Digital Signal processing

Our Concern is discrete time signals only

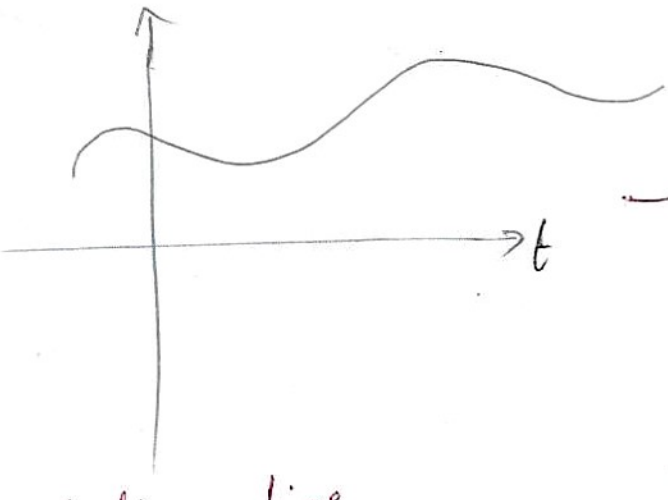
Continuous time signal ⟶ Discrete time signal
(Analog signal) ↓ ??
digital

Signal: function of one or more independent Variables

$X(t)$ ← independent Variable
↑
function (dependent)

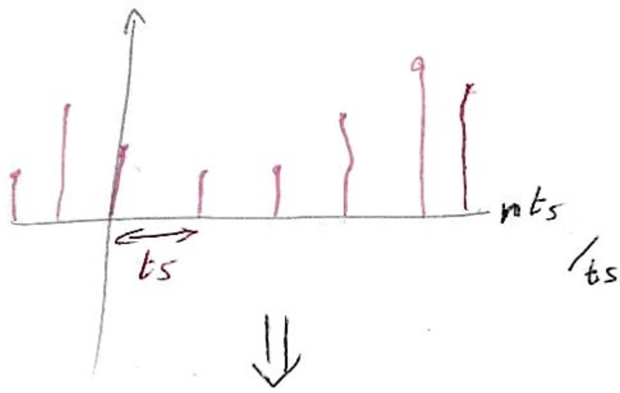
$Z(X, y)$ image

$f(x, y, t)$ video



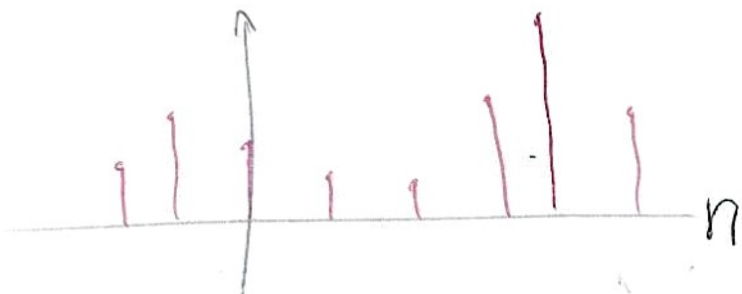
Continuous time

→



⇓

→ discrete time

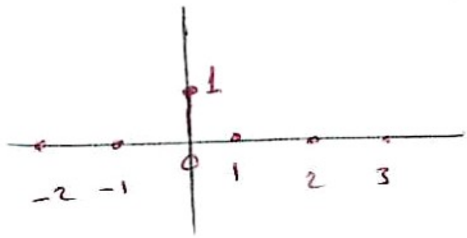


2.1 Basic sequence

① The unit sample sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$\delta[0]$



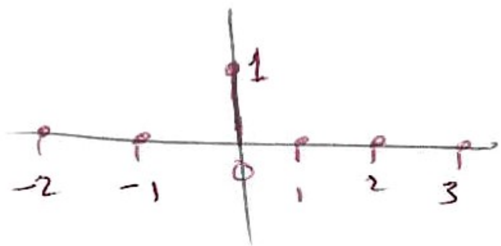
$\delta[n]$ is an even function

$\delta[n-1]$

$$\delta[kn] = \delta[n]$$

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta\boxed{[n-n_0]}$$

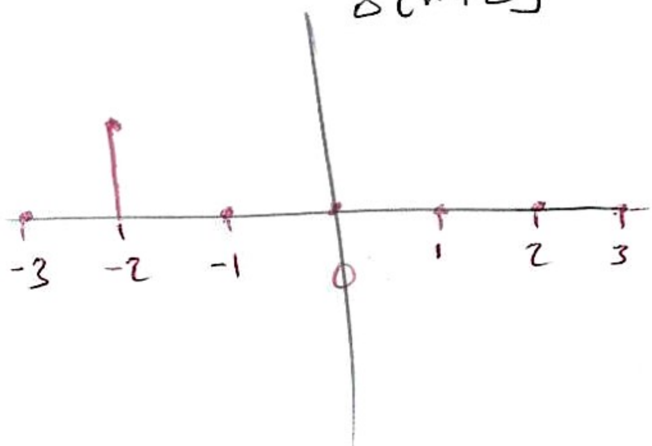
$\delta[1]$



$$x[n] * \delta[n-n_0] = x[n-n_0]$$

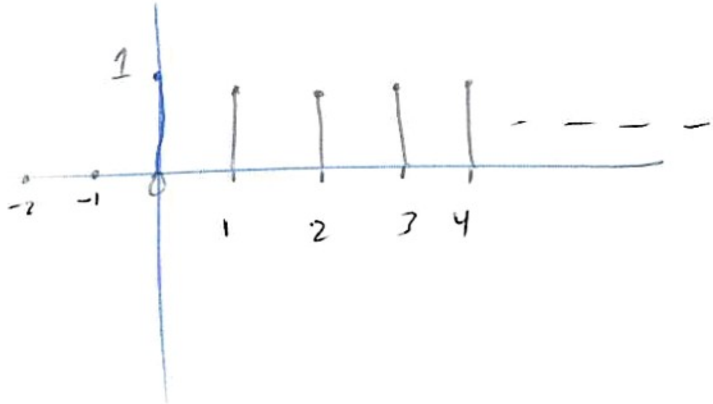
$$\sum_{k=-\infty}^{n} \delta[k] = u[n]$$

$\delta[n+2]$



(2) The unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



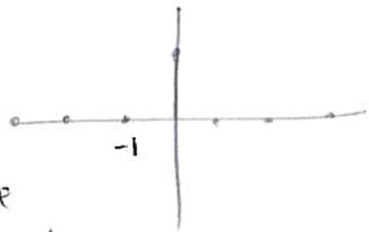
$$u[n] = \sum_{k=0}^{k=\infty} \delta[n-k]$$

using change of variables if $r = n - k$

the definition

$$u[n] = \sum_{r=\infty}^{-\infty} \delta[r] = \sum_{r=-\infty}^{n} \delta(r)$$

$\delta[-1] = 0$



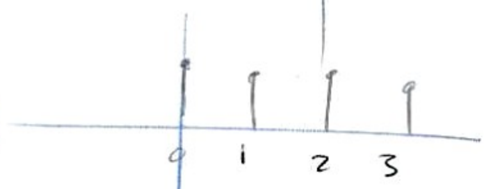
sum of all the previous value upto -1

$\delta[0] = 1$

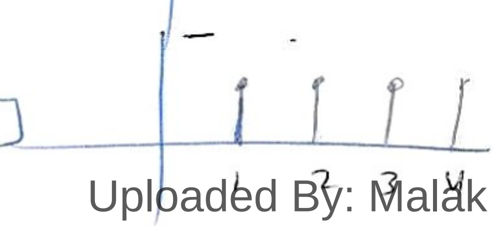
$\delta[1] = 1$

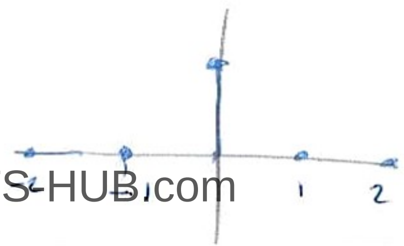
$$\delta[n] = \frac{u[n] - u[n-1]}{\Delta n}$$

$u[n]$



$u[n-1]$





$\Leftarrow$

④ Sinusoidal sequence

$$x[n] = \underset{\text{amplitude}}{A} \cos(\omega_0 n + \underset{\text{phase shift}}{\phi})$$

ω_0 → digital frequency (Ratio) of the original and sampling frequency

$$x(t) = A\cos(\Omega t + \phi) \qquad \Omega = \frac{2\pi K}{T}$$

K ← integer, T ← real

$t = n t_s$

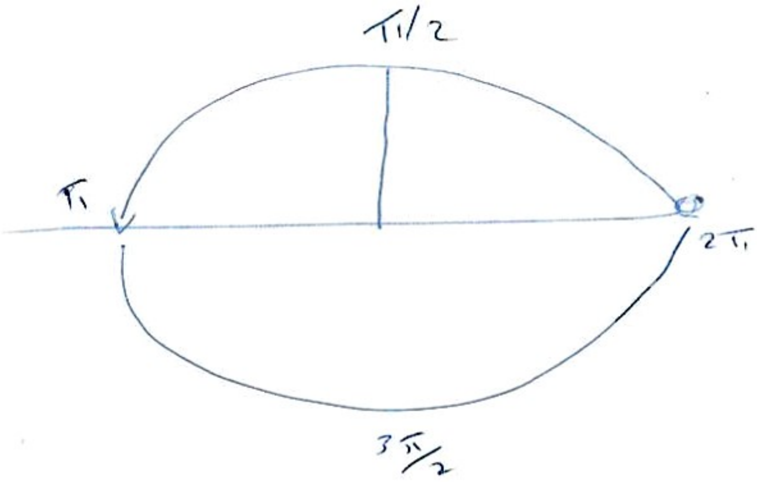
$$x(t) = A\cos(\Omega n t_s + \phi$$

$\omega_0 = \Omega t_s$

rad/s (s) = rad only so has no physical meaning

$$\omega_0 = \frac{2\pi K}{N}$$

K ← integer, N ← integer



so $\cos(\omega_0) = \cos(\omega_0 + 2\pi)$

$x[n] = A\cos((\omega_0 + 2\pi r)n) + \phi)$

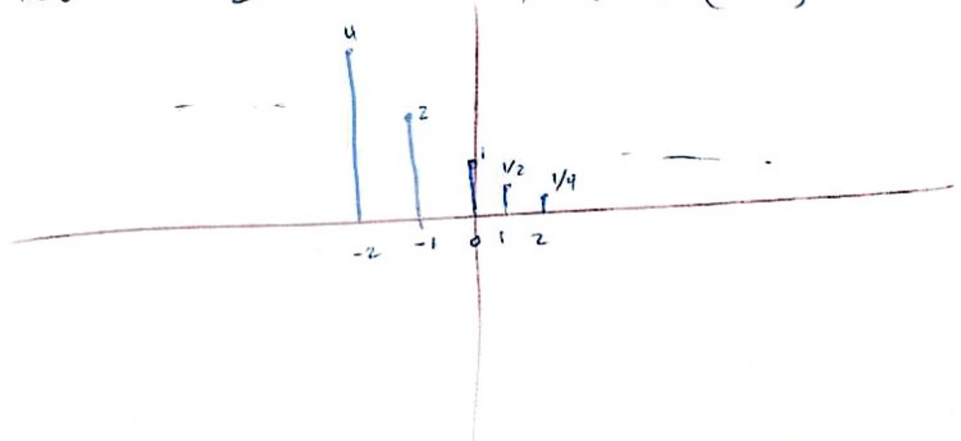
$= A\cos(\omega_0 n + \phi$

③ Exponantial sequence

$$x[n] = A\alpha^{n}$$

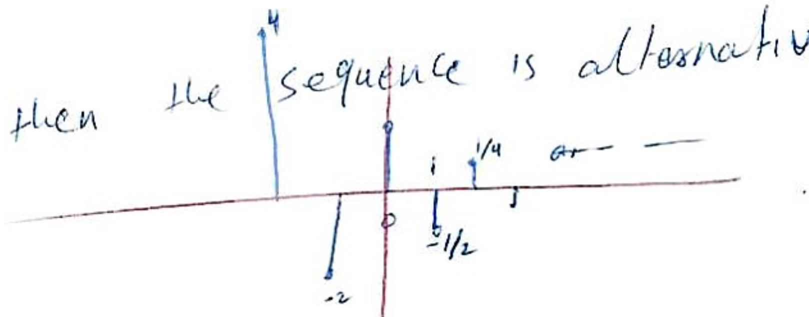
① If A & α are real

If $0<\alpha<1$ and A is positive, then the sequence is decaying exponantial ~~decay~~ $x[n] = (1/2)^{n}$



If $\alpha > 1$ and A is positive, the the sequence is growing exponantial

If $-1<\alpha<0$ then the sequence is alternative



$$x[n] = A\alpha^{n}\, u[n]$$ ← to used as Causal Signal

$$= \begin{cases} A\alpha^{n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

If A & α are complex

$\alpha = a + jb \qquad \alpha = \sqrt{a^2+b^2} \qquad w_0 = \tan\frac{b}{a}$

$\Downarrow$

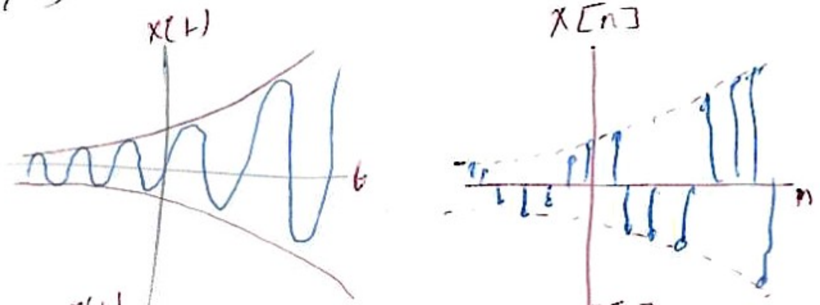
$\alpha = |\alpha| e^{jw_0} \qquad A = |A| e^{j\phi}$

$$x[n] = A\alpha^n = |A| e^{j\phi} |\alpha|^n e^{jw_0 n}$$

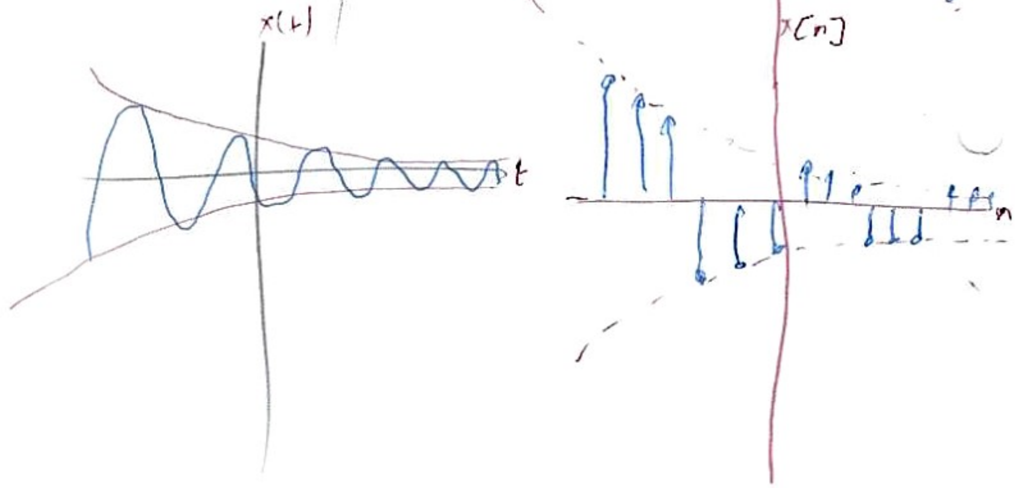
$$= |A| |\alpha|^n e^{j(\phi + w_0 n)} \qquad e^{j\theta} = \cos\theta + j\sin\theta$$

$$= |A| |\alpha|^n \cos(w_0 n + \phi) + j |A| |\alpha|^n \sin(w_0 n + \phi)$$

If $|\alpha| > 1 \longrightarrow$



If $|\alpha| < 1$



periodicity

$$x[n] = A e^{j(w_0 + 2\pi)n}$$

$$= A e^{jw_0 n} \underbrace{e^{j2\pi n}}_{1} = A e^{jw_0 n}$$

$$e^{j2\pi n} = \underbrace{\cos(2\pi n)}_{1} + j \underbrace{\sin(2\pi n)}_{0}$$

periodicity

$$x[n] = \{1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4\}$$

$$x[n] = A\cos(w_0 n + \phi)$$

is periodic if and only if

$$A\cos(w_0(n+N) + \phi) = A\cos(w_0 n + \phi)$$

$w_0 N = 2\pi K$, K is integer

same for complex exponential

$$e^{jw_0(n+N)} = e^{jw_0 n} = e^{jw_0 n} e^{jw_0 N}$$

$e^{jw_0 N} = 1$, $w_0 N = 2\pi K$

$$w_0 = \frac{2\pi \overset{\text{integer}}{K}}{\underset{\text{integer}}{N}} , \quad N = \frac{2\pi K}{w_0}$$

Example:- let $x_1[n] = \cos(\pi n/4)$, is this signal periodic? show the fundamental period?

$$w_0 = \frac{\pi}{4} = \frac{2\pi K}{N}$$

$$N = \frac{2\pi K}{\frac{\pi}{4}} = 8K$$

If K integer then N integer so the signal is periodic

$N_0 = 8$ samples

$\hookrightarrow$ to convert this to seconds, I need the <u>sampling period</u>.

Let us consider anothe signal

$$X_2[n] = \cos\left(\frac{3\pi n}{8}\right)$$

$$w_0 = \frac{3\pi}{8}$$

at $K=3$

$$N = \frac{2\pi K}{\frac{3\pi}{8}} = \frac{16}{3} K$$ periodic signal

$N_0 = 16$ samples

please note:-

$f = \frac{1}{t}$ Continuous

If f increased $\Rightarrow$ t decreased

but for discrete signals the relation is not correct

$\omega_{01} = \frac{\pi}{4}$ $\omega_{02} = 3\frac{\pi}{8}$

$= \frac{2\pi}{8}$

$\underset{\frac{3\pi}{8}}{\omega_{02}} > \underset{\frac{\pi}{4} = \frac{2\pi}{8}}{\omega_{0}1}$

$\underset{16}{N_{0}2} > \underset{8}{N_{0}1}$

this is clear from the previous example, the physical period here depends on the sampling rate.

let $X_3[n] = \cos(n)$

$\omega_0 = 1$

$N = \frac{2\pi K}{\omega_0} = 2\pi K$

there is No integer to make N integer

so $X_3[n]$ is not periodic (aperiodic)

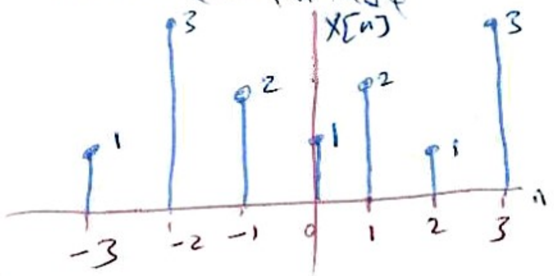
Operations on Signals

- amplitude scaling

$2X[n] \rightarrow$ amplification

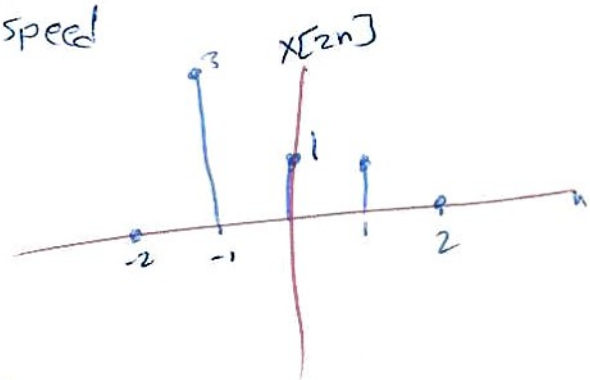
$\frac{1}{2}X[n] \rightarrow$ attenuation

$-2X[n] \leftarrow$ Reflection
on the amplitude



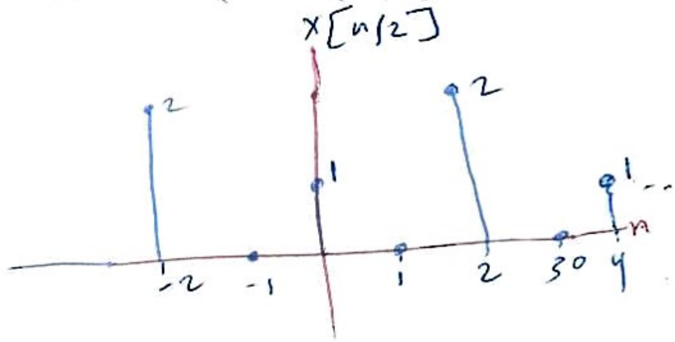
- time scaling

$X[2n] \leftarrow$ compression
double the speed



$X[\frac{1}{2}n] \leftarrow$ expansion

$X[-n] \leftarrow$ Reflection on the time axis



- Amplitud shifting

$X[n]+2$ shift up

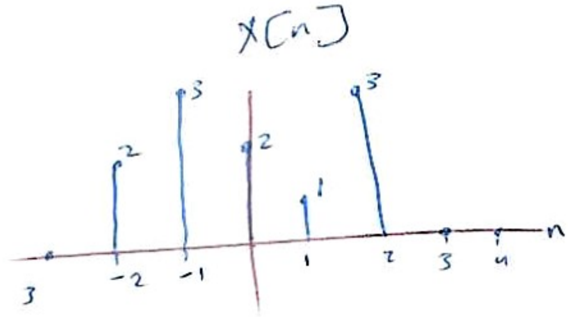
$X[n]-2$ shift down

Continuous شغالين عليه

4- time shifting

$x[n-1] \leftarrow$ delayed

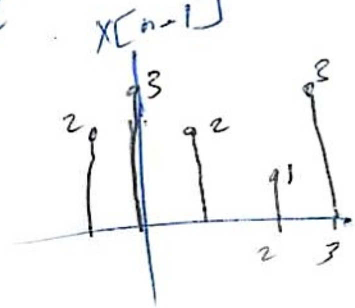
$x[n+1] \leftarrow$ advanced



$x[n] = \{2, 3, \underset{\uparrow}{2}, 1, 3\}$

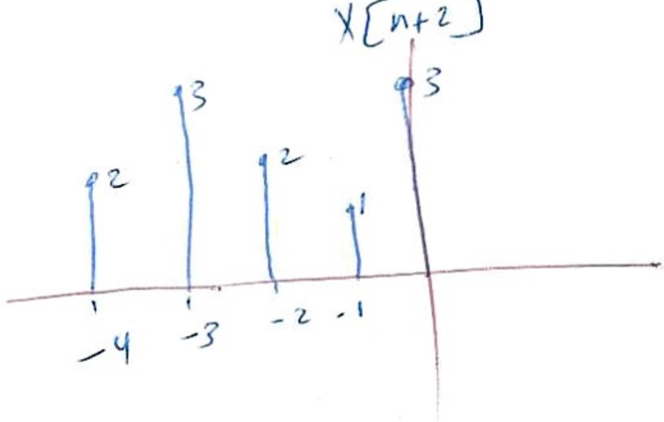
$x[2n-4]$ the best way to solve such aproblem is to take $x[2(n-2)]$

time scaling to shifted signal



$x[2-n] \equiv x[-(n-2)]$

$x[n-1] = \{2, \underset{\uparrow}{3}, 2, 1, 3\}$



Discrete time systems

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

$$y[n] = T\{x[n]\}$$

$$x[n] \longrightarrow y[n]$$

* linear

if $x_1[n] \longrightarrow y_1[n]$

and $x_2[n] \longrightarrow y_2[n]$

then $a\,x_1[n] + b\,x_2[n] \longrightarrow a\,y_1[n] + b\,y_2[n]$

$y_3 = \alpha y_1 + \alpha y_2$

$y_4 = \cancel{f}\,[\alpha x_1 + \alpha x_2]$

$y_3 \overset{??}{=} y_4$

if yes then linear

Example $y[n] = 3x[n] + 5$

* Time inveriant (shift inveriant)

$x[n] \longrightarrow y[n]$

$x[n-n_0] \longrightarrow y[n-n_0]$

$y_1[n] = f(x_1[n]) = x[n-n_0]$

$y(n-n_0) \overset{?}{=} y_1[n]$

If yes $\Rightarrow$ inveriant

$$y[n] = 3x[n] + 5$$

$$y[n] = n\,x[n]$$

So for any LTI system

$$\delta[n] \longrightarrow \boxed{LTI} \longrightarrow y[n] = h[n]$$

$$\delta[n - \overset{k}{n_0}] \longrightarrow \boxed{LTI} \longrightarrow h[n-k]$$

if $x[n] = \sum_{n=-\infty}^{\infty} x[k]\, \delta[n-k]$

$$y[n] = \sum_{n=-\infty}^{\infty} x[k]\, h[n-k]$$ by property of LTI systems

same

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

Convolution Sum

$$y[n] = x[n] * h[n]$$
$$= h[n] * x[n]$$ its Comulative

if we have series of system (cascade), the order is not important

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$$

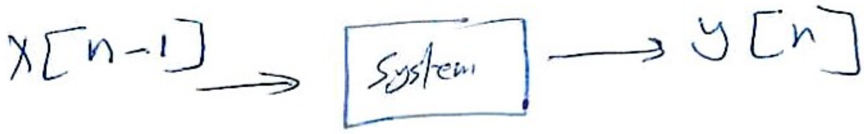
$$h_1 * h_2[n] = h_2[n] * h_1[n]$$

Examples- Ideal delay system

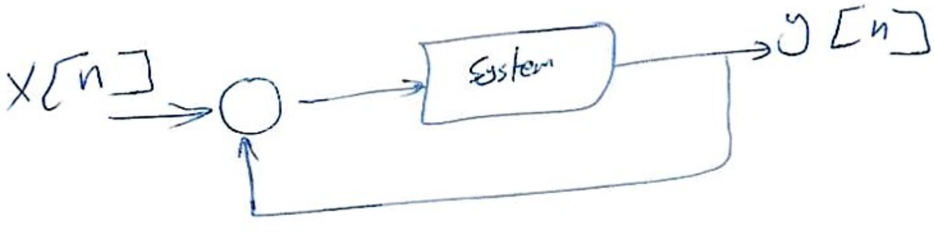
$$y[n] = x[n - n_d] \qquad -\infty < n < \infty$$

n_d: constant positive integer, called the delay of the system

the system could be dependent on the present input or on past input (difference equation)



with out feed back or with feed back



delay system is applicable but the advanced system not

Examples- moving averager

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

$$= \frac{1}{M_1 + M_2 + 1} \{ x[n + M_1] + x[n + M_1 - 1] + \dots + x[n]$$

$$+ x[n-1] + \dots + x[n - M_2] \}$$

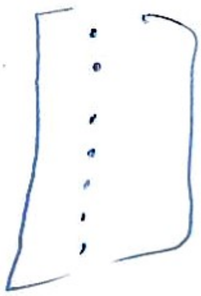
For $M_1 = 0$, $M_2 = 5$

$$y[7] = \frac{1}{6}\left[x[7] + x[6] + x[5] + x[4] + x[3] + x[2]\right]$$

$$y[8] = \frac{1}{6}\left[x[8] + x[7] + x[6] + x[5] + x[4] + x[3]\right]$$

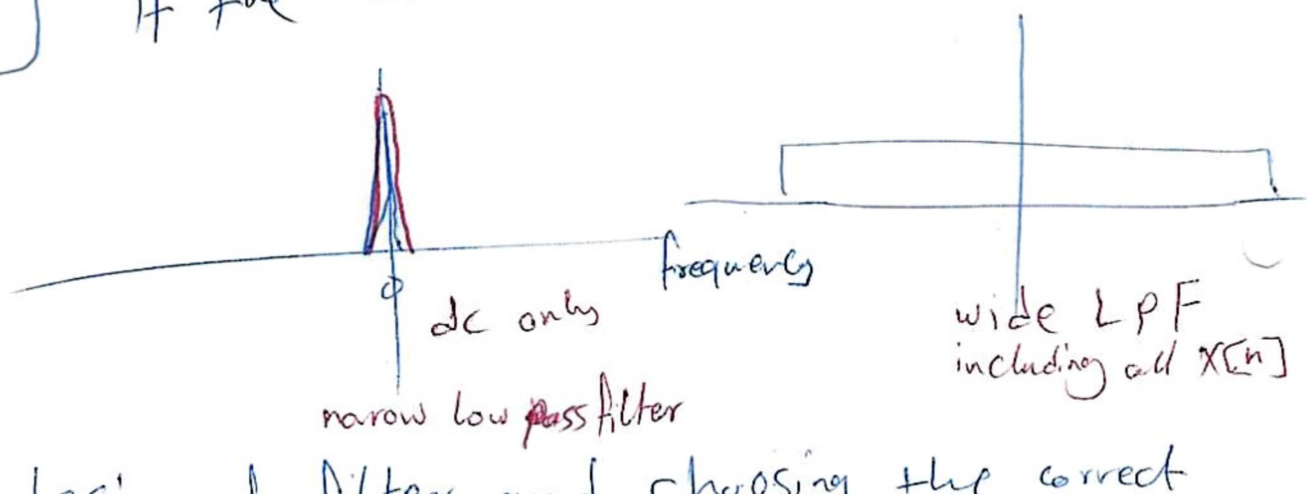
and so on

relation of the window size and filter bandwidth



when the window size = all the data then $y[n]$ will be one value only (dc)

If the window size = 1 $\longrightarrow$ then $y[n] = x[n]$



the design of filter and choosing the correct window will be discussed later in chapters 7 and 9

Example on matlab

$\Omega = 2\pi f$

$w_o = \Omega t_s = \frac{\Omega}{f_s} = \frac{2\pi f}{f_s}$

```
n=0:1000
x = cos(2*pi*500*n/8000) + cos(2*pi*1000*n/8000) + cos(2*pi*4000*n/8000)
plot(n,x)
stem(n,x)
```

increase the window size to cancel frequencies

* Memoryless / Memory

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n

Example $y[n] = (x[n])^2$ for each value of n

memoryles system

$y[n] = x[n^2]$

memory

Ideal delay system / Moving average filter
memory / memory

Example $y[n] = \sum_{k=-\infty}^{n} x[k]$ is it linear?

linear

Example :- $w[n] = \log(|x[n]|)$

non linear

Example :- $y[n] = \sum_{k=-\infty}^{n} x(k)$, is it time inveriant?

time inverian → chang of variables

Example :- "compressor system" Time Variant system

$$y[n] = x[Mn] \quad -\infty < n < \infty$$

* Causality :- The output at any time depends on the instant value of the input or at past value

$$y[n] = x[n] - x[n-1]$$

$$y[n] = \sum_{n=-\infty}^{\infty} x[n]$$

$$y[n] = \sum_{n=-\infty}^{n+4} x[n]$$

$$y[n] = x[n^2]$$

* stability :- [bounded - Input Bounded - output]

if $x[n]$ bounded i.e $|x[n]| < B_x < \infty$, all n

then $y[n]$ is bounded i.e $|y[n]| < B_y < \infty$, all n

LTI $\quad \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ (absolutely summable)

e.g $\quad h[n] = 2^n u[n] \longrightarrow$ unstable

$h[n] = (\frac{1}{2})^2 u[n] \longrightarrow$ stable

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

Converge

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \quad |\alpha| < 1$$

$y(t) = \sum_{k=n_0}^{n} x(k)$ check the stability

stable?

$$\left|\sum_{k=n_0}^{n} x[k]\right| = \sum_{k=n_0}^{n} |x| \le M < \infty$$

$$= (n - n_0)\, M < \infty$$

finite $\longrightarrow$ stable

Causality = ?? depends on n_0

Example :- $x[n] = 4\cos(0.2\pi n) - 3\sin(0.3\pi n)$
$+ 5\cos(0.4\pi n)$

$\omega_1 N_1 = 2\pi K_1$

$0.2\pi N_1 = 2\pi K_1$

$$\frac{N_1}{K_1} = \frac{2}{0.2} = \frac{10}{1} \Rightarrow N_1 = 10 \text{ Samples}$$

$\omega_2 N_2 = 2\pi K_2$

$0.3\pi N_2 = 2\pi K_2$

$$\frac{N_2}{K_2} = \frac{2\cancel{\pi}}{0.3\cancel{\pi}} = \frac{2}{0.3} = \frac{20}{3} \Rightarrow N_2 = 20 \text{ Samples}$$

$\omega_3 N_3 = 2\pi K_3$

$0.4\pi N_3 = 2\pi K_3$

$$\frac{N_3}{K_3} = \frac{2\pi}{0.4\pi} = \frac{20}{4} = 5 \Rightarrow N_3 = 5 \text{ Samples}$$

$\therefore N_0 = LCM(N_1, N_2, N_3) = LCM(10, 20, 5)$

$= 20$ Samples

* Note :- The Summation of two periodic discrete signal is always periodic

Example :- Is the signal $4\cos(0.2\pi n)\,u[n]$ a periodic signal

It is not because It is not periodic for $n<0$, periodic signals sould be periodic for the intire interval from $-\infty$ to ∞

Example :- is the following signal periodic

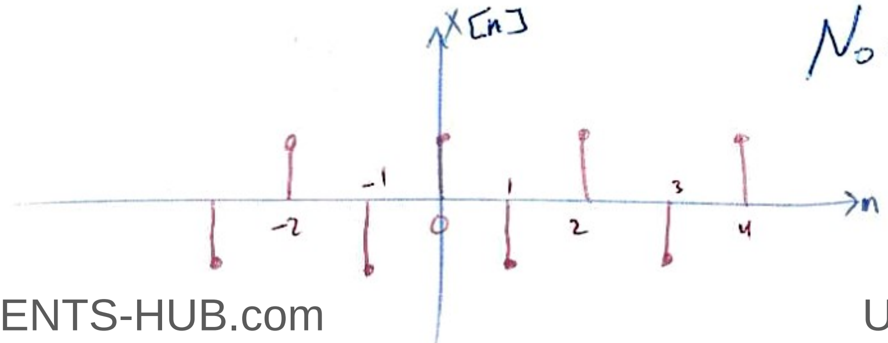
$x[n] = e^{j2n}$ $\quad A_0 e^{j\omega_0 n}$

$\omega_0 N = 2\pi K$

$2N = 2\pi K$

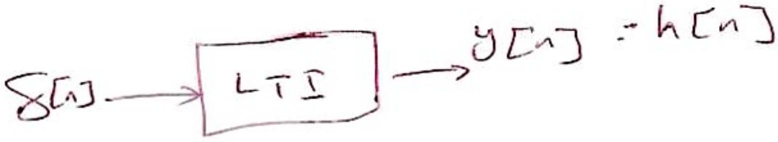
$N = \frac{2\pi k}{2} = \pi k$ not periodic

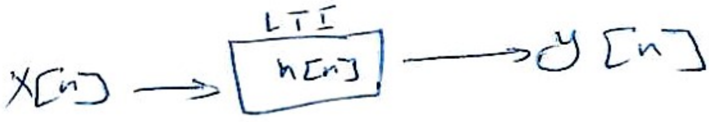
Example :- $x[n] = \sum_{k=-\infty}^{\infty} (-1)^k \delta(n-k)$



$N_0 = 2$

* linear time invariant system



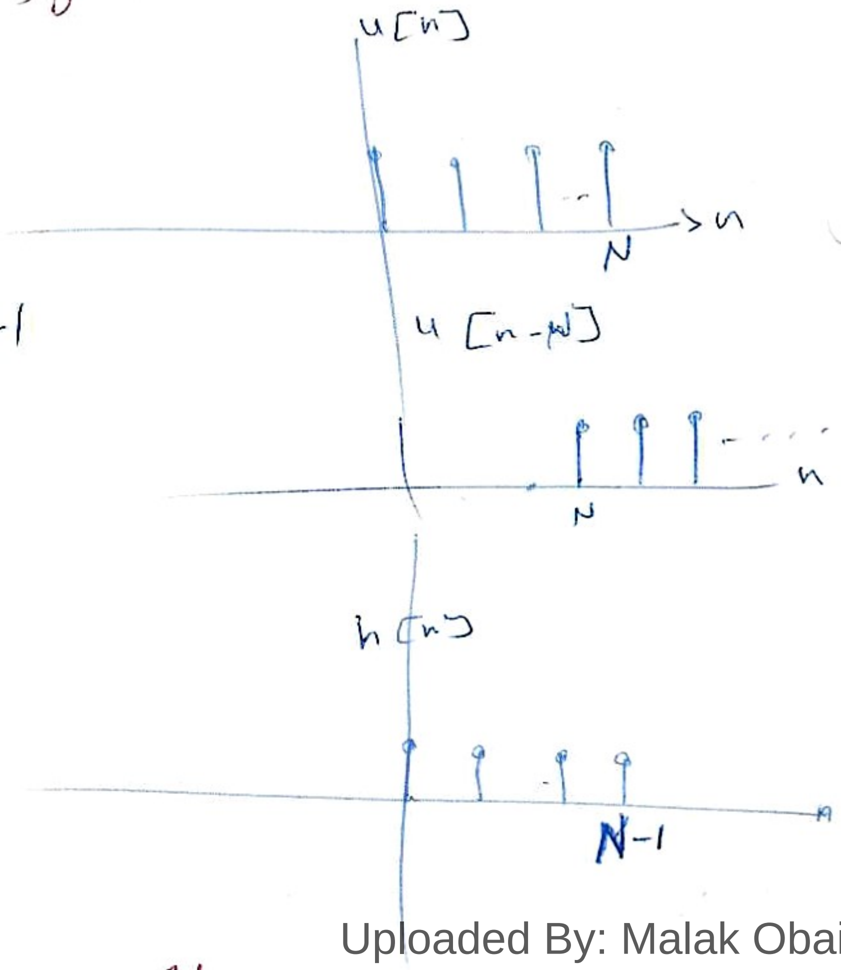


$$y[n] = x[n] * h[n]$$

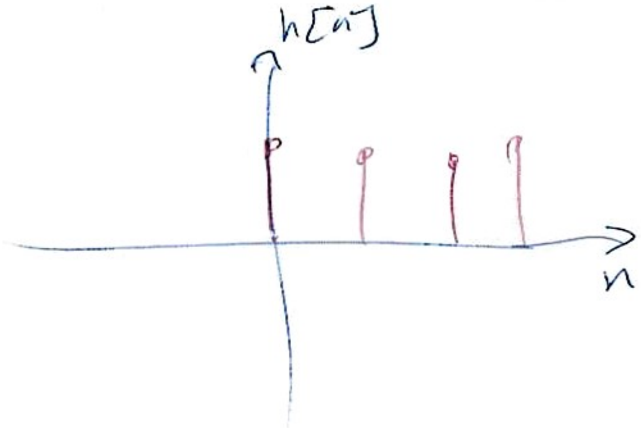
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

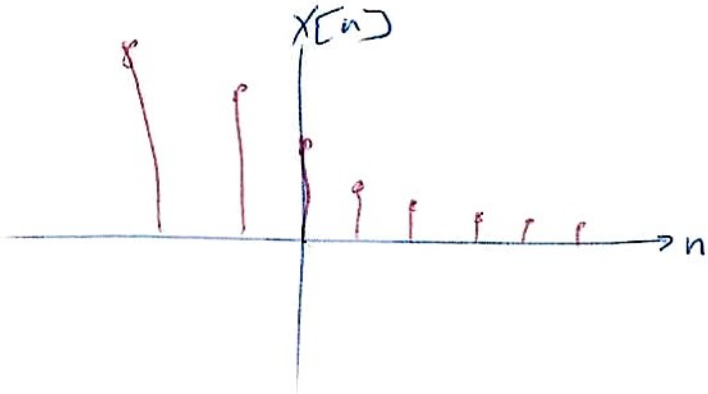
Example: Consider LTI system with $h[n] = u[n] - u[n-N]$
and the input signal $x[n] = a^n u[n] \quad 0 < a < 1$

Find $y[n]$

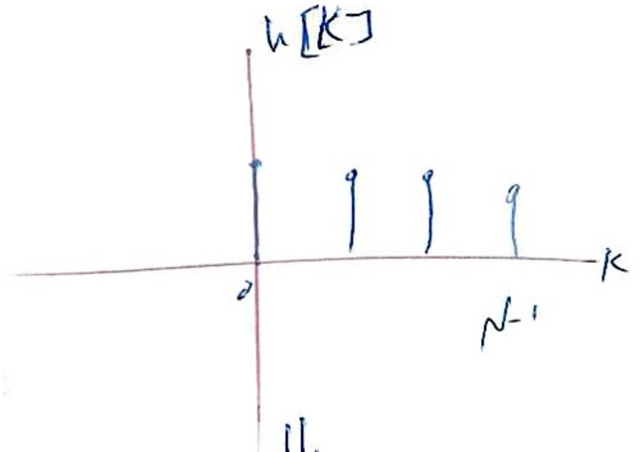


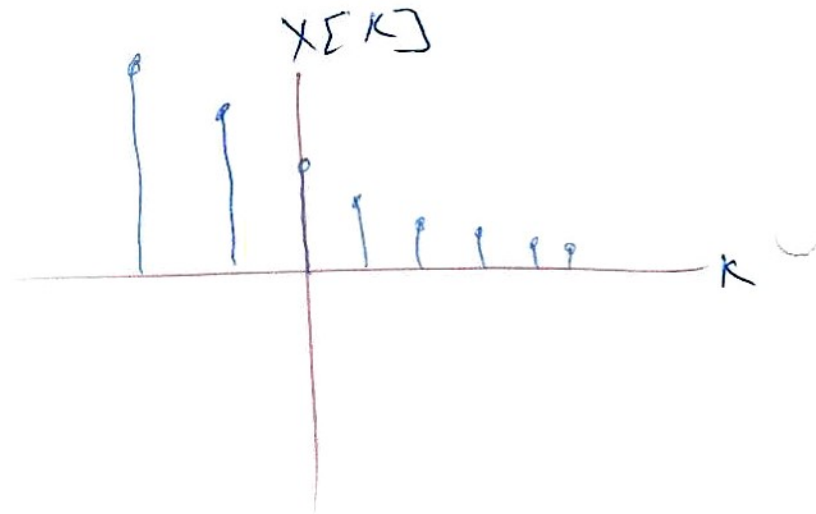
$$h[n] = \begin{cases} 1 & 0 < n < N-1 \\ 0 & o.w \end{cases}$$



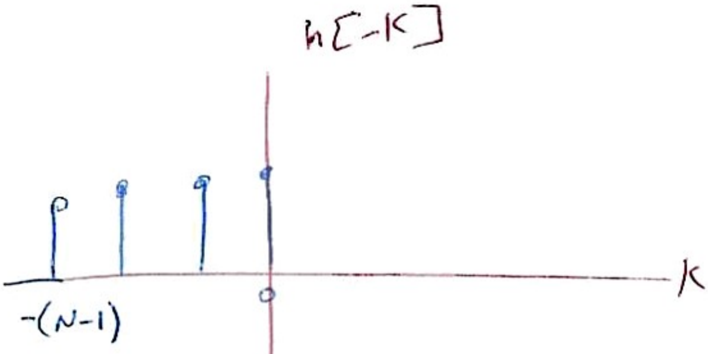


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$



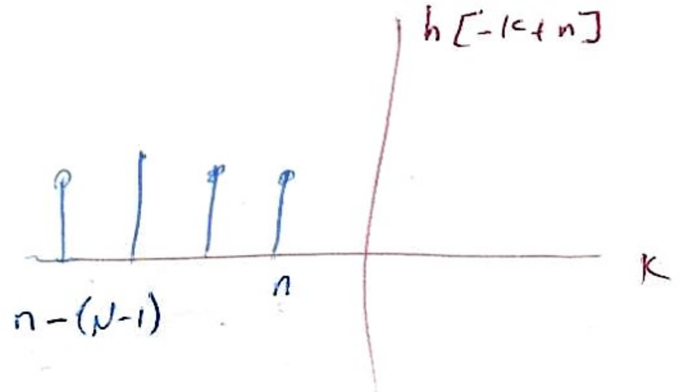


$\Downarrow$



shift t n

+ n



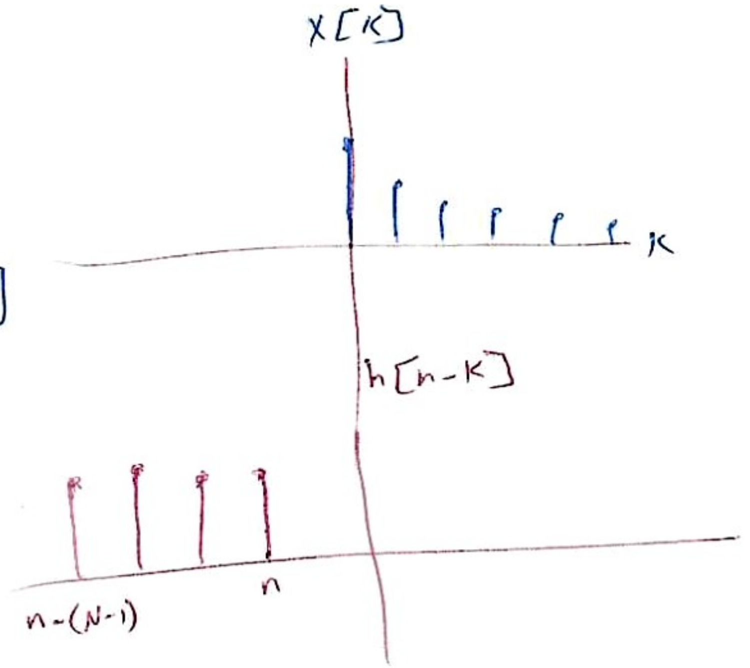
$\Longrightarrow$ do the summation

① for $n < 0$

$[0, \infty]$ + $[0, N-1]$

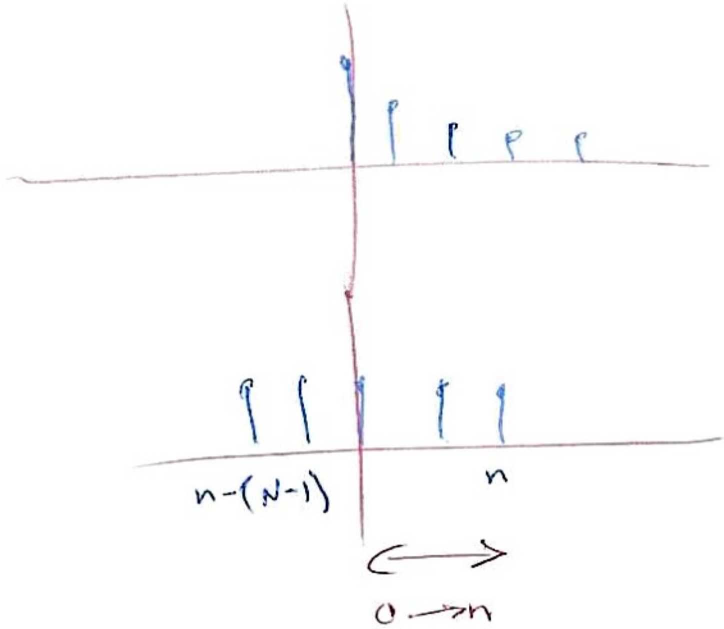
$y[n] = 0$

$[0, N-1, \infty]$



② for $0 \leq n < N-1$

$\therefore y[n] = \sum_{K=0}^{n} (1)\, a^K$

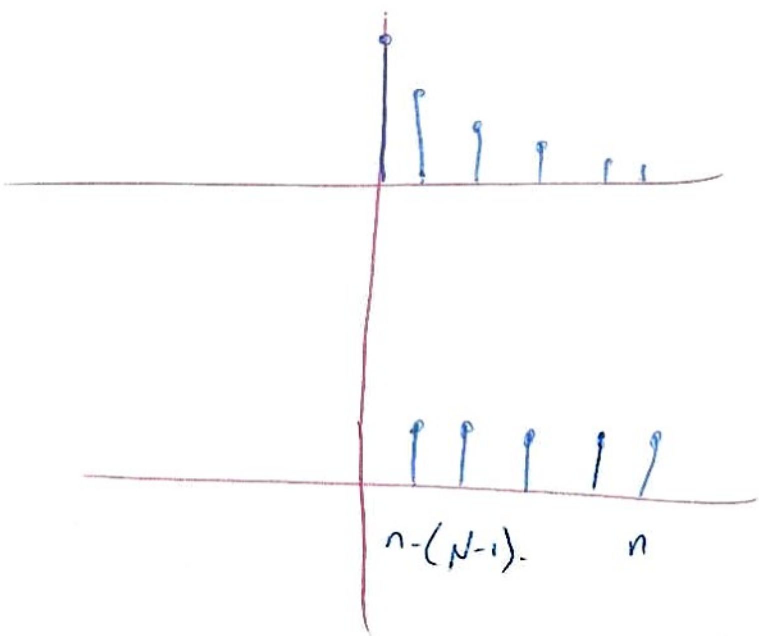


$N_2 > N_1$

$$\sum_{K=N_1}^{N_2} a^K = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$\sum_{K=0}^{n} a^k = \frac{a^0 - a^{n+1}}{1-a}$$

for $n \geq N-1$



$$y[n] = \sum_{K=n-(N-1)}^{n} a^K$$

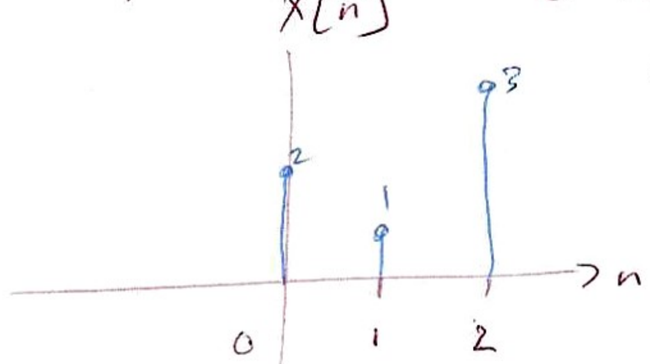
$$= \frac{a^{n-(N-1)} - a^{n}}{1-a}$$

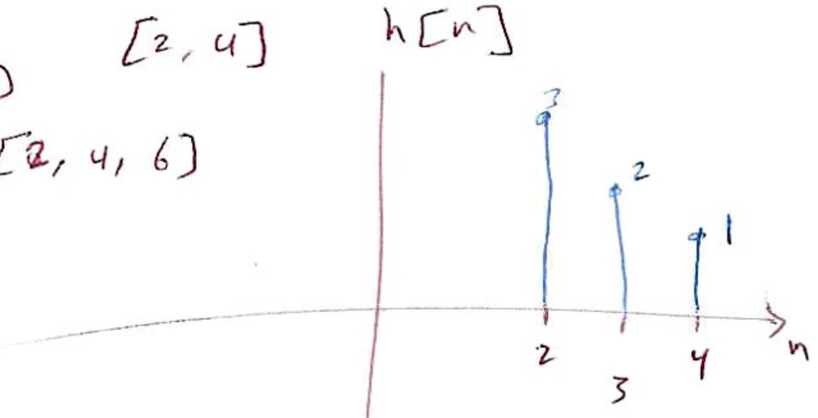
$$y[n] = \begin{cases} 0 & n<0 \\ \dfrac{1-a^{n+1}}{1-a} & 0 \le n < N \\ \dfrac{a^{n-(N-1)} - a^{n}}{1-a} & \end{cases}$$

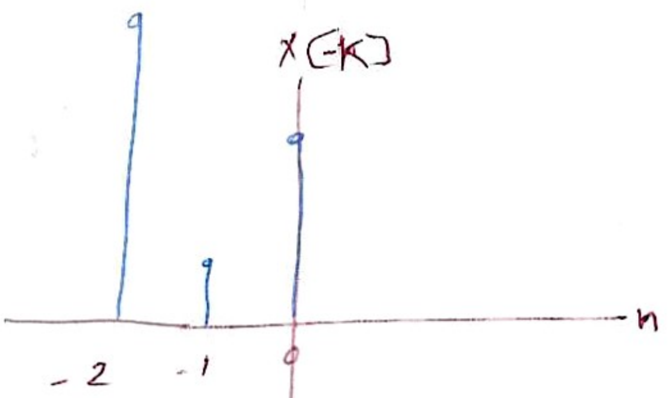
Example:

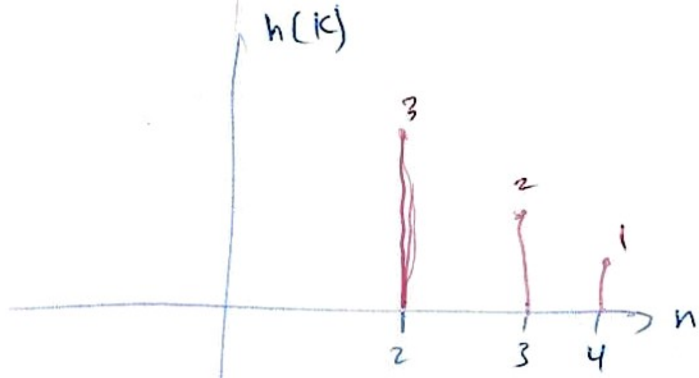
$[0,2)$ $[2,4]$

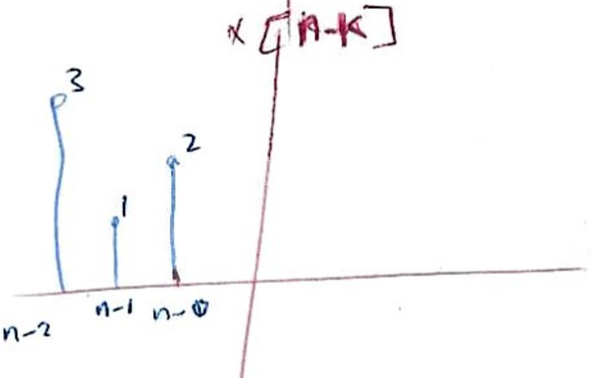
$[2,4,6]$

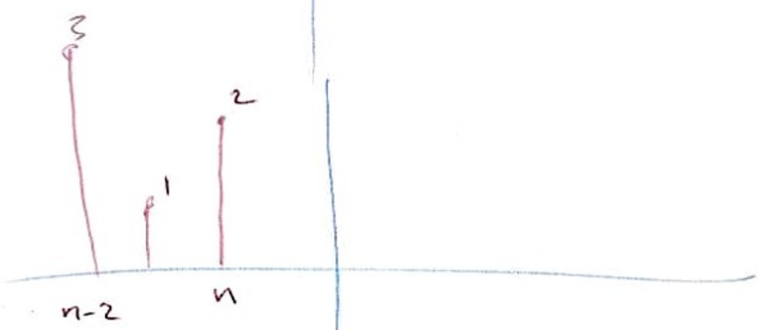












for $n < 2$ $y[n] = 0$

for $n = 2$ $y[2] = 6$

for $n = 3$

$y[3] = 1*3 + 2*2 = 7$

for $n = 4$, $y[4] = 9 + 2 + 2 = 13$

for $n = 5$ $y[5] = 6 + 1 = 7$

for $n = 6$ $y[6] = 3$

for $n > 6$ $y[n] = 0$

$x[n]$ \ $h[n]$	3	2	1
2	6	4	2
1	3	2	1
3	9	6	3

$$y[n] = \{6, 7, 13, 7, 3\}$$

Example:
consider the following system with impulse response $h[n]$

=

n	0	1	2	3	4	5	6	7
$x[n]$	$x[0]$	$x[1]$	$x[2]$	0	0	0	0	0
$h[n]$	0	0	$h[2]$	$h[3]$	$h[4]$	0	0	0
	0	0	$h[2]x[0]$	$h[2]x[0]$	$h[2]x[0]$			
	0	0	0	$h[3]x[0]$	$h[3]x[1]$	$h[3]x[2]$		
	0	0	0	0	$h[4]x[0]$	$h[4]x[1]$	$h[4]x[2]$	0
Σ			6	7	13	7	8	0

Example: Consider the system with $h[n] = \{-1, \underset{\uparrow}{2}, 0, 1\}$

and $x[n] = \{3, 1, \underset{\uparrow}{0}, -1\}$, find $y[n]$

$$x[n] \longrightarrow \boxed{\;h[n]\;} \longrightarrow y[n]$$

$y[n] = x[n] * h[n]$

x[n] \ h[n]	-1	2	0	1
3	-3	6	0	3
1	-1	2	0	1
0	0	0	0	0
-1	1	-2	0	-1

$$y[n] = \{-3, 5, 2, \underset{\uparrow}{4}, -1, 0, -1\}$$

<u>exercise:</u> $h[n] = \{1, -1, \underset{\uparrow}{1}\}$, $x[n] = \{\underset{\uparrow}{2}, -1, 1\}$

find $y[n] = ??$

Example:- Determine the impulse Response of an LTI System $y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$

To find the impulse response $x[n] = \delta[n]$

So in the equation above replace each $x[n]$ by $\delta[n]$

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow y[n]$$

$$\delta[n] \longrightarrow \boxed{LTI} \longrightarrow h[n]$$

So $h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$

$$= \{\underset{\uparrow}{a_1}, a_2, a_3, a_4\}$$

Example:- Consider the LTI discrete system with $h[n] = \{1, \underset{\uparrow}{2}, 0, -3\}$, find and plot $y[n]$ for

① $x_1[n] = \delta[n]$

② $x_2[n] = \delta[n+1] + \delta[n-2]$

③ $x_3[n] = \{\underset{\uparrow}{1}, 1, 1\}$

④ $x_4[n] = \{2, 1, \underset{\uparrow}{-1}, -2, -3\}$

Solution :-

$$x[n] \longrightarrow \overset{h[n]}{\boxed{\text{LTI}}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

① $x_1[n] = \delta[n]$

$$y[n] = x_1[n] * h[n]$$
$$= \delta[n] * h[n]$$
$$= h[n]$$
$$= \{1, \underset{\uparrow}{2}, 0, -3\}$$

$h[n]$	1	2	0	-3
1	1	2	0	-3

② $x_2[n] = \delta[n+1] + \delta[n-2]$

$$y[n] = x_2[n] * h[n]$$
$$= [\delta[n+1] + \delta[n-2]] * h[n]$$
$$= \delta[n+1] * h[n] + \delta[n-2] * h[n]$$
$$= h[n+1] + h[n-2]$$
$$= \{1, 2, \underset{\uparrow}{0}, -3\} + \{\underset{\uparrow}{0}, 1, 2, 0, -3\}$$

$$y[n] = \{1, 2, 0, -2, 2, 0, -3\}$$

③ $x_3[n] = \{\underset{\uparrow}{1}, 1, 1\}$

$y_3[n] = \cancel{x}[n] * h[n]$

$= x_3[n] * h[n]$

$x_3[n]$ \ $h[n]$	1	2 ↓	0	-3
→ 1	1	2	0	-3
1	1	2	0	-3
1	1	2	0	-3

$y[n] = \{1, \underset{\uparrow}{3}, 3, -1, -3, -3\}$

or $x_3[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$y[n] = [\delta[n] + \delta[n-1] + \delta[n-2]] * h[n]$

$= h[n] + h[n-1] + h[n-2]$

$= \{1, \underset{\uparrow}{2}, 0, -3\} + \{\underset{\uparrow}{1}, 2, 0, -3\} + \{\underset{\uparrow}{0}, 1, 2, 0, -3\}$

$= \{1, \underset{\uparrow}{3}, 3, -1, -3, -3\}$

④ $x_4[n] = \{2, 1, \underset{\uparrow}{-1}, -2, 3\}$

$y[n] = x[n] * h[n]$

$= x_4[n] * h[n]$

$x[n]$ \ $h[n]$	1	2 ↓	0	-3
2	2	4	0	-6
1	1	2	0	-3
→ -1	-1	-2	0	3
-2	-2	-4	0	6
-3	-3	-6	0	+9

$y[n] = \{2, 5, 1, \underset{\uparrow}{-10}, -10, -9, 6, +9\}$

or

$$x_4[n] = 2\delta[n+2] + \delta[n+1] - \delta[n] - 2\delta[n-1] - 3\delta[n-2]$$

$$y[n] = \left[2\delta[n+2] + \delta[n+1] - \delta[n] - 2\delta[n-1] - 3\delta[n-2]\right] * h[n]$$

$$= 2h[n+2] + h[n+1] - h[n] - 2h[n-1] - 3h[n-2]$$

$$= 2\{1, 2, 0, \underset{\uparrow}{-3}\} + \{1, 2, \underset{\uparrow}{0}, -3\} - \{1, \underset{\uparrow}{2}, 0, -3\} - 2\{\underset{\uparrow}{1}, 2, 0, -3\} - 3\{\underset{\uparrow}{0}, 1, 2, 0, -3\}$$

	-3	-2	-1	0	1	2	3	4	5
$-h[n]$			-1	-2	0	3	0	0	
$h[n+1]$		1	2	0	-3	0	0	0	
$2h[n+2]$	2	4	0	-6	0	0	0	0	
$-2h[n-1]$	0	0	0	-2	-4	0	6	0	
$-3h[n-2]$	0	0	0	0	-3	-6	9	9	
	2	5	1	$\underset{\uparrow}{-10}$	-10	-3	6	9	

* Connections of LTI System

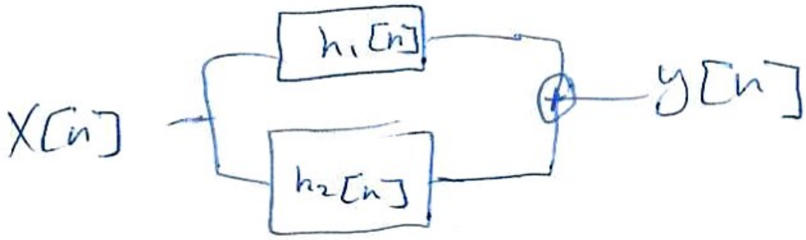
Series (Cascade) connection

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$

$$x[n] * h_1[n] \times h_2[n] = y[n]$$

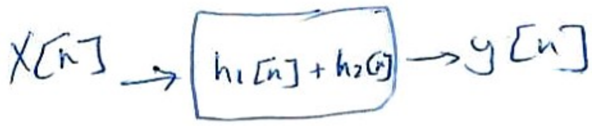
$$x[n] \longrightarrow \boxed{h_1 * h_2} \longrightarrow y[n]$$

Parallel connection

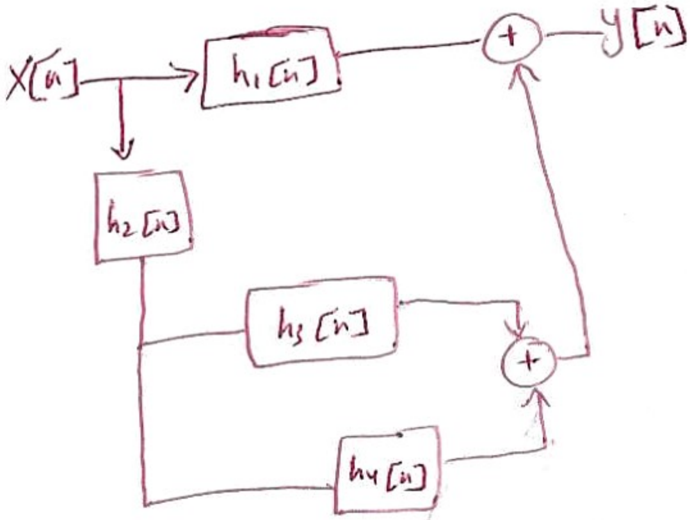


$$y[n] = X[n] * [h_1[n] + h_2[n]]$$

$$= X[n] * h_1[n] + X[n] * h[n]$$



Example: Consider the following system



If $h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$

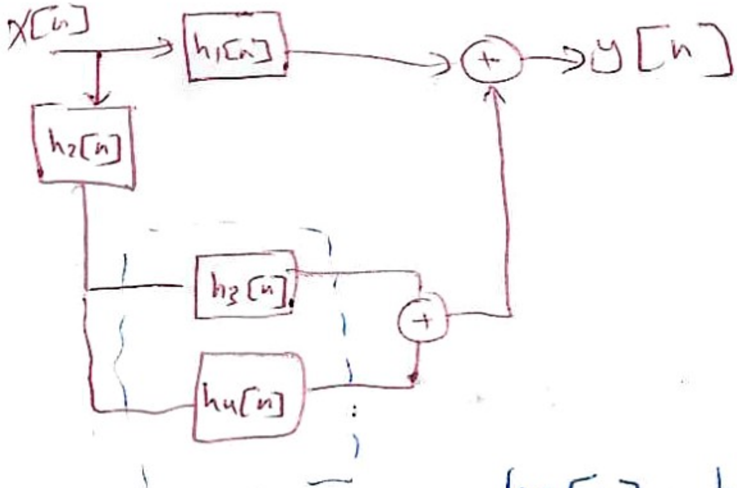
$h_3[n] = 2\delta[n]$

$h_4[n] = -2(\frac{1}{2})^n u[n]$

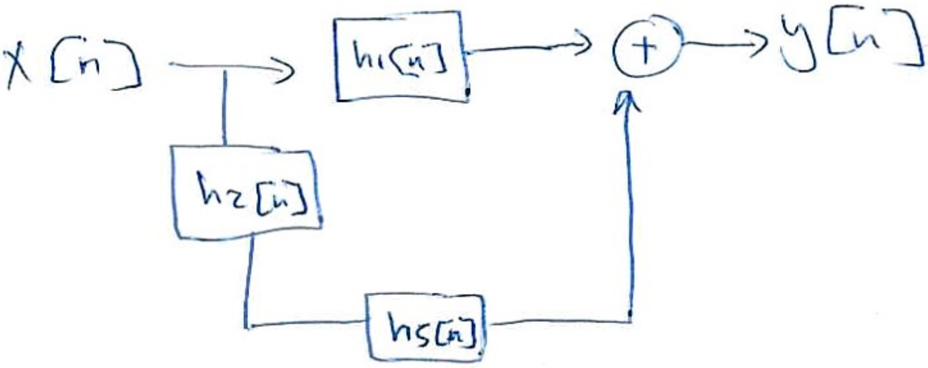
Evaluate the overall impulse response $h[n]$

—

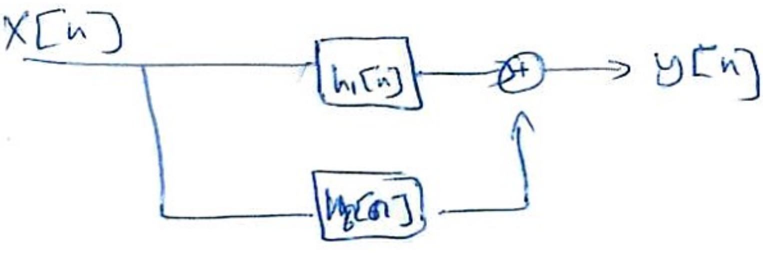
Solution



$$h_5[n] = h_3[n] + h_4[n]$$



$$h_6 = h_2[n] * h_5[n]$$



$$y[n] = \left[h_1[n] + h_6[n]\right] * x[n]$$

$$x[n] \rightarrow \boxed{h_7[n]} \rightarrow y[n]$$

$$h_7[n] = h_1[n] + h_6[n]$$

$$y[n] = \underbrace{\Big[h_1[n] + h_2[n] * \big(h_3[n] + h_4[n]\big)\Big]}_{h[n]} * x[n]$$

$$h[n] = h_1[n] + h_2[n] * \left[h_3[n] + h_4[n]\right]$$

$$= h_1[n] + h_2[n] * h_3[n] + h_2[n] * h_4[n]$$

$$h_2[n] * h_4[n] = \left(\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]\right) * \left(-2\left(\tfrac{1}{2}\right)^n u[n]\right)$$

$$= \left(\tfrac{1}{2}\right)(-2)\left(\tfrac{1}{2}\right)^n u[n] * \delta[n] - \left(\tfrac{1}{4}\right)(-2)\left(\tfrac{1}{2}\right)^n u[n] * \delta[n-1]$$

$$= -1\left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= -1\left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{2}\right)^n u[n-1]$$

$$= -1\left(\tfrac{1}{2}\right)^n \left[u[n] - u[n-1]\right]$$

$$= -\left(\tfrac{1}{2}\right)^n \delta[n] \qquad = -1\left(\tfrac{1}{2}\right)^0 \delta[n] = -\delta[n]$$

$$h_2[n] * h_3[n] = \left[\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]\right] * 2\delta[n]$$

$$= \left(\tfrac{1}{2}\right)(2)\,\delta[n]*\delta[n] - \left(\tfrac{1}{4}\right)(2)\,\delta[n-1]*\delta[n]$$

$$= \delta[n] - \tfrac{1}{2}\delta[n-1]$$

$$h[n] = h_1[n] + [h_2 * h_4] + [h_2 * h_3]$$

$$= \left[\delta[n] + \tfrac{1}{2}\delta[n-1]\right] + \left[-\delta[n]\right] + \left[\delta[n] - \tfrac{1}{2}\delta[n-1]\right]$$

$$= \delta[n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$
$$= x[n] * \delta[n]$$
$$= x[n]$$

$$y[n] = x[n] * \delta[n - n_0]$$ convolution theorem

$$= x[n - n_0]$$

$$y[n] = x[n]\,\delta[n - n_0]$$

$$= x[n_0]\,\delta[n - n_0]$$

* Correlation of Signals :-

measure the Similarity of two Signals

- Cross Correlation : two different processing signals
- Auto Correlation : the signal and shifted version of the same signal

$$r_{x,y}(L) = \sum_{n=-\infty}^{\infty} x[n]\, y[n-L]$$

- Convolution → reflect + shifted
- Correlation → only delay

$L = 0, \pm 1, \pm 2, \ldots$

↳ shift value

$$r_{x,y} \neq r_{y,x}$$

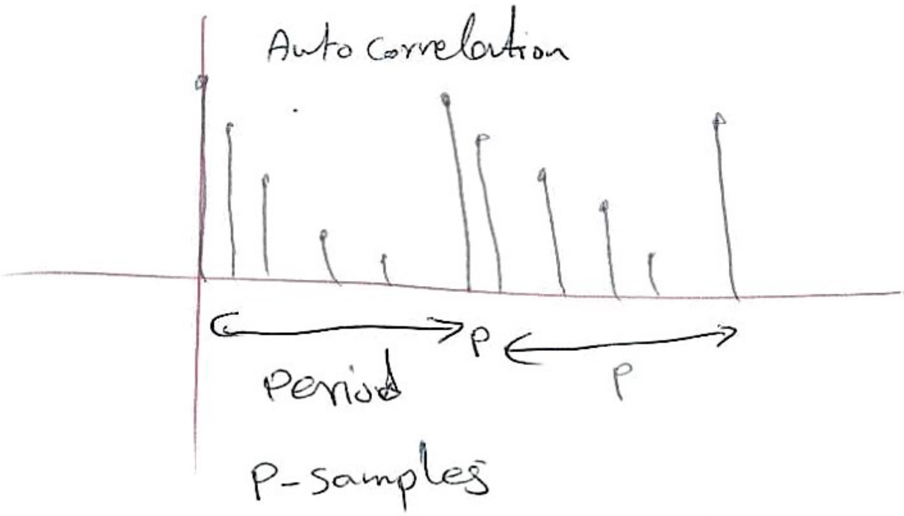
$$r_{y,x}[L] = \sum_{n=-\infty}^{\infty} y[n]\, x[n-L]$$

if $n - L = m$

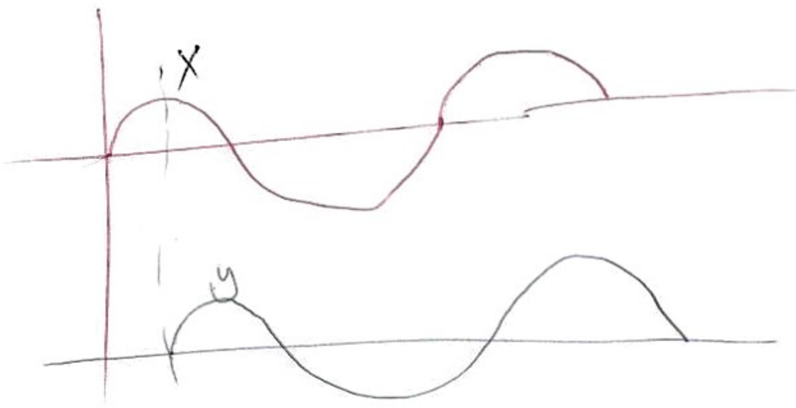
$$= \sum_{n=-\infty}^{\infty} y[m+L]\, x[m] = r_{x,y}[-L]$$

time reversing

Auto-correlation is a technique to find the periodicity of the signal



Cross-correlation : to find the delay between two signals



$$r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2[n] = \text{Energy of the signal}$$

$r_{xx}[L] = r_{xx}[-L]$, r_{xx} is an even function

$$r_{xy}[L] = \sum_{n=-\infty}^{\infty} x[n]\, y[-L-n] = x[L] * y[-L]$$

Convolution

$= conv(X, y)$

$r_{xy}(L) = conv(X, fliplr(y))$

$r_{xy}[L] = Xcorr(X,y)$ Cross-correlation

$r_{xx}[L] = Xcorr(X)$ Auto-correlation

Correlation ~~Coefficient~~ Normalization

C $\rho_{xx}(L) = \frac{r_{xx}[L]}{r_{xx}[0]}$ $|\rho_{xx}[L]| \leq 1$

$\rho_{xy}(L) = \frac{r_{xy}[L]}{\sqrt{r_{xx}[0] \; r_{yy}[0]}}$ $|\rho_{xy}[L]| \leq 1$

Auto correlation

$X[n] = \{-1, 2, 1\}$

$X[-n]$ \ $X[n]$	-1	2	1
+1	-1	2	1
2	-2	4	2
-1	1	-2	-1

		-1	2	1	
-1	2	1			
		-1			

-1

	-1	2	1	
-1	2	1		
	-2	2		

0

	-1	2	1	
	-1	2	1	
	1	4	1	

6

	-1	2	1	
		-1	2	1
		-2	2	

0

	-1	2	1			
			-1	2	1	
			-1			

-1

$-1, 0, \underset{\uparrow}{6}, 0, -1$

$$\rho_{xx}(l) = \frac{\{-1, 0, 6 - 0, -1\}}{\{6\}}$$

$\rho_{xx}[0] = 1$

$\rho_{xx}[-1] = 0$

$\rho_{xx}[L] = \{-0.166, 0, 1, 0, -0.166\}$

Cross Correlation

Example:
Given that $x[n] = \{1, 1, 2, 2\}$, $y[n] = \{1, 2, 3, 4\}$

Find $r_{xy}[L]$, and $\rho_{xy}[L]$

$r_{xy} = \{4, 7, 13, 17, 11, 6, 2\}$

y[-n] \ x[n]	1	1	2	2
4	4	4	8	8
3	3	3	6	6
2	2	2	4	4
1	1	1	2	2

$r_{xx}[0] = 10$

$r_{yy}[0] = 30$

$$\rho_{xy}[L] = \frac{\{4, 7, 13, 17, 11, 6, 2\}}{\sqrt{10 * 30}}$$

$$= \frac{\{4, 7, 13, 17, 11, 6, 2\}}{17.32}$$

Example:

$x[n] = \{1, 3, -2, 4\}$

$y[n] = \{2, 3, -1, 3\}$

$z[n] = \{-2, 4, -1, 2\}$

Find r_{xy}

r_{xz}

ρ_{xy}

ρ_{xz}

$Z[-n]$ \ $X[n]$	1	3	-2	4
2	2	6	-4	8
-1	-1	-3	2	-4
4	4	12	-8	16
-2	-2	-6	4	-8

$y[-n]$ \ $X[n]$	1	3	-2	4
3	3	9	-6	12
-1	-1	-3	+2	-4
3	3	9	-6	12
2	2	6	-4	8

$$r_{xz} = \{2, 5, -3, \underset{\uparrow}{20}, -18, 20, -8\}$$

$$r_{xy}[l] = \{3, 8, -6, \underset{\uparrow}{25}, -4, 8, 8\}$$

$$r_{xx}[0] = 30$$

$$r_{zz}[0] = 25$$

$$r_{xx}[0] = 30$$

$$r_{yy}[0] = 23$$

$$\rho_{xz}[l] = \frac{\{2, 5, -3, 20, -18, 20, -8\}}{\{\sqrt{30 \times 25}\}} = \frac{\{2, 5, -3, 20, -18, 20, -8\}}{\{27.39\}}$$

$$\rho_{xy}[l] = \frac{\{3, 8, -6, 25, -4, 8, 8\}}{\{\sqrt{30 \times 23}\}} = \frac{\{3, 8, -6, 25, -4, 8, 8\}}{\{26.27\}}$$

we can conclude that y is more similar to X than Z