Note that small change in f_0 can result in a large change in the period!

(This is because N should be integer)

$\boxed{P.3}$ Discrete-time sinusoids whose frequencies are seperated by an integer multiple of 2π are identical (indistinguishable).

$$A\cos[(w_0 + 2\pi r)n + \theta] = A\cos(w_0 n + \theta)$$

As a result, all sinusoidal sequencies

$$x_r[n] = A\cos(w_r n + \theta) \qquad r = 0, 1, 2, \ldots$$

where

$$w_r = w_0 + 2\pi r \qquad -\pi \leq w_0 \leq \pi$$

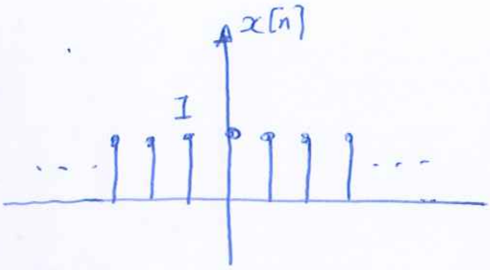
are identical

- Any sequence resulting from a sinusoid with a frequency $|w| > \pi$, or $|f| > 1/2$ is identical to a sequence obtained from a sinusoidal signal with frequency $|w| < \pi$, or $|f| < 1/2$.

- we call the sinusoid having the frequency $|w| > \pi$ an alias of a corresponding sinusoid with frequency $|w| < \pi$.

- we regard frequencies in the range $-\pi < w < \pi$, or $-\frac{1}{2} < f < \frac{1}{2}$, as unique and all frequencies $|w| > \pi$, or $f > \frac{1}{2}$, as aliases.

$\boxed{P.4}$ The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi$ (or $\omega = -\pi$) or, equivalently, $f = \frac{1}{2}$ (or $f = -\frac{1}{2}$).

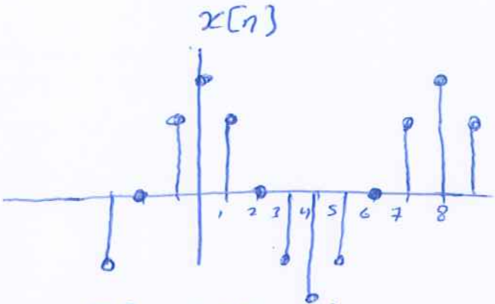


$x[n] = \cos(w_0 n)$

$w_0 = 0$ or 2π

$f = 0$

$N = \infty$

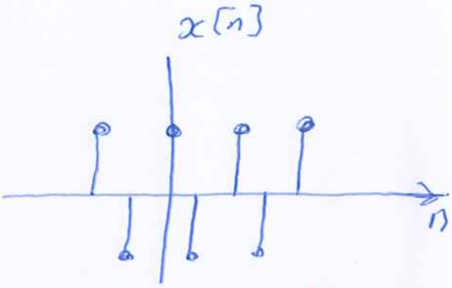


$x[n] = \cos(w_0 n)$

$w_0 = \pi/4$ or $\frac{7\pi}{4}$

$f = 1/8$

$N = 8$



$x[n] = \cos(w_0 n)$

$w_0 = \pi$

$f = 1/2$

$N = 2$

* we note that the rate of oscillation increases as the frequency increases.

* To see what happens for $\pi \leq w_0 \leq 2\pi$

let $w_2 = 2\pi - w_0$

then $x_2[n] = \cos(w_2 n) = \cos(2\pi n - w_0 n)$

$= \cos(-w_0 n) = \cos(w_0 n)$

Hence w_2 is an alias of w_0.

As we increase from 0 toward π, $x[n]$ oscillates more rapidly.

× As w_0 increases from π toward 2π, $x[n]$ oscillation become slower.

* For sinusoidal (and complex exponential signals) values of w in the vacinity of $W = 2\pi k$ are low frequencies

= = w = = = = $w = \pi + 2\pi k$ = high = .

Oscillation

Note Similar statements apply for the complex exponential sequence

$$x[n] = Ce^{jw_0 n}$$

$$x[n+N] = Ce^{jw_0 n} e^{jw_0 N} = x[n]$$

$$\text{if} \quad w_0 N = 2\pi k \Rightarrow N = \frac{2\pi k}{w_0}$$

Examples :

① $x_1[n] = \cos(\frac{\pi}{4} n)$

$$N_1 = \frac{2\pi k}{w_0} = \frac{2\pi}{\frac{\pi}{4}} k = 8k$$

choose $k = 1 \Rightarrow N = 8$ samples

② $x_2[n] = \cos(\frac{3}{8}\pi n)$

$$N_2 = \frac{2\pi k}{w_0} = \frac{2\pi}{\frac{3}{8}\pi} k = \frac{16}{3} k \qquad \text{choose } k = 3$$

$\Rightarrow N = 16$ samples

Note that, Contrary to continuous-time sinusoids, increasing the value of w_0 for a discrete-time sinusoid does not necessarily decrease the period of the signal.

$$w_1 = \frac{2}{8}\pi < w_2 = \frac{3}{8}\pi$$

$$N_1 = 8 < N_2 = 16$$

This occurs because discrete-time signals are defined only for integer indices n.

③ $x_3[n] = \cos(n)$, $\omega_0 = 1$

$$N = \frac{2\pi}{\omega_0} k = \frac{2\pi}{1} k$$

it is not periodic , there is no integer N such that $x_3[n]$ satisfies the condition $x_3[n+N] = x_3[n]$ for all n.

④ $x_4[n] = \{-1, 3, 6, 4, 2, -1, 3, 6, 4, 2, -1, \cdots\}$

$N = 5$ samples , x_4 is a periodic sequence.

Note

The physical period depends on the sampling Rate (sampling period T_s).

To find the period in seconds, we need to know T_s.

Example :-

$$x[n] = \cos(0.2\pi n) + 2\sin(0.3\pi n) + 3\cos(0.4\pi n)$$

A) check the periodicity of $x[n]$?

B) Find N_0 for $x[n]$?

$$\omega N = 2\pi k \Rightarrow N = \frac{2\pi k}{\omega}$$

for $\omega = 0.2\pi \Rightarrow N = \frac{2\pi}{0.2\pi} k = 10k$

for $k=1 \Rightarrow N_1 = 10$

for $\omega = 0.3\pi \Rightarrow N_2 = \frac{2\pi}{0.3\pi} k = \frac{20}{3} k$

for $k=3 \Rightarrow N_2 = 20$

for $\omega = 0.4\pi \Rightarrow N_3 = \frac{2\pi}{0.4\pi} k = \frac{\cancel{20}^{5}}{\cancel{4}_{1}} k$

for $k=1 \Rightarrow N_3 = 5$ samples

N for $x[n] = \mathrm{LCM}(N_1, N_2, N_3)$

$= \mathrm{LCM}(10, 20, 5) = 20$ samples

5 : 5 10 15 $\boxed{20}$ 25 . . .

10 : 10 $\boxed{20}$ 30 40 . . . -

20 : $\boxed{20}$ 40 60 . . .

Note: Sum of of periodic discrete-time signals is always periodic

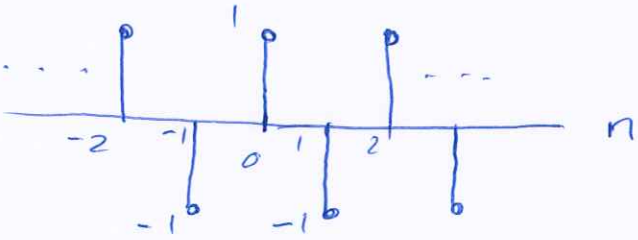
Example: $x[n] = 4\cos(0.1\pi n)\, u[n]$ is not periodic for $n<0$

periodicity is defined from $(-\infty, \infty)$

Example :- $z[n] = 5e^{j2n}$

$$N = \frac{2\pi}{\omega}k = \frac{2\pi}{2}k = \pi k$$ Not periodic

Example: $$y[n] = \sum_{k=-\infty}^{\infty} (-1)^k \delta[n-k]$$



periodic with $N = 2$