→ Chapter ② : Dynamic Models & Analogies

→ Important relations used in models :

TABLE 2.1
Some Linear Constitutive Relations

System Type	Energy Storage Elements: A-type (Across) Element	Energy Storage Elements: T-type (Through) Element	Energy Dissipating Elements: D-type (Dissipative) Element
Translatory-Mechanical v = velocity f = force	*Mass* $m\frac{dv}{dt} = f$ (Newton's second law) m = mass	*Spring* $\frac{df}{dt} = kv$ (Hooke's law) k = stiffness	*Viscous Damper* $f = bv$ b = damping constant
Electrical v = voltage i = current	*Capacitor* $C\frac{dv}{dt} = i$ C = capacitance	*Inductor* $L\frac{di}{dt} = v$ L = inductance	*Resistor* $Ri = v$ R = resistance
Thermal T = temperature difference Q = heat transfer rate	*Thermal Capacitor* $C_t\frac{dT}{dt} = Q$ C_t = thermal capacitance	*None*	*Thermal Resistor* $R_tQ = T$ R_t = thermal resistance
Fluid P = pressure difference Q = volume flow rate	*Fluid Capacitor* $C_f\frac{dP}{dt} = Q$ C_f = fluid capacitance	*Fluid Inertor* $I_f\frac{dQ}{dt} = P$ I_f = inertance	*Fluid Resistor* $R_fQ = P$ R_f = fluid resistance

→ Force / Current analogy :

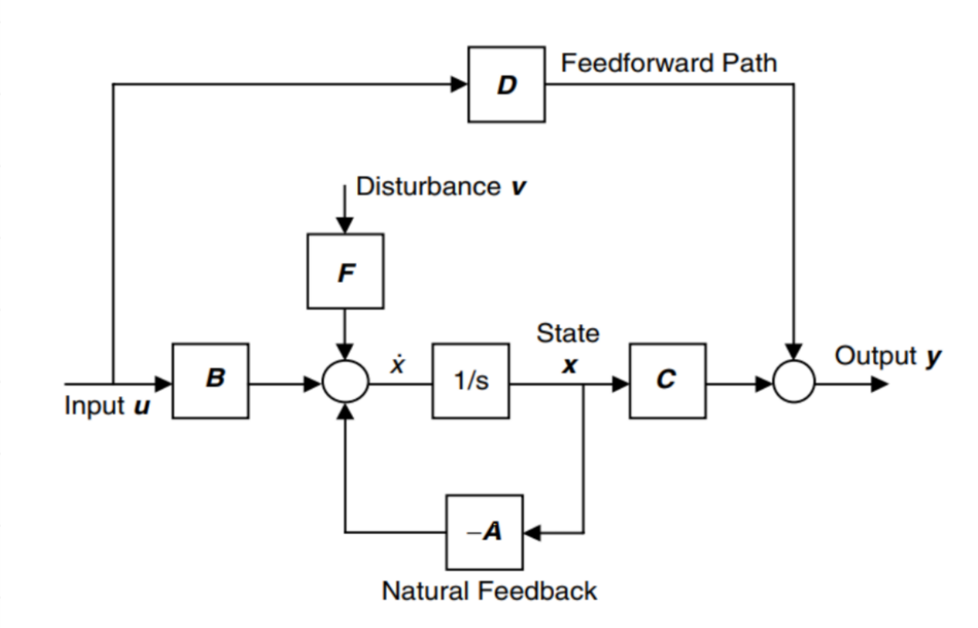
Force (f) → Current (i)
Velocity (V) → Voltage (V)
System parameters : m → C
K → $1/L$
b → $1/R$

→ State space representation :

$$\underset{n\times 1}{\dot{x}} = \underset{n\times n}{A}x + \underset{n\times r}{B}u + Fv$$
$$\underset{p\times 1}{y} = \underset{p\times n}{C}x + \underset{p\times r}{D}u$$

n = # of states
r = # of inputs
p = # of outputs

→ Linear Model for SSR:



→ Electrical systems modelling:

$$\boxed{* \, i_c(t) = C \frac{dV_c}{dt} = C \dot{V}_c}$$

$$\boxed{* \, V_L = L \frac{di}{dt} = L \dot{I}}$$

* In electric circuit modelling:
\# of states = # of C & L in the circuit
- V_c is a state
- I_L is a state (or difference in currents Ex: I_{12})

→ Nonlinear state space model:

* Let the scalar system $\dot{x} = f(x)$, $x \in R$
* Define $x = x_o + \Delta x \xrightarrow{d/dt} \dot{x} = \dot{x}_o + \Delta\dot{x}$
where x_o = operating point

$\dot{x} = f(x, u)$
operating point (x_o, u_o)
$x = x_o + \Delta x$
$u = u_o + \Delta u$

$$A' = \begin{bmatrix} df_1/dx_1 & \cdots & df_1/dx_n \\ \vdots & \ddots & \\ df_n/dx_1 & \cdots & df_n/dx_n \end{bmatrix}_{n\times n} \Bigg|_{(x_0,u_0)}$$

$$B' = \begin{bmatrix} df_1/du_1 & \cdots & df_1/du_r \\ \vdots & \ddots & \\ df_n/du_n & \cdots & df_n/du_r \end{bmatrix}_{n\times r} \Bigg|_{(x_0,u_0)}$$

$$C' = \begin{bmatrix} dy_1/dx_1 & \cdots & dy_1/dx_n \\ \vdots & \ddots & \\ dy_p/dx_1 & \cdots & dy_p/dx_n \end{bmatrix}_{p\times n} \Bigg|_{(x_0,u_0)}$$

$$D' = \begin{bmatrix} dy_1/du_1 & \cdots & dy_1/du_r \\ \vdots & \ddots & \\ dy_p/du_1 & \cdots & dy_p/du_r \end{bmatrix}_{p\times r} \Bigg|_{(x_0,u_0)}$$

→ **Convert the (SSR) to Transfer Matrix:**

$$\boxed{G(s) = C(sI - A)^{-1} \cdot B + D}$$

→ **Lagrange's Equations:**

- Indirect approach that can be applied for other types of systems.
- Used when forces at interconnections are not important.
- Considers the energies in the system.

– The equation is:

$$\boxed{\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_i}\right) - \frac{\delta T}{\delta q_i} + \frac{\delta R}{\delta \dot{q}_i} + \frac{\delta U}{\delta q_i} = Q_i}$$

Outputs of the system $(x, \theta \dots)$

<u>where</u>: q_i = independent coordinates necessary to describe the system's motion.

Q_i = corresponding loading in each coordinate.

$\frac{1}{2}Kx^2 = U = f_1(q_i)$ = potential energy

$\frac{1}{2}mv^2 \; (\frac{1}{2}J\dot{\theta}^2) = T = f_2(\dot{q}_i^2)$ = Kinetic energy in term of system masses, inertias, linear/angular velocities.

$\frac{1}{2}b\dot{x}^2 = R = f_3(\dot{q}_i^2)$ = energy dissipation due to viscous friction.

– <u>Deriving EOM via Lagrange</u>:

1. Select an independent system of coordinates. (q_i)
2. Identify loading Q_i.
3. Derive T, U, R.
4. Substitute results in Lagrange.

→ <u>Translational & Rotational Motion</u>:

Table 2.1: Comparison of Translational and Rotational Mechanics

Translational (linear)		Rotational (in a plane)	
Quantity	**Unit**	**Quantity**	**Unit**
Displacement (s)	m	Rotation (θ)	rad
Velocity (v)	[m/s]	Rotational Speed (ω)	[rad/s]
Acceleration (a)	[m/s²]	Rotational Acceleration (α)	[rad/s²]
Jerk (j)	[m/s³]	-	-
Jounce (also called snap)	[m/s⁴]	-	-
Force (F)	[N]	Torque (T)	[Nm]
Mass (m)	[kg]	Mass Moment of Inertia (I)	[kg.m²]
Kinetic Energy KE = $\frac{1}{2}mv^2$	[J]	Kinetic Energy KE = $\frac{1}{2}I\omega^2$	[J]
F = ma		T = Iα	

$$\sum F = ma \iff \sum T = I\alpha$$

→ **Mechanisms for motion transmission:**

1. Rotary to rotary motion. (gears, belts, pulleys).
2. Rotary to translational motion (lead screw, rack-pinion)
3. Cyclic motion transmission (cams).

①: R → R

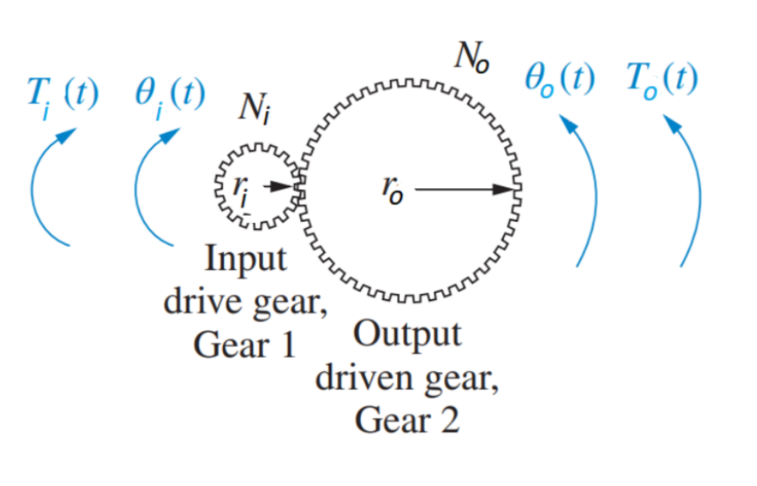
* **Gears:**

- Most gearboxes are used as reduction gearboxes, thus speed ↓ & torque ↑
- Let r_g = gearbox reduction ratio.
  Let (HSS) be the drive shaft side.
  Let (LSS) be the load side shaft
- When there is no power loss ($\eta = 100\%$):

$$\boxed{P_i = P_o} \quad , \quad \boxed{T_i \omega_i = T_o \omega_o}$$

$$\boxed{r_g = \frac{\omega_i}{\omega_o} = \frac{\omega_{HSS}}{\omega_{LSS}}}$$

$$\boxed{\frac{r_o}{r_i} = \frac{N_o}{N_i} = \frac{T_o}{T_i} = \frac{\omega_i}{\omega_o}}$$



thus: $\boxed{T_o = r_g T_i}$

- Note that the efficiency will not be 100%, thus denote the forward efficiency as (η_f)

- Rewriting the previous equations :

$$\boxed{\begin{aligned} P_o &= \eta_f P_i \qquad (2.31) \\ T_o\omega_o &= \eta_f T_i \omega_i \qquad (2.32) \end{aligned}}$$

$$\boxed{\begin{aligned} r_g &= \frac{\omega_i}{\omega_o} = \frac{\omega_{HSS}}{\omega_{LSS}} \qquad (2.33) \\ r_g &= \frac{\omega_i}{\omega_o} = \frac{T_o}{\eta_f T_i} \qquad (2.34) \end{aligned}}$$

$$\boxed{\begin{aligned} T_o &= \eta_f r_g T_i \qquad (2.35) \\ T_{LSS} &= \eta_f r_g T_{HSS} \qquad (2.36) \end{aligned}}$$

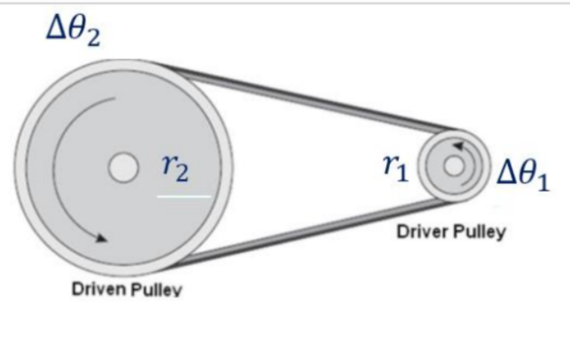
- Spur, helical, bevel gears have $\eta = 98 - 99\%$
- Worm has $\eta = 20 - 98\%$
- Crossed helical has $\eta = 70 - 98\%$

- The square of gearing ratio (r_g^2) is used when we want to find the equivalent inertia at the (HSS) due to the fact that the Kinetic energy must be the same.

$$\boxed{I_{HSS} = \frac{I_{LSS}}{r_g^2}} \quad [2.40]$$

+ Pulleys :

- Assuming no slipping between the belt and pulleys, the linear displacement along the belt and both pulleys = r_i



$$\boxed{X = \Delta\theta_1 r_1 = \Delta\theta_2 r_2} \quad [2.41]$$

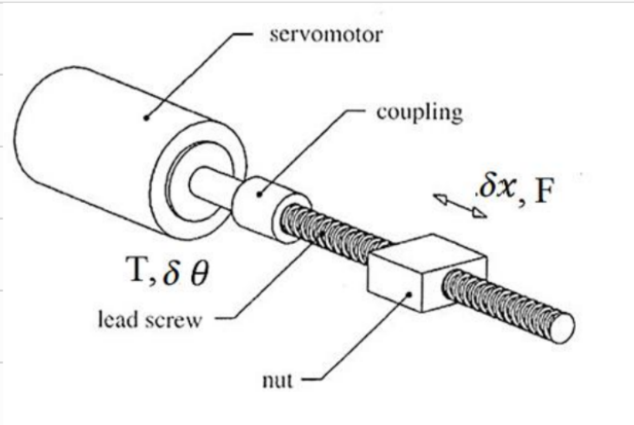
$$\boxed{r_g = \frac{\Delta\theta_1}{\Delta\theta_2} = \frac{r_2}{r_1} = \frac{d_2}{d_1} = \frac{N_2}{N_1}} \quad [2.42]$$

- Inertia & torque reflection between (HSS) and (LSS) has the same relationship as the gear mechanism.

Q: R → T

+ Lead screw mechanism: (Power screw)

- Suppose that the screw rotation by (θ) and the nut moves by (X).
- The lead of the screw is:



$$\boxed{l = 2\pi r} \quad [3.1]$$

where: r = axial distance moved per one rad. of screw rotation.

$$\boxed{l = np} \quad [3.2]$$

where: p = the pitch
n = number of teeth. [single, double, triple]

- The net force available from the nut to drive the load is:

$$\boxed{F = \frac{e}{r} T_R = \frac{2\pi e}{l} T_R} \quad [3.3]$$

where: T_R = the net torque.
e = lead screw efficiency.

→ Self locking condition:

where: $\boxed{d_m = d - \frac{p}{2}}$

$$\boxed{\pi \mu d_m > l}$$

→ Useful tables for power screw design:

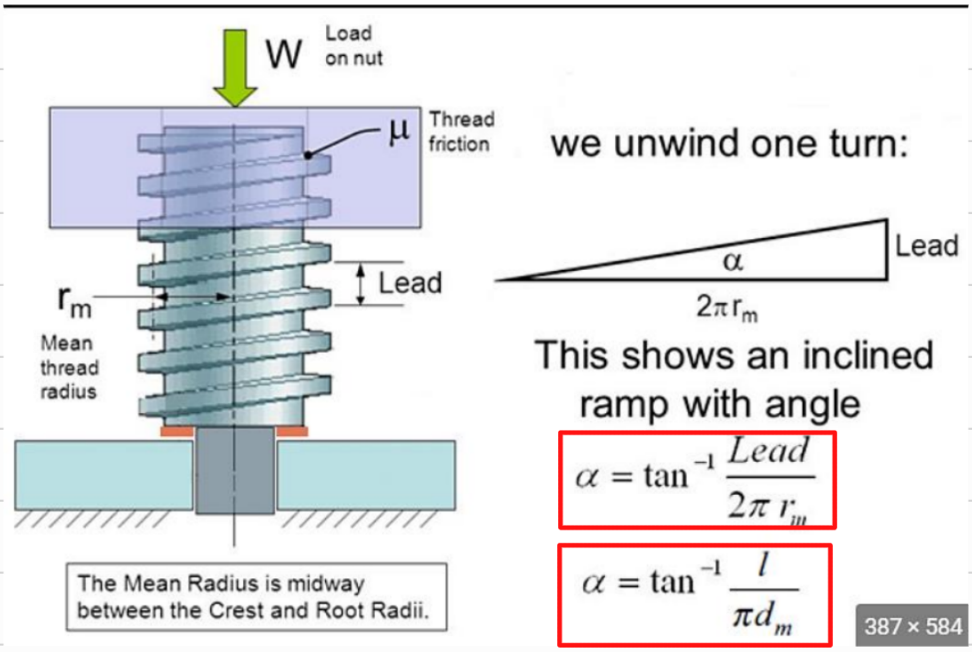
Table 8–5

Coefficients of Friction f for Threaded Pairs

Source: H. A. Rothbart and T. H. Brown, Jr., *Mechanical Design Handbook,* 2nd ed., McGraw-Hill, New York, 2006.

Screw Material	Nut Material: Steel	Bronze	Brass	Cast Iron
Steel, dry	0.15–0.25	0.15–0.23	0.15–0.19	0.15–0.25
Steel, machine oil	0.11–0.17	0.10–0.16	0.10–0.15	0.11–0.17
Bronze	0.08–0.12	0.04–0.06	—	0.06–0.09

d, in	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{2}$	3
p, in	$\frac{1}{16}$	$\frac{1}{14}$	$\frac{1}{12}$	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$



- The helix angle of the screw is:

$$\boxed{\tan\alpha = \frac{l}{\pi d_m}}$$

where: d_m = mean diameter of the screw

* **Assuming square threads:**

- The frictional torque is:

$$\boxed{T_f = \frac{\mu d}{2r} T_R}$$ [3.5]

where: μ = Coeff. of friction

- The screw efficiency is:

$$\boxed{e = \frac{T_R - T_f}{T_R} = 1 - \frac{\mu d}{2r} = 1 - \frac{\mu}{\tan\alpha}}$$ [3.6]

- The rotational displacement $\Delta\theta$ [rad] and the linear displacement ΔX are related by:

$$\boxed{\Delta\theta = \frac{2\pi\,\Delta x}{n p}} \quad , \quad \boxed{\Delta x = \frac{n p}{2\pi}\,\Delta\theta}$$

- The effective gear ratio is :

$$\boxed{r_g = \frac{n p}{2\pi}}$$

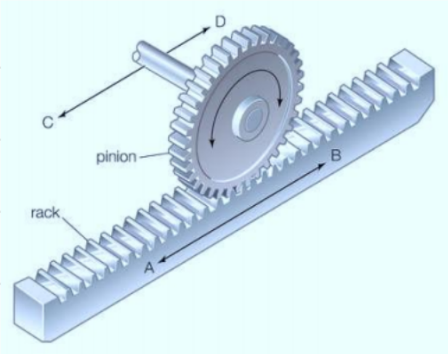
+ **Rack & Pinion mechanism :**

- Since there is no slip :

$$\Delta x = r\Delta\theta \tag{2.55}$$

$$\Delta \dot{x} = r\Delta\dot{\theta} \tag{2.56}$$

$$v = r\omega \tag{2.57}$$



- The effective gear ratio is :

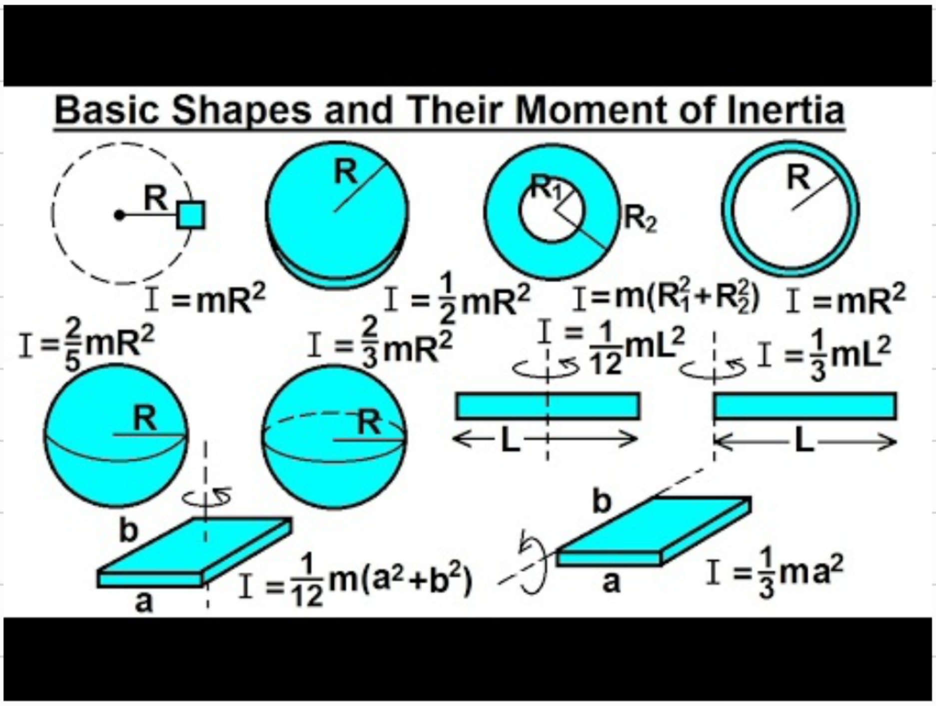
$$\boxed{N = \frac{1}{r}}$$ [2.58] used when $\Delta\theta$ [rad] or ω [rad/s]

$$\boxed{N = \frac{1}{2\pi r}}$$ [2.59] used when $\Delta\theta$ [rev] or ω [rev/s]

- Mass & force relations for lead screws apply for rack and pinions, in eqs : [3.3]

+ **Pulley & belt :**
- Rack pinion relations apply here (when used for R→T motion).

→ (I) values for different shapes:



* For any object : $\boxed{I = mr^2}$

* For any system : (With respect to HSS) → Output is located on HSS

$$\boxed{I_{sys} = \sum I_{HSS} + \frac{\sum I_{LSS}}{r_g^2}}$$

$$r_g = \frac{r_2}{r_1} = \frac{r_{LSS}}{r_{HSS}} = \frac{\omega_1}{\omega_2} = \frac{\omega_{HSS}}{\omega_{LSS}} = \frac{N_{LSS}}{N_{HSS}}$$

$$C_{eq} = \frac{\sum C_{LSS}}{r_g^2}$$

* For any system with respect to LSS : → Output is located on LSS

$$\boxed{I_{sys} = r_g^2 \sum I_{HSS} + \sum I_{LSS}}$$

$$C_{eq} = r_g^2 \sum C_{HSS}$$

Remember: $T_{LSS} = \eta_f \, r_g \, T_{HSS}$

- To solve any problem in this Chapter:

① Specify inputs and Outputs. (Outputs are normally things that we want to measure / record in a system)

② Simplify the system by finding the equivalent inertia / damping with respect to where the output is located.

③ Draw the equivalent system and specify the equivalent Forces / Torques.

④ Use $\Sigma T = J_{eq} \ddot{\theta}$

or $\Sigma F = m_{eq} \ddot{x}$

⑤ Find the transfer function/s of the system.

For a system that Contains a damper:

* $C_{HSS} = \frac{C_{LSS}}{rg^2}$

⇒ Chapter ③ : Motion Planning

- Path : a sequence of configurations in a particular order regardless of the timing.

- Trajectory : Conserned about when each part of the path must be attained.

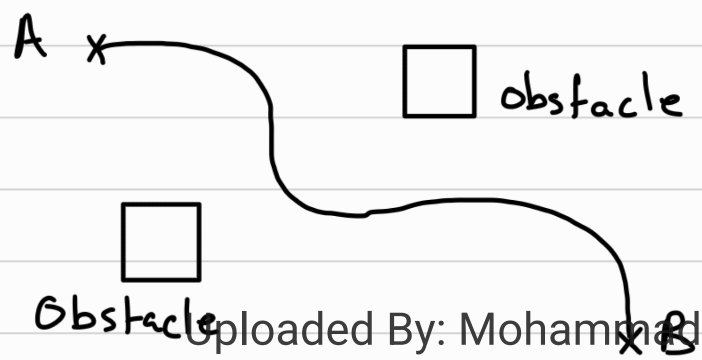
- Trajectory planning contains : ① Path profile.
  ② Velocity profile.
  ③ Acceleration profile.

- Requirements of trajectory planning :

* Initial position : position (given)
  velocity (given, normally zero)
  acceleration (given, normally zero)

* Final position : position (given)
  velocity (given, normally zero)
  acceleration (given, normally zero)

→ It is required to find a trajectory that connects the initial and final points while satisfying the constraints at the end points.



– Trajectories for point to point motion:

→ We need to determine $q(t)$ which is a scalar variable

→ Suppose that at time (t_0), the joint variable satisfies:
$q(t_0) = q_0$, $\dot{q}(t_0) = V_0$

and we want to attain the values at time (t_f):
$q(t_f) = q_f$, $\dot{q}(t_f) = V_f$

+ Acceleration equations are: $\ddot{q}(t_0) = \alpha_0$
$\ddot{q}(t_f) = \alpha_f$

– Cubic polynomial trajectories:

→ Its form: $\boxed{q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3}$

→ The desired velocity is: $\boxed{\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2}$

→ Substitute t_0, t_f in each equation to get q_0, V_0, q_f, V_f

+ This method gives discontinuities in the acceleration, which leads to an impulsive jerk. (this causes vibrations in the system).

- The four equations can be combined into a single matrix:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix}$$

- Quintic Polynomial Trajectories:

$$\boxed{q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5}$$

$$\begin{aligned} q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 + a_5 t_0^5 \\ v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 + 4a_4 t_0^3 + 5a_5 t_0^4 \\ \alpha_0 &= 2a_2 + 6a_3 t_0 + 12a_4 t_0^2 + 20a_5 t_0^3 \\ q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5 \\ v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4 \\ \alpha_f &= 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3 \end{aligned}$$

which can be written as

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ \alpha_0 \\ q_f \\ v_f \\ \alpha_f \end{bmatrix}$$

- PTP motion in the joint space:

→ It is the basic trajectory given by a linear motion from a point to another point in the joint space.

→ The manipulator must move from $q(t_0) = q_0$ to $q(t_f) = q_f$ while satisfying the max. velocity and acceleration constraints,

→ The PTP in the joint space is expressed as a combination of q_0 & q_f :

$$\boxed{q(t) = q_0 + s(t)\underbrace{\Delta q}_{q_f - q_i}}$$

Where : $0 = S(t_0) \leq S(t) \leq S(t_f) = 1$

- Planning of $S(t)$:

→ The following constraints must be presented :

$$|\dot{S}(t)| \leq \dot{S}_{max} \quad , \quad \dot{S}_{max} > 0$$

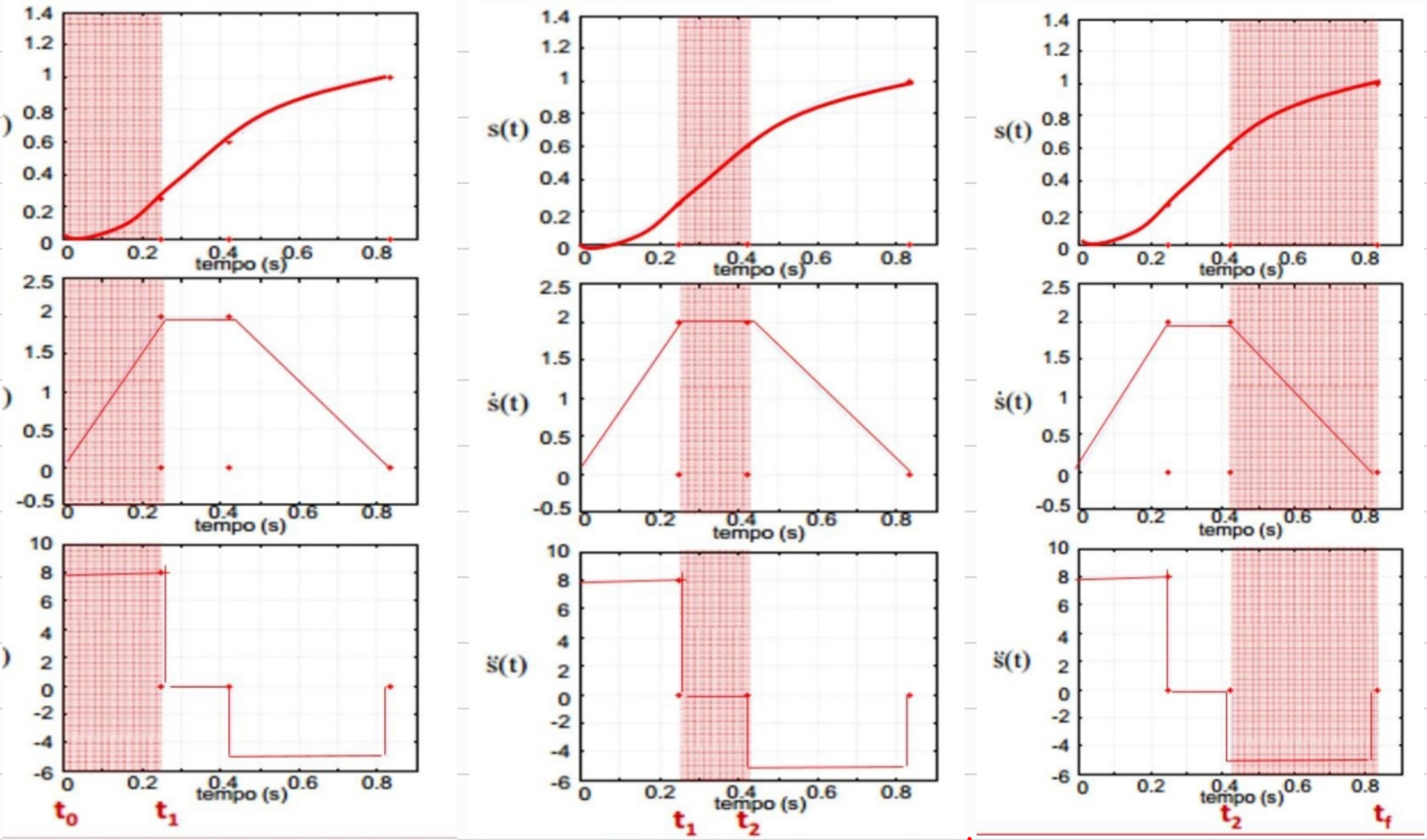
$$-\ddot{S}_{max} \leq \ddot{S}(t) \leq \ddot{S}^{+}_{max} \quad , \quad \ddot{S}^{-}_{max} , \ddot{S}^{+}_{max} > 0$$

- The complete equations of a PTP motion in the joint space from $\mathbf{q}_0$ a $\mathbf{q}_f$ are expressed as functions of the planned variable s(t) as:

$$\begin{aligned}
\mathbf{q}(t) &= \mathbf{q}_0 + s(t)(\mathbf{q}_f - \mathbf{q}_0) = \mathbf{q}_0 + s(t)\Delta\mathbf{q} \\
\dot{\mathbf{q}}(t) &= \dot{s}(t)(\mathbf{q}_f - \mathbf{q}_0) = \dot{s}(t)\Delta\mathbf{q} \\
\ddot{\mathbf{q}}(t) &= \ddot{s}(t)(\mathbf{q}_f - \mathbf{q}_0) = \ddot{s}(t)\Delta\mathbf{q}
\end{aligned}$$

– 2-1-2 Profile :

→ The min. time trajectory satisfying the imposed constraints (trapezoidal velocity profile).



* The max. acceleration allowed is applied until max velocity is reached
* $S(t)$ is 2nd order polynomial

* Velocity is kept equal to the max. value allowed, hence $a=0$.
* $S(t)$ is 1st order polynomial.

* max. deceleration allowed is applied until the final value $S(t_f)=1$ is reached with zero velocity.
* $S(t)$ is 2nd order polynomial function.

- Equations of 2-1-2 profile:

→ Position:

$$S(t)=\begin{cases}\frac{1}{2}\ddot{S}^{+}_{max}(t-t_0)^2+\dot{S}_0(t-t_0)+S_0 \quad ① & t\in[t_0,t_1]\\ \dot{S}_{max}(t-t_1)+S_1 \quad ② & t\in[t_1,t_2]\\ -\frac{1}{2}\ddot{S}^{-}_{max}(t-t_2)^2+\dot{S}_{max}(t-t_2)+S_2 \quad ③ & t\in[t_2,t_f]\end{cases}$$

- Velocity

$$\dot{S}(t)=\begin{cases}\ddot{S}^{+}_{max}(t-t_0)+\dot{S}_0 & t\in[t_0,t_1]\\ \dot{S}_{max} & t\in[t_1,t_2]\\ \dot{S}_{max}-\ddot{S}^{-}_{max}(t-t_2) & t\in[t_2,t_f]\end{cases}$$

- Acceleration

$$\ddot{S}(t)=\begin{cases}\ddot{S}^{+}_{max} & t\in[t_0,t_1]\\ 0 & t\in[t_1,t_2]\\ \ddot{S}^{-}_{max} & t\in[t_2,t_f]\end{cases}$$

* Initial Conditions:

$$(+)\ \boxed{\begin{aligned} S_0 &= S(t_0)=0\\ \dot{S}_0 &= \dot{S}(t_0)=0\end{aligned}}\ (+)$$

→ Note that there is continuity at switching time instants such that ① = ② at $[t_1]$ and ② = ③ at $[t_2]$

+ Final Conditions :

$$\boxed{\begin{aligned} s_f &= s(t_f) = 1 \\ \dot{s}_f &= \dot{s}(t_f) = 0 \end{aligned}}$$

→ At the first interval : $\boxed{t_1 - t_0 = \dfrac{\dot{s}_{max}}{\ddot{s}^{+}_{max}}}$

→ At the final interval : $\boxed{t_f - t_2 = \dfrac{\dot{s}_{max}}{\ddot{s}^{-}_{max}}}$

→ By the imposition of the final constraints :

$$\boxed{t_f - t_0 = (t_f - t_1) + (t_1 - t_0) = \frac{1}{\dot{s}_{max}} + \frac{1}{2}\left(\frac{\dot{s}_{max}}{\ddot{s}^{-}_{max}} + \frac{\dot{s}_{max}}{\ddot{s}^{+}_{max}}\right)}$$

$$\boxed{t_2 - t_1 = \frac{1}{\dot{s}_{max}} - \frac{1}{2}\left(\frac{\dot{s}_{max}}{\ddot{s}^{-}_{max}} + \frac{\dot{s}_{max}}{\ddot{s}^{+}_{max}}\right)}$$

− Bang Bang Profile :

- The time interval at constant velocity actually exists only if

$$t_2 - t_1 > 0 \Rightarrow \frac{1}{\dot{s}_{max}} > \frac{1}{2}\left(\frac{\dot{s}_{max}}{\ddot{s}^{-}_{max}} + \frac{\dot{s}_{max}}{\ddot{s}^{+}_{max}}\right)$$

- On the contrary if

$$\frac{1}{\dot{s}_{max}} \leq \frac{1}{2}\left(\frac{\dot{s}_{max}}{\ddot{s}^{-}_{max}} + \frac{\dot{s}_{max}}{\ddot{s}^{+}_{max}}\right)$$

the profile is divided in two intervals only:

- for $t \in \mathcal{J}' = [t_0, t^*]$ the maximum acceleration $\ddot{s}^{+}_{max}$ is applied
- for $t \in \mathcal{J}'' = [t^*, t_f]$ the maximum deceleration $\ddot{s}^{-}_{max}$ is applied

⇒ The profile is "bang-bang" in the acceleration, corresponding to a triangular velocity profile

→ Equations of Bang Bang Profile :

- **Acceleration:** $\ddot{s}(t) = \begin{cases} \ddot{s}^{+}_{max} & t \in \mathcal{J}' \\ -\ddot{s}^{-}_{max} & t \in \mathcal{J}'' \end{cases}$

- **Velocity:** $\dot{s}(t) = \begin{cases} \ddot{s}^{+}_{max}(t - t_0) & t \in \mathcal{J}' \\ \dot{s}^{*} - \ddot{s}^{-}_{max}(t - t^{*}) & t \in \mathcal{J}'' \end{cases}$

- **Position:** $s(t) = \begin{cases} \frac{1}{2}\ddot{s}^{+}_{max}(t - t_0)^2 & t \in \mathcal{J}' \\ -\frac{1}{2}\ddot{s}^{-}_{max}(t - t^{*})^2 + \dot{s}^{*}(t - t^{*}) + s^{*} & t \in \mathcal{J}'' \end{cases}$

N.B.: The initial conditions have been already imposed:

$\boxed{s_0 = s(t_0) = 0}$ $\boxed{\dot{s}_0 = \dot{s}(t_0) = 0}$

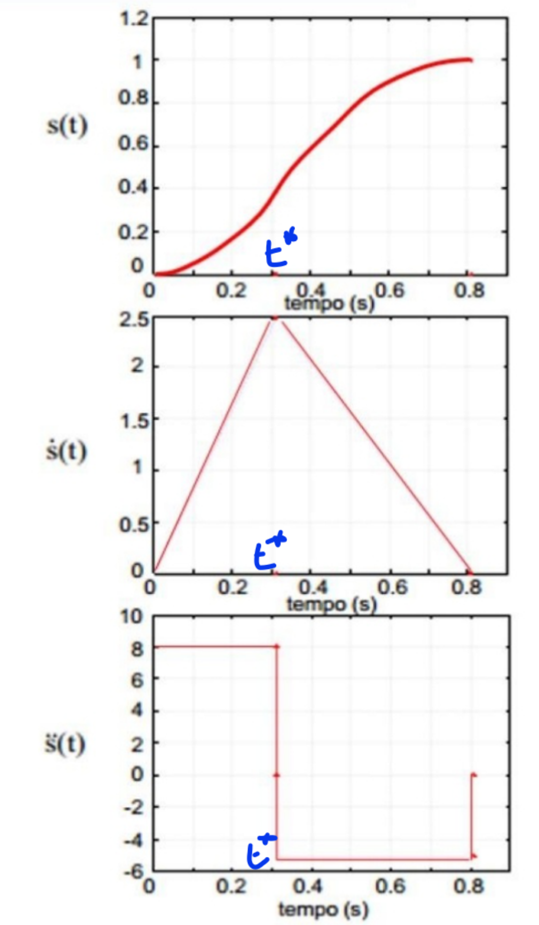
+ At initial conditions :

$$\boxed{\begin{aligned} S^{*} &= S(t^{*}) = \frac{1}{2}\ddot{S}^{+}_{max}(t^{*} - t_0)^2 \\ \dot{S}^{*} &= \dot{S}(t^{*}) = \ddot{S}^{+}_{max}(t^{*} - t_0) \end{aligned}}$$

$$S^{*} = \frac{1}{2}\frac{\dot{S}^{*2}}{\ddot{S}^{+}_{max}}$$

+ At final Conditions :

(+) $$\boxed{\begin{aligned} S_f &= S(t_f) = 1 \\ \dot{S}_f &= \dot{S}(t_f) = 0 \end{aligned}}$$ (+)



+ At the first interval: (+) $$\boxed{\underbrace{t^{*} - t_0}_{(*)} = \frac{\dot{S}^{*}}{\ddot{S}^{+}_{max}}}$$ (+)

* At the final interval: $\boxed{t_f - t^* = \frac{\dot{S}^*}{\ddot{S}^+_{max}}}$

→ The max. velocity is: $\boxed{\dot{S}^* = \sqrt{\frac{2\,\ddot{S}^+_{max}\,\ddot{S}^-_{max}}{(\ddot{S}^+_{max} - \ddot{S}^-_{max})}}}$

→ The duration of the complete motion is:

$$\boxed{(t_f - t_0) = \sqrt{\frac{2(\ddot{S}^+_{max} + \ddot{S}^-_{max})}{\ddot{S}^+_{max}\,\ddot{S}^-_{max}}}}$$

Motion planning with a constant acceleration

$$\boxed{v_f - v_i = a(t_f - t_i)} \tag{2.1}$$

The distance covered can be calculated as the area of the trapezoid:

$$\boxed{s_f - s_i = (\frac{v_i + v_f}{2})(t_f - t_i)} \tag{2.2}$$

Rearranging (2.2):

$$s_f - s_i = (\frac{2v_i + v_f - v_i}{2})(t_f - t_i) = v_i(t_f - t_i) + \frac{(v_f - v_i)(t_f - t_i)}{2} \tag{2.3}$$

Substituting (2.1) in (2.3) gives:

$$\boxed{s_f - s_i = v_i(t_f - t_i) + \frac{1}{2}a(t_f - t_i)^2} \tag{2.4}$$

Rearranging (2.1) gives:

$$(t_f - t_i) = \frac{(v_f - v_i)}{a}, \tag{2.5}$$

Substituting (2.5) in (2.2) gives:

$$s_f - s_i = (\frac{v_i + v_f}{2})(\frac{v_f - v_i}{a}) = \frac{v_f^2 - v_i^2}{2a} \tag{2.6}$$

Re-arranging provides the following important equation:

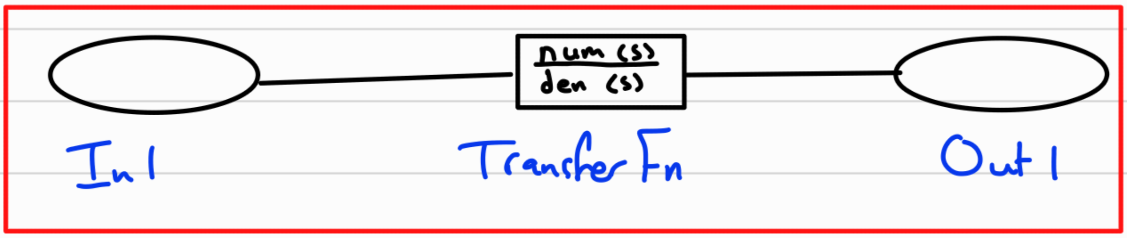
$$\boxed{v_f^2 - v_i^2 = 2a(s_f - s_i)} \tag{2.7}$$

- Parameters Estimation using MatLab :

* Parameter estimation is used to estimate certain unknown parameters in a grey box.

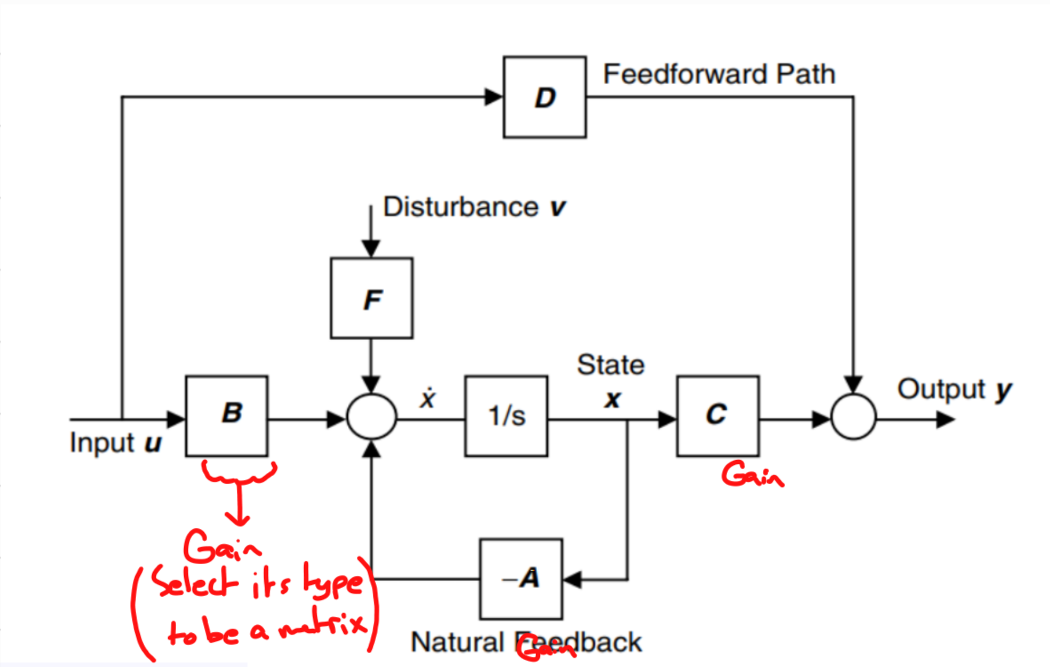
Simulink → Analysis → Parameter estimation
→ Select parameters → Specify min. & max. values for each parameter
→ New exmeriment → specify outputs [time, out1]
→ Specify inputs [time, In1] → Plot & Simulate
→ Press (more options) → Modify parameter tolerance and functional tolerance (0.000001)

→ For a SISO system :



→ For a MIMO System :

Simulink → Implement SSR as follows :



→ Follow the same procedure of the SISO system.

- Ident tool:

Load the measured data file in the command window (C.W)

→ Write "ident" in C.W to open the ident tool

→ Import data → Time domain data

→ Specify inputs and outputs from the measured data → Click import

→ Data becomes visible in the box

→ Drag it to the work space → Click on the time plot box in order to see it

→ Go to estimate → select transfer function models

→ specify the number of poles and zeros

→ Click estimate

→ The transfer function model should appear → Select it

→ to see it click on "Model output"

⇒ Chapter (4) : Motor Selection Criteria

- 4.1 : Introduction

→ Actuators are the most critical parts in the mechatronics sytem design, thus selecting actuators with the right specs is an important task.

- 4.2 : Actuator Selection

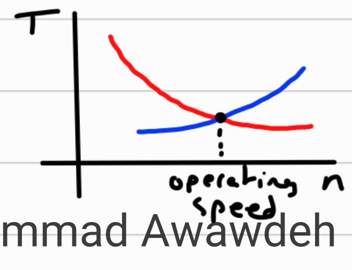
→ Things to consider when selecting an actuator:

① The starting torque, where the motor must generate a sufficient torque to overcome friction & load torques.

+ The acceleration of the motor and load at any instant is:

$$\boxed{\alpha = \frac{T_{motor} - T_{Load}}{J_{eq}}}$$

② The max. speed the motor can produce
③ Motor's operating duty cycle.
④ The required power for the desired load.
⑤ Available power sources.
⑥ The load's inertia.
⑦ The possibility that the load will be driven at a constant speed.
⑧ The purpose of application (Position Control or Speed Control)
⑨ If the system requires transmission or gearbox.
⑩ The possibility of motor's direction reversal.
⑪ Weight or size restrictions of the system.
⑫ Matching torque-speed curve and motor load



— Torque-speed curve
— load line

- 4.4 : Components of the actuator system

① Driver : usually a power electronic device.

② The actuator → Linear / → Rotary

* Actuators can be electromagnetic, hydraulic, Piezo electric

③ Mechanical Drive : placed on the output of the actuator.

- DC Motor Sizing :

→ Shaft speed : it is the no load speed which is the max. speed the motor can reach when no torque is applied.

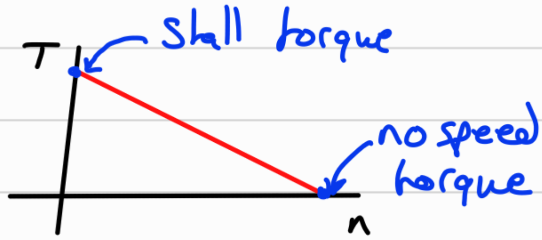
→ Output torque : the generated torque from the motor (N.m)

→ Stall torque : the torque when the shaft speed is zero.

→ Continuous torque : the max. torque at normal running conditions.

* Torque and current relation in a DC motor :

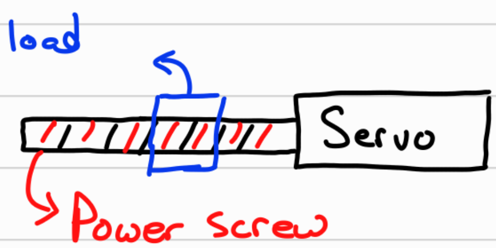
$$\boxed{T = K_t \, I}$$



→ Servo Motor Sizing :

+ For the rotational motion : $\Sigma M = J_{eq} \, \alpha$

+ For the linear motion : $\Sigma F = m a$



→ Sizing Factors:

① Motion profile
② Inertia
③ Speed
④ Torque
⑤ Regeneration capacity
⑥ Cost
⑦ Resolution
⑧ Environment
⑨ Power requirements
⑩ Physical size limitations.

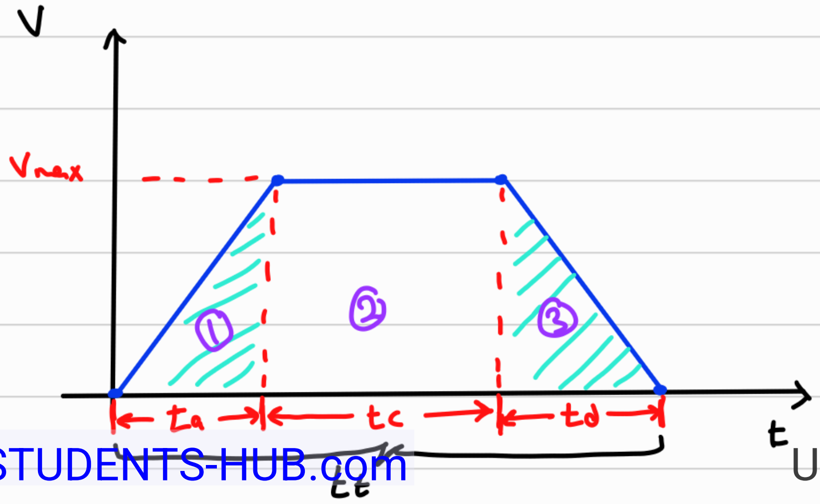
→ The main sizing factors are: ① Inertia ratio
② Speed
③ Max torque
④ RMS torque

* $\underbrace{\text{Inertia ratio}} = \dfrac{J_{load}}{J_{motor}}$

→ Typical values: 5:1 typical
2:1 (or less) for high performance
10:1 (or higher) when performance is not critical

→ If inertia ratio is too high: * Select a motor with higher inertia
* Decrease the load inertia
* Add a gearbox

- Speed - Torque Curves:



Standard trapezoidal profile where 1/3 of t_t is t_a, 1/3 for t_c, $\frac{1}{3}$ for t_d

$\Rightarrow t_a = t_c = t_d = \frac{1}{3} t_t$

$$\boxed{V_{avg} = \frac{d_t}{t_t}}$$

(*) $\boxed{V_{max} = \frac{3}{2} V_{avg}}$ → depends on the assumption (derived from our assumption)

$$\boxed{a = \frac{V_{max}}{t_a}}$$

* The total distance formula (based on the areas of the rectangular portions:

$$\boxed{d_t = \underbrace{\frac{1}{2} t_a V_{max}}_{\text{area①}} + \underbrace{t_c V_{max}}_{\text{area②}} + \underbrace{\frac{1}{2} t_d V_{max}}_{\text{area③}}}$$

* From our assumptions $(t_a = t_c = t_d)$

$$d_t = 2\, t_a V_{max} \Rightarrow d_t = 2 \cdot \frac{1}{3} t_t \cdot V_{max}$$

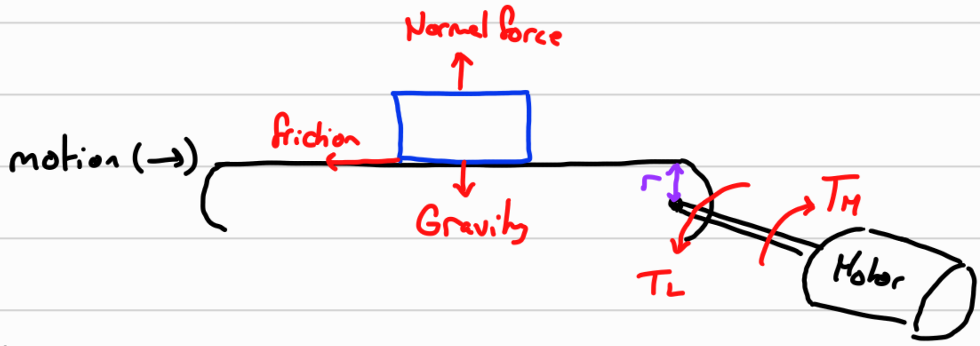
$$\Rightarrow V_{max} = \frac{3}{2} V_{avg}$$

- The load torque :

→ It is the amount of torque required for the application and it includes the friction & gravitational load.

→ It can be calculated using the equation : $\boxed{\Sigma M = 0}$

Ex: (for a conveyor)



→ Examples on another applications : * Elevator

* Motor attached to a power screw.

– The total required torque:

$$\boxed{T_{req.} = (T_L + T_a) \cdot S_f}$$

where: S_f = factor of safety
T_a = acceleration torque
T_L = Load torque

– The acceleration torque:

$$\boxed{T_a = J_{eq}\,\alpha}$$

→ It is the required torque to reach the constant speed after startup and acceleration.

→ Torque during acceleration (T_{Ma}) = $T_a + T_L$

– Constant speed torque:

→ It is the torque required to move the load at a constent speed.

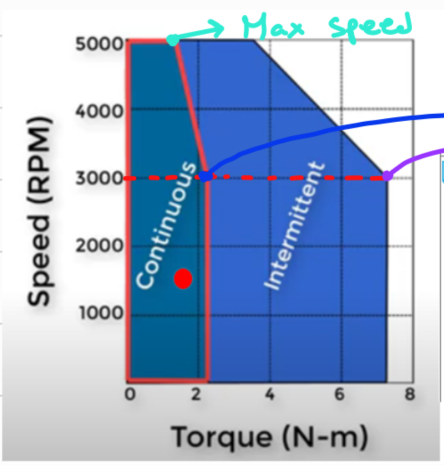
→ $T_M = T_L$

– Deceleration torque:

→ The torque required for deceleration and stopping.

→ $T_{Md} = -T_a + T_L$

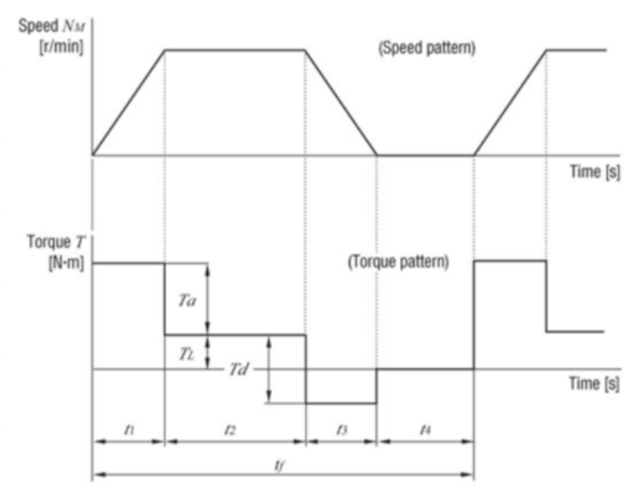
– Motor Ratings:

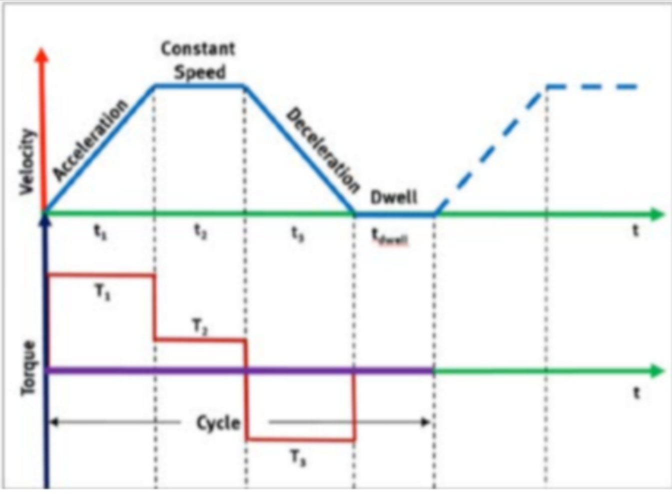


❑ If the torque lies on the continuous region that means the motor can works with any torque and with any speed without the motor is over heat

* If the torque - speed point lies in the intermittent region, the motor can work for 3 seconds or a little longer and provide $T = 3T_{rated}$ (peak torque)

– RMS Torque:





* Method ①:

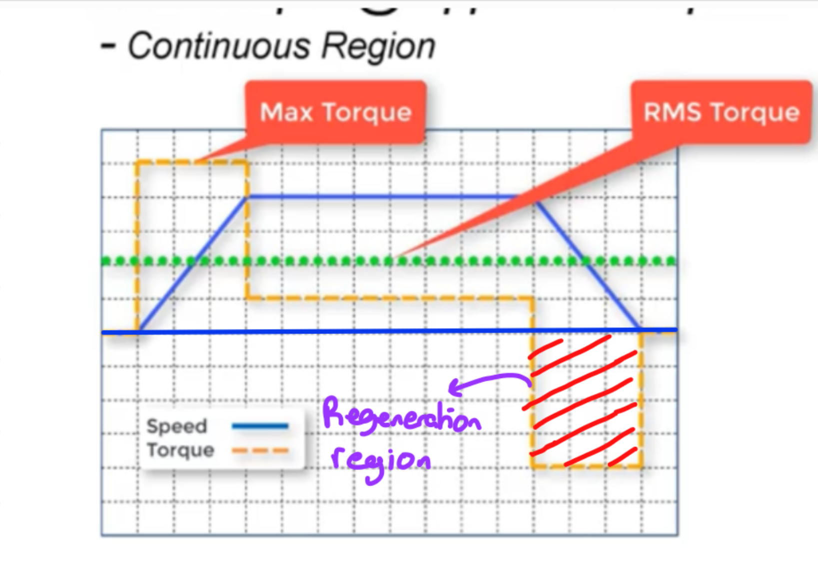
$$\boxed{T_{rms} = \sqrt{\frac{(T_a+T_L)^2 \cdot t_1 + T_L^2 \cdot t_2 + (T_d - T_L)^2 \cdot t_3}{t_f}}}$$

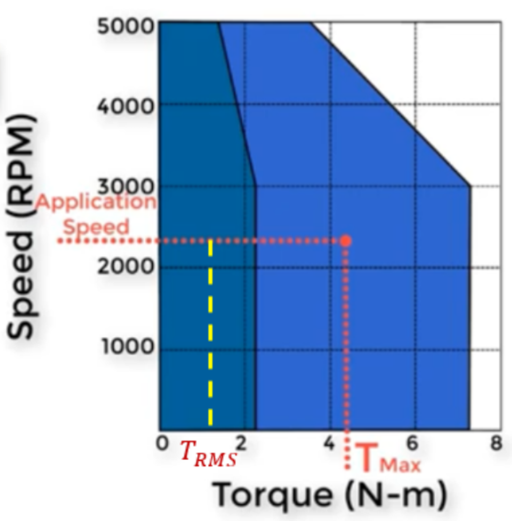
* Method ②:

$$\boxed{(T_{rms})^2 = \frac{T_1^2 t_1 + T_2^2 t_2 + T_3^2 t_3 + \dots + T_n^2 t_n}{t_1 + t_2 + t_3 + \dots + t_n}}$$

* Note that the RMS torque must lie in the continuous region.

– Speed & Torque Profiles:



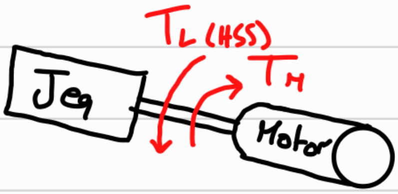


→ For bang-bang / quintic / cubic profiles, no T_{RMS} can be Calculated

- <u>To Solve Problems and properly select a suitable motor</u>:

① Reduce the given system to a simple system that consists of a motor, J_{eq} and a load torque with respect to the high speed shaft

* <u>Remember</u>:



$$J_{eq} = J_{HSS} + \frac{J_{LSS}}{r_g^2}$$

$$r_g = \frac{N_o}{N_i} \quad , \quad r_g = \frac{\theta_{HSS}}{\theta_{LSS}}$$

② Use any desired motion profile to find acceleration and torque profiles for the motor (trapezoidal profile is preffered). (Find dt, V_{max}, V_{avg}, a)

③ Find the load torque of the system which can be calculated from the analysis of the FBD of the entire system.

* Remember to convert T_L when calculated with respect to Hss where: $T_{L(HSS)} = \frac{T_{L(LSS)}}{r_g}$ → $= \frac{F\,r}{\eta}$ ↳ if given for the system

④ Find the acceleration torque of the system

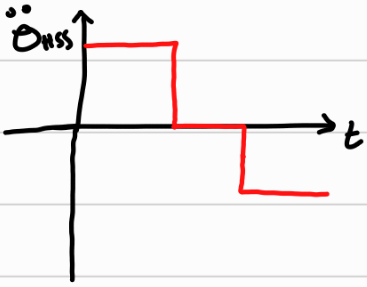
$$T_a = J_{eq}\, \ddot{\theta}_{HSS}$$

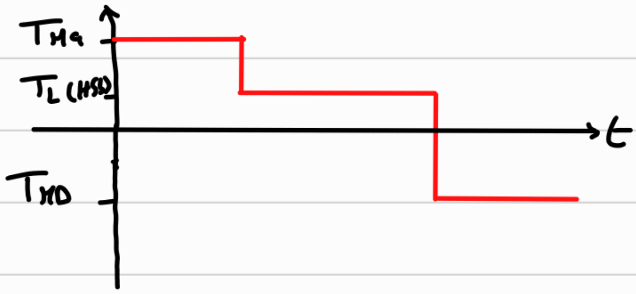
⑤ Find the motor torque at acceleration / deceleration phases:

$$T_{Ma} - T_{L(HSS)} = T_a \quad \text{(acceleration)}$$

$$T_{Md} - T_{L(HSS)} = -T_a \quad \text{(deceleration)}$$

⑥ Draw the acceleration / torque curve of the motor





⑦ Find the total required torque

$$T_{req} = (T_L + T_a)\, S_F$$

⑧ To find Pout of motor : $P_{out} = T \dot{\theta} \eta$

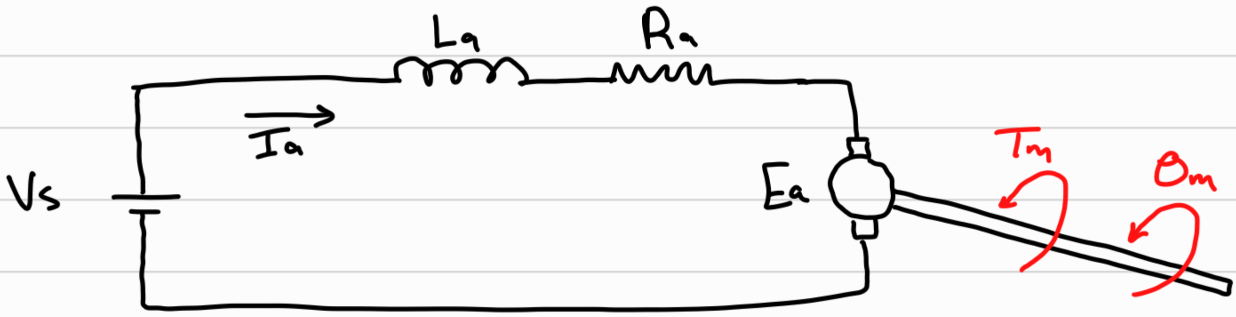
⑨ Find the RMS torque if possible, and show its value on the previous torque profile
(Only possible in 212 profile) and bang-bang profile

⑩ Check if T_{Ma}, T_{rms} lie in the continuous area of the torque - speed curve to see if the motor can be used in the application.

→ To convert speed from (rad/s) to (rpm)

$$\dot{\theta}_{HSS}\ (rpm) = \dot{\theta}_{HSS}\ (rad/s) \cdot \frac{60}{2\pi}$$

- Permanent magnet DC Motor :



Rotor Circuit

$$\boxed{E_a = K\,\omega_m = K\,\frac{d\theta_m}{dt}}$$

$$\boxed{T = K\,I_a}$$

$$E_a = V_s - I_a R_a$$

→ Equivalent mechanical model :