exercises of chapter 16.

Q3: show that x^2+3x+2 has four zeros in $\mathbb{Z}_6$.

$\rightarrow$ $x^2+3x+2 = (x+2)(x+1) = 0$

$\Rightarrow$ $x = -2 \mod 6 = 4$

$x = -1 \mod 6 = 5$

But $\mathbb{Z}_6$ Not integral domain and so there are other sol. to $ab=0$

i.e, $4.3 = 0$ in $\mathbb{Z}_6$

$\Rightarrow$ $(x+2)(x+1) = 4.3$

$\Rightarrow$ $x = 2$ is solution.

also $3.2 = 0$ in $\mathbb{Z}_6$

$\Rightarrow$ $(x+2)(x+1) = 3.2$

$\Rightarrow$ $x=1$ is solution.

So x^2+3x+2 has 4 root $\{1,2,4,5\}$.

Q4: if R is commutative Ring, show that the characteristic of $R[x]$ is the same as the characteristic of R.

let R be a commutative Ring with char. K Then $Kr=0$ $\forall r\in R$.

let $f(x)\in R[x]$. then $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for some $a_i \in R$, $n\in \mathbb{Z}^+$

then $K f(x) = (K a_n) x^n + (K a_{n-1}) x^{n-1} + \cdots + (K a_1) x + K a_0 = 0+0+\cdots+0=0$

Hence the char. of $R[x]$ is at most K.

However, since for all $r\in R$, $r\in R[x]$, the char. of $R[x]$ must be at least K. Thus, the char. is exactly K.

Q5: on Note

Q10: let R be a commutative Ring, show that $R[x]$ has a subring isomorphic to R.

Proof: let R be a commutative Ring and consider $R[x]$.

Define $\phi : R \to R[x]$ By $r \to r$

clearly ϕ is 1-1 and homomorphism

Now $\phi(R)$ is a subring of $R[x]$ since it is the image of a hom.

then $\phi(R)$ is a subring of $R[x]$ isomorphic to R.

Q11:

Q18: let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$. determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

$$\begin{array}{r|l} & 4x^2 + 3x + 6 \\ \hline 3x^2+2x+1 & 5x^4 + 3x^3 + 1 \\ & -(5x^4 + x^3 + 4x^2) \\ \hline & 2x^3 - 4x^2 + 1 \\ & -(2x^3 + 6x^2 + 3x) \\ \hline & 4x^2 - 3x + 1 \\ & -(4x^2 + 5x + 6) \\ \hline & 6x - 5 \quad \text{mod } 7 \\ & = 6x + 2 \end{array}$$

$\rightarrow$ $5 \div 3 = 5 . 3^{-1} = 5.5 = 25 \mod 7 = 4$

3^{-1} on $\mathbb{Z}_7$: $3 . a \equiv 1 \pmod 7$.

$a = 5$.

$\rightarrow$ $2 \div 3 = 2 . 3^{-1} = 2.5 = 10$

3^{-1} on $\mathbb{Z}_7 = 5$

$\rightarrow$ $4 \div 3 = 4 . 3^{-1} = 4.5 = 20$

$$\Longrightarrow \quad 5x^4 + 3x^3 + 1 = (3x^2 + 2x + 1)(4x^2 + 3x + 6) + (6x + 2).$$

Q19: let D be an integral domain and $f(x), g(x) \in D[x]$. prove that $\deg(f(x) . g(x)) = \deg f(x) + \deg g(x)$, show by example, that for commutative ring R it is possible that $\deg f(x)g(x) < \deg f(x) + \deg g(x)$, where $f(x), g(x)$ nonzero elements in $R[x]$.

let D be an integral domain and $f(x), g(x) \in D[x]$.

suppose that $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ so that $\deg f(x) = n$, $\deg g(x) = m$

we know $f(x) g(x) = \sum_{i=0}^{n+m} c_{n+m} x^{n+m}$ where $c_{n+m} = a_0 b^{n+m} + a_1 b^{n+m-1} + \cdots + a_{n+m-1} b^1 + a_{n+m} b^0$.

since a_i and b_i are in integral domain, $a_i b_i \neq 0$ when $a_i \neq 0$, $b_i \neq 0$

in particular, we know that a_n and b_m are nonzero so $a_n b_m \neq 0$

Now, all other terms in the sum of c_{n+m} are zero because either a_i has $i > n$ or b_j has $j > m$.

Thus, $c_{n+m} = a_n b_m$

Thus, c_{n+m} is not zero and the $\deg(f(x) g(x)) = n+m$

Q2: Prove that the ideal $<x>$ in $Q[x]$ is maximal.

$\phi[x] / <x>$ → this quotient Ring contains cosets that look like $a + <x>$ where $a \in Q$.

Thus, using the map $Q[x] / <x> \to Q$ defined by $a + <x> = a$ is an isomorphisim.

Thus, $Q[x] / <x> \cong Q$.

Now, Q is field so $<x>$ is maximal.