Instructor's Solutions Manual

An Introduction to Analysis

Fourth Edition

William R. Wade
University of Tennessee, Knoxville

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SOLUTIONS TO EXERCISES

CHAPTER 1

1.2 Ordered field axioms.

1.2.0. a) False. Let $a = 2/3$, $b = 1$, $c = -2$, and $d = -1$.

b) False. Let $a = -4$, $b = -1$, and $c = 2$.

c) True. Since $a \le b$ and $b \le a + c$, $|a - b| = b - a \le a + c - a = c$.

d) True. No $a \in \mathbf{R}$ satisfies $a < b - \varepsilon$ for all $\varepsilon > 0$, so the inequality is vacuously satisfied. If you want a more constructive proof, if $b \le 0$ then $a < b - \varepsilon < 0 + 0 = 0$. If $b > 0$, then for $\varepsilon = b$, $a < b - \varepsilon = 0$.

1.2.1. a) If $a < b$ then $a + c < b + c$ by the Additive Property. If $a = b$ then $a + c = b + c$ since $+$ is a function. Thus $a + c \le b + c$ holds for all $a \le b$.

b) If $c = 0$ then $ac = 0 = bc$ so we may suppose $c > 0$. If $a < b$ then $ac < bc$ by the Multiplicative Property. If $a = b$ then $ac = bc$ since $\cdot$ is a function. Thus $ac \le bc$ holds for all $a \le b$ and $c \ge 0$.

1.2.2. a) Suppose $0 \le a < b$ and $0 \le c < d$. Multiplying the first inequality by c and the second by b, we have $0 \le ac \le bc$ and $bc < bd$. Hence by the Transitive Property, $ac < bd$.

b) Suppose $0 \le a < b$. By (7), $0 \le a^2 < b^2$. If $\sqrt{a} \ge \sqrt{b}$ then $a = (\sqrt{a})^2 \ge (\sqrt{b})^2 = b$, a contradiction.

c) If $1/a \le 1/b$, then the Multiplicative Property implies $b = ab(1/a) \le ab(1/b) = a$, a contradiction. If $1/b \le 0$ then $b = b^2(1/b) \le 0$ a contradiction.

d) To show these statements may not hold when $a < 0$, let $a = -2$, $b = -1$, $c = 2$ and $d = 5$. Then $a < b$ and $c < d$ but $ac = -4$ is not less than $bd = -5$, $a^2 = 4$ is not less than $b^2 = 1$, and $1/a = -1/2$ is not less than $1/b = -1$.

1.2.3. a) By definition,

$$a^+ - a^- = \frac{|a| + a}{2} - \left(\frac{|a| - a}{2}\right) = \frac{2a}{2} = a$$

and

$$a^+ + a^- = \frac{|a| + a}{2} + \left(\frac{|a| - a}{2}\right) = \frac{2|a|}{2} = |a|.$$

b) By Definition 1.1, if $a \ge 0$ then $a^+ = (a + a)/2 = a$ and if $a < 0$ then $a^+ = (-a + a)/2 = 0$. Similarly, $a^- = 0$ if $a \ge 0$ and $a^- = -a$ if $a < 0$.

1.2.4. a) $|2x + 1| < 7$ if and only if $-7 < 2x + 1 < 7$ if and only if $- - 4 < x < 3$.

b) $|2 - x| < 2$ if and only if $-2 < 2 - x < 2$ if and only if $-4 < -x < 0$ if and only if $0 < x < 4$.

c) $|x^3 - 3x + 1| < x^3$ if and only if $-x^3 < x^3 - 3x + 1 < x^3$ if and only if $3x - 1 > 0$ and $2x^3 - 3x + 1 > 0$. The first inequality is equivalent to $x > 1/3$. Since $2x^3 - 3x + 1 = (x - 1)(2x^2 + 2x - 1)$ implies that $x = 1, (-1 \pm \sqrt{3})/2$, the second inequality is equivalent to $(-1 - \sqrt{3})/2 < x < (-1 + \sqrt{3})/2$ or $x > 1$. Therefore, the solution is $(1/3, (\sqrt{3} - 1)/2) \cup (1, \infty)$.

d) We cannot multiply by the denominator $x - 1$ unless we consider its sign.

Case 1. $x - 1 > 0$. Then $x < x - 1$ so $0 < -1$, i.e., this case is empty.

Case 2. $x - 1 < 0$. Then by the Second Multiplicative Property, $x > x - 1$ so $0 > -1$, i.e., every number from this case works. Thus the solution is $(-\infty, 1)$.

e) *Case 1.* $4x^2 - 1 > 0$. Cross multiplying, we obtain $4x^2 < 4x^2 - 1$, i.e., this case is empty.

Case 2. $4x^2 - 1 < 0$. Then by the Second Multiplicative Property, $4x^2 > 4x^2 - 1$, i.e., $0 > -1$. Thus the solution is $(-1/2, 1/2)$.

1.2.5. a) Suppose $a > 2$. Then $a - 1 > 1$ so $1 < \sqrt{a - 1} < a - 1$ by (6). Therefore, $2 < b = 1 + \sqrt{a - 1} < 1 + (a - 1) = a$.

b) Suppose $2 < a < 3$. Then $0 < a - 2 < 1$ so $0 < a - 2 < \sqrt{a - 2} < 1$ by (6). Therefore, $0 < a < 2 + \sqrt{a - 2} = b$.

c) Suppose $0 < a < 1$. Then $0 > -a > -1$, so $0 < 1 - a < 1$. Hence $\sqrt{1 - a}$ is real and by (6), $1 - a < \sqrt{1 - a}$. Therefore, $b = 1 - \sqrt{1 - a} < 1 - (1 - a) = a$.

d) Suppose $3 < a < 5$. Then $1 < a - 2 < 3$ so $1 < \sqrt{a - 2} < a - 2$ by (6). Therefore, $3 < 2 + \sqrt{a - 2} = b < a$.

1.2.6. $a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \geq 0$ for all $a, b \in [0, \infty)$. Thus $2\sqrt{ab} \leq a + b$ and $G(a, b) \leq A(a, b)$. On the other hand, since $0 \leq a \leq b$ we have $A(a, b) = (a + b)/2 \leq 2b/2 = b$ and $G(a, b) = \sqrt{ab} \geq \sqrt{a^2} = a$. Finally, $A(a, b) = G(a, b)$ if and only if $2\sqrt{ab} = a + b$ if and only if $(\sqrt{a} - \sqrt{b})^2 = 0$ if and only if $\sqrt{a} = \sqrt{b}$ if and only if $a = b$.

1.2.7. a) Since $|x + 2| \leq |x| + 2$, $|x| \leq 2$ implies $|x^2 - 4| = |x + 2|\,|x - 2| \leq 4|x - 2|$.
b) Since $|x + 3| \leq |x| + 3$, $|x| \leq 1$ implies $|x^2 + 2x - 3| = |x + 3|\,|x - 1| \leq 4|x - 1|$.
c) Since $|x - 2| \leq |x| + 2$, $-3 \leq x \leq 2$ implies $|x^2 + x - 6| = |x + 3|\,|x - 2| \leq 6|x - 2|$.
d) Since the minimum of $x^2 + x - 1$ on $(-1, 0)$ is -1.25, $-1 < x < 0$ implies $|x^3 - 2x + 1| = |x^2 + x - 1|\,|x - 1| < 5|x - 1|/4$.

1.2.8. a) Since $(1 - n)/(1 - n^2) = 1/(1 + n)$, the inequality is equivalent to $1/(n + 1) < .01 = 1/100$. Since $1 + n > 0$ for all $n \in \mathbf{N}$, it follows that $n + 1 > 100$, i.e., $n > 99$.
b) By factoring, we see that the inequality is equivalent to $1/(2n + 1) < 1/40$, i.e., $2n + 1 > 40$. Thus $n > 39/2$, i.e., $n \geq 20$.
c) The inequality is equivalent to $n^2 + 1 > 500$. Thus $n > \sqrt{499} \approx 22.33$, i.e., $n \geq 23$.

1.2.9. a) $mn^{-1} + pq^{-1} = mqq^{-1}n^{-1} + pq^{-1}nn^{-1} = (mq + pn)n^{-1}q^{-1}$. But $n^{-1}q^{-1}nq = 1$ and uniqueness of multiplicative inverses implies $(nq)^{-1} = n^{-1}q^{-1}$. Therefore, $mn^{-1} + pq^{-1} = (mq + pn)(nq)^{-1}$. Similarly, $mn^{-1}(pq^{-1}) = mpn^{-1}q^{-1} = mp(nq)^{-1}$. By what we just proved and (2),

$$\frac{m}{n} + \frac{-m}{n} = \frac{m - m}{n} = \frac{0}{n} = 0.$$

Therefore, by the uniqueness of additive inverses, $-(m/n) = (-m)/n$. Similarly, $(m/n)(n/m) = (mn)/(mn) = mn(mn)^{-1} = 1$, so $(m/n)^{-1} = n/m$ by the uniqueness of multiplicative inverses.
b) Any subset of $\mathbf{R}$ which contains 0 and 1 will satisfy the Associative and Commutative Properties, the Distributive Law, and have an additive identity 0 and a multiplicative identity 1. By part a), $\mathbf{Q}$ satisfies the Closure Properties, has additive inverses, and every nonzero $q \in \mathbf{Q}$ has a multiplicative inverse. Therefore, $\mathbf{Q}$ satisfies Postulate 1.
c) If $r \in \mathbf{Q}$, $x \in \mathbf{R} \setminus \mathbf{Q}$ but $q := r + x \in \mathbf{Q}$, then $x = q - r \in \mathbf{Q}$, a contradiction. Similarly, if $rx \in \mathbf{Q}$ and $r \neq 0$, then $x \in \mathbf{Q}$, a contradiction. However, the product of any irrational with 0 is a rational.
d) By the First Multiplicative Property, $mn^{-1} < pq^{-1}$ if and only if $mq = mn^{-1}qn < pq^{-1}nq = np$.

1.2.10. $0 \leq (cb - ad)^2 = c^2b^2 - 2abcd + a^2d^2$ implies $2abcd \leq c^2b^2 + a^2d^2$. Adding $a^2b^2 + c^2d^2$ to both sides, we conclude that $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$.

1.2.11. Let $\mathbf{P} := \mathbf{R}^+$.
a) Let $x \in \mathbf{R}$. By the Trichotomy Property, either $x > 0$, $-x > 0$, or $x = 0$. Thus $\mathbf{P}$ satisfies i). If $x > 0$ and $y > 0$, then by the Additive Property, $x + y > 0$ and by the First Multiplicative Property, $xy > 0$. Thus $\mathbf{P}$ satisfies ii).
b) To prove the Trichotomy Property, suppose $a, b \in \mathbf{R}$. By i), either $a - b \in \mathbf{P}$, $b - a = -(a - b) \in \mathbf{P}$, or $a - b = 0$. Thus either $a > b$, $b > a$, or $a = b$.
To prove the Transitive Property, suppose $a < b$ and $b < c$. Then $b - a, c - b \in \mathbf{P}$ and it follows from ii) that $c - a = b - a + c - b \in \mathbf{P}$, i.e., $c > a$.
Since $b - a = (b + c) - (a + c)$, it is clear that the Additive Property holds.
Finally, suppose $a < b$, i.e., $b - a \in \mathbf{P}$. If $c > 0$ then $c \in \mathbf{P}$ and it follows from ii) that $bc - ac = (b - a)c \in \mathbf{P}$, i.e., $bc > ac$. If $c < 0$ then $-c \in \mathbf{P}$, so $ac - bc = (b - a)(-c) \in \mathbf{P}$, i.e., $ac > bc$.

1.3 The Completeness Axiom.

1.3.0. a) True. If $A \cap B = \emptyset$, then $\sup(A \cap B) := -\infty$ and there is nothing to prove. If $A \cap B \neq \emptyset$, then use the Monotone Property.
b) True. If $x \in A$, then $x \leq \sup A$. Since $\varepsilon > 0$, we have $\varepsilon x \leq \varepsilon \sup A$, so the latter is an upper bound of B. It follows that $\sup B \leq \varepsilon \sup A$. On the other hand, if $x \in A$, then $\varepsilon x \in B$, so $\varepsilon x \leq \sup B$, i.e., $\sup B/\varepsilon$ is an upper bound for A. It follows that $\sup A \leq \sup B/\varepsilon$.
c) True. If $x \in A$ and $y \in B$, then $x + y \leq \sup A + \sup B$, so $\sup(A + B) \leq \sup A + \sup B$. If this inequality is strict, then $\sup(A + B) - \sup B < \sup A$, and it follows from the Approximation Property that there is an $a_0 \in A$ such that $\sup(A + B) - \sup B < a_0$. This implies that $\sup(A + B) - a_0 < \sup B$, so by the Approximation Property again, there is a $b_0 \in B$ such that $\sup(A + B) - a_0 < b_0$. We conclude that $\sup(A + B) < a_0 + b_0$, a contradiction.

d) False. Let $A = B = [0,1]$. Then $A - B = [-1,1]$ so $\sup(A - B) = 1 \neq 0 = \sup A - \sup B$.

1.3.1. a) Since $x^2 + 2x - 3 = 0$ implies $x = 1, -3$, $\inf E = -3$, $\sup E = 1$. b) Since $x^2 - 2x + 3 > x^2$ implies $x < 3/2$, $\inf E = 0$, $\sup E = 3/2$. c) Since $p^2/q^2 < 5$ implies $p/q < \sqrt{5}$, $\inf E = 0$, $\sup E = \sqrt{5}$. d) Since $1 + (-1)^n/n = 1 - 1/n$ when n is odd and $1 + 1/n$ when n is even, $\inf E = 0$ and $\sup E = 3/2$. e) Since $1/n + (-1)^n = 1/n + 1$ when n is even and $1/n - 1$ when n is odd, $\inf E = -1$ and $\sup E = 3/2$. f) Since $2 - (-1)^n/n^2 = 2 - 1/n^2$ when n is even and $2 + 1/n^2$ when n is odd, $\inf E = 7/4$ and $\sup E = 3$.

1.3.2. Since $a - 1/n < a + 1/n$, choose $r_n \in \mathbf{Q}$ such that $a - 1/n < r_n < a + 1/n$, i.e., $|a - r_n| < 1/n$.

1.3.3. $a < b$ implies $a - \sqrt{2} < b - \sqrt{2}$. Choose $r \in \mathbf{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then $a < r + \sqrt{2} < b$. By Exercise 1.2.9c, $r + \sqrt{2}$ is irrational. Thus set $\xi = r + \sqrt{2}$.

1.3.4. If m is a lower bound of E then so is any $\widetilde{m} \leq m$. If m and $\widetilde{m}$ are both infima of E then $m \leq \widetilde{m}$ and $\widetilde{m} \leq m$, i.e., $m = \widetilde{m}$.

1.3.5. Suppose that E is a bounded, nonempty subset of $\mathbf{Z}$. Since $-E$ is a bounded, nonempty subset of $\mathbf{Z}$, it has a supremum by the Completeness Axiom, and that supremum belongs to $-E$ by Theorem 1.15. Hence by the Reflection Principle, $\inf E = -\sup(-E) \in -(-E) = E$.

1.3.6. a) Let $\epsilon > 0$ and $m = \inf E$. Since $m + \epsilon$ is not a lower bound of E, there is an $a \in E$ such that $m + \epsilon > a$. Thus $m + \epsilon > a \geq m$ as required.

b) By Theorem 1.14, there is an $a \in E$ such that $\sup(-E) - \epsilon < -a \leq \sup(-E)$. Hence by the Second Multiplicative Property and Theorem 1.20, $\inf E + \epsilon = -(\sup(-E) - \epsilon) > a > -\sup(-E) = \inf E$.

1.3.7. a) Let x be an upper bound of E and $x \in E$. If M is any upper bound of E then $M \geq x$. Hence by definition, x is the supremum of E.

b) The correct statement is: If x is a lower bound of E and $x \in E$ then $x = \inf E$.

PROOF. $-x$ is an upper bound of $-E$ and $-x \in -E$ so $-x = \sup(-E)$. Thus $x = -\sup(-E) = \inf E$.

c) If E is the set of points x_n such that $x_n = 1 - 1/n$ for odd n and $x_n = 1/n$ for even n, then $\sup E = 1$, $\inf E = 0$, but neither 0 nor 1 belong to E.

1.3.8. Since $A \subseteq E$, any upper bound of E is an upper bound of A. Since A is nonempty, it follows from the Completeness Axiom that A has a supremum. Similarly, B has a supremum. Moreover, by the Monotone Property, $\sup A, \sup B \leq \sup E$.

Set $M := \max\{\sup A, \sup B\}$ and observe that M is an upper bound of both A and B. If $M < \sup E$, then there is an $x \in E$ such that $M < x \leq \sup E$. But $x \in E$ implies $x \in A$ or $x \in B$. Thus M is not an upper bound for one of the sets A or B, a contradiction.

1.3.9. By induction, $2^n > n$. Hence by the Archimedean Principle, there is an $n \in \mathbf{N}$ such that $2^n > 1/(b-a)$. Let $E := \{k \in \mathbf{N} : 2^n b \leq k\}$. By the Archimedean Principle, E is nonempty. Hence let m_0 be the least element in E and set $q = (m_0 - 1)/2^n$. Since $b > 0$, $m_0 \geq 1$. Since m_0 is least in E, it follows that $m_0 - 1 < 2^n b$, i.e., $q < b$. On the other hand, $m_0 \in E$ implies $2^n b \leq m_0$, so

$$a = b - (b - a) < \frac{m_0}{2^n} - \frac{1}{2^n} = \frac{m_0 - 1}{2^n} = q.$$

1.3.10. Since $|x_n| \leq M$, the set $E_n = \{x_n, x_{n+1}, \dots\}$ is bounded for each $n \in \mathbf{N}$. Thus $s_n := \sup E_n$ exists and is finite by the Completeness Axiom. Moreover, since $E_{n+1} \subseteq E_n$, it follows from the Monotone Property, $s_n \geq s_{n+1}$ for each $n \in \mathbf{N}$. Thus $s_1 \geq s_2 \geq \dots$.

By the Reflection Principle, it follows that $t_1 \leq t_2 \leq \cdots$.

Or, if you prefer a more direct approach, $\sigma_n := \sup\{-x_n, -x_{n+1}, \dots\}$ satisfies $\sigma_1 \geq \sigma_2 \geq \dots$. Since $t_n = -\sigma_n$ for $n \in \mathbf{N}$, it follows from the Second Multiplicative Property that $t_1 \leq t_2 \leq \dots$.

1.3.11. Let $E = \{n \in \mathbf{Z} : n \leq a\}$. If $a \geq 0$, then $0 \in E$. If $a < 0$, then by the Archimedean Principle, there is an $m \in \mathbf{N}$ such that $m > -a$, i.e., $n := -m \in E$. Thus E is nonempty. Since E is bounded above (by a), it follows from the Completeness Axiom and Theorem 1.15 that $n_0 = \sup E$ exists and belongs to E.

Set $k = n_0 + 1$. Since $k > \sup E$, k cannot belong to E, i.e., $a < k$. On the other hand, since $n_0 \in E$ and $b - a > 1$,

$$k = n_0 + 1 \leq a + 1 < a + (b - a) = b.$$

We conclude that $a < k < b$.

1.4 Mathematical Induction.

1.4.0. a) False. If $a = -b = 1$ and $n = 2$, then $(a+b)^n = 0$ is NOT greater than $b^2 = 1$.

b) False. If $a = -3$, $b = 1$, and $n = 2$, then $(a+b)^n = 4$ is not less than or equal to $b^n = 1$.

c) True. If n is even, then $n-k$ and k are either both odd or both even. If they're both odd, then $a^{n-k}b^k$ is the product of two negative numbers, hence positive. If they're both even, then $a^{n-k}b^k$ is the product of two positive numbers, hence positive. Thus by the Binomial Formula,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k = a^n + na^{n-1}b + \sum_{k=2}^{n} \binom{n}{k} a^{n-k}b^k =: a^n + na^{n-1}b + C.$$

Since C is a sum of positive numbers, the promised inequality follows at once.

d) True. By the Binomial Formula,

$$\frac{1}{2^n} = \left(\frac{1}{a} + \frac{a-2}{2a}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{a^k} \frac{(a-2)^{n-k}}{2^{n-k}a^{n-k}} = \sum_{k=0}^{n} \binom{n}{k} \frac{(a-2)^{n-k}}{a^n 2^{n-k}}.$$

1.4.1. a) By hypothesis, $x_1 > 2$. Suppose $x_n > 2$. Then by Exercise 1.2.5a, $2 < x_{n+1} < x_n$. Thus by induction, $2 < x_{n+1} < x_n$ for all $n \in \mathbf{N}$.

b) By hypothesis, $2 < x_1 < 3$. Suppose $2 < x_n < 3$. Then by Exercise 1.2.5b, $0 < x_n < x_{n+1}$. Thus by induction, $0 < x_n < x_{n+1}$ for all $n \in \mathbf{N}$.

c) By hypothesis, $0 < x_1 < 1$. Suppose $0 < x_n < 1$. Then by Exercise 1.2.5c, $0 < x_{n+1} < x_n$. Thus by induction this inequality holds for all $n \in \mathbf{N}$.

d) By hypothesis, $3 < x_1 < 5$. Suppose $3 < x_n < 5$. Then by Exercise 1.2.5d, $3 < x_{n+1} < x_n$. Thus by induction this inequality holds for all $n \in \mathbf{N}$.

1.4.2. a) $0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-1)^k = \sum_{k=0}^{n} \binom{n}{k}(-1)^k$.

b) $(a+b)^n = a^n + na^{n-1}b + \cdots + b^n \geq a^n + na^{n-1}b$.

c) By b), $(1+1/n)^n \geq 1^n + n1^{n-1}(1/n) = 2$.

d) $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$ so $\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$. On the other hand $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ by induction.

1.4.3. a) This inequality holds for $n = 3$. If it holds for some $n \geq 3$ then

$$2(n+1) + 1 = 2n + 1 + 2 < 2^n + 2 < 2^n + 2^n = 2^{n+1}.$$

b) The inequality holds for $n = 1$. If it holds for n then

$$n + 1 < 2^n + 1 \leq 2^n + n < 2^n + 2^n = 2^{n+1}.$$

c) Now $n^2 \leq 2^n + 1$ holds for $n = 1, 2,$ and 3. If it holds for some $n \geq 3$ then by a),

$$(n+1)^2 = n^2 + 2n + 1 < 2^n + 2^n = 2^{n+1} < 2^{n+1} + 1.$$

d) We claim that $3n^2 + 3n + 1 \leq 2 \cdot 3^n$ for $n = 3, 4, \ldots$. This inequality holds for $n = 3$. Suppose it holds for some n. Then

$$3(n+1)^2 + 3(n+1) + 1 = 3n^2 + 3n + 1 + 6n + 6 \leq 2 \cdot 3^n + 6(n+1).$$

Similarly, induction can be used to establish $6(n+1) \leq 4 \cdot 3^n$ for $n \geq 1$. (It holds for $n = 1$, and if it holds for n then $6(n+2) = 6(n+1) + 6 \leq 4 \cdot 3^n + 6 < 4 \cdot 3^n + 8 \cdot 3^n = 4 \cdot 3^{n+1}$.) Therefore,

$$3(n+1)^2 + 3(n+1) + 1 \leq 2 \cdot 3^n + 6(n+1) \leq 2 \cdot 3^n + 4 \cdot 3^n = 2 \cdot 3^{n+1}.$$

Thus the claim holds for all $n \geq 3$.

Now $n^3 \leq 3^n$ holds by inspection for $n = 1, 2, 3$. Suppose it holds for some $n \geq 3$. Then

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 \leq 3^n + 2 \cdot 3^n = 3^{n+1}.$$

1.4.4. a) The formula holds for $n = 1$. If it holds for n then

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + n + 1 = (n+1)(\frac{n}{2} + 1) = \frac{(n+1)(n+2)}{2}.$$

b) The formula holds for $n = 1$. If it holds for n then

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n+1}{6}(n(2n+1) + 6(n+1)) = \frac{(n+1)(n+2)(2n+3)}{6}.$$

c) The formula holds for $n = 1$. If it holds for n then

$$\sum_{k=1}^{n+1} \frac{a-1}{a^k} = 1 - \frac{1}{a^n} + \frac{a-1}{a^{n+1}} = 1 - \frac{1}{a^{n+1}}.$$

d) The formula holds for $n = 1$. If it holds for n then

$$\begin{aligned}\sum_{k=1}^{n+1} (2k-1)^2 &= \frac{n(4n^2-1)}{3} + (2n+1)^2 = \frac{2n+1}{3}(2n^2 + 5n + 3) \\ &= \frac{2n+1}{3}(2n+3)(n+1) = \frac{(n+1)(4n^2+8n+3)}{3} \\ &= \frac{(n+1)(4(n+1)^2-1)}{3}.\end{aligned}$$

1.4.5. $0 \le a^n < b^n$ holds for $n = 1$. If it holds for n then by (7), $0 \le a^{n+1} < b^{n+1}$.

By convention, $\sqrt[n]{b} \ge 0$. If $\sqrt[n]{a} < \sqrt[n]{b}$ is false, then $\sqrt[n]{a} \ge \sqrt[n]{b} \ge 0$. Taking the nth power of this inequality, we obtain $a = (\sqrt[n]{a})^n \ge (\sqrt[n]{b})^n = b$, a contradiction.

1.4.6. The result is true for $n = 1$. Suppose it's true for some odd number ≥ 1, i.e., $2^{2n-1} + 3^{2n-1} = 5\ell$ for some $\ell, n \in \mathbf{N}$. Then

$$2^{2n+1} + 3^{2n+1} = 4 \cdot 2^{2n-1} + 9 \cdot 3^{2n-1} = 4 \cdot 5\ell + 5 \cdot 3^{2n-1}$$

is evidently divisible by 5. Thus the result is true by induction.

1.4.7. We first prove that $2n! + 2 \le (n+1)!$ for $n = 2, 3, \dots$. It's true for $n = 2$. Suppose that it's true for some $n \ge 2$. Then by the inductive hypothesis,

$$2(n+1)! + 2 = 2(n+1)n! + 2 = 2n! + 2 + 2n \cdot n! \le (n+1)! + 2n \cdot n!.$$

But $2 < n+1$ so we continue the inequality above by

$$2(n+1)! + 2 < (n+1)! + n \cdot (n+1)! = (n+2) \cdot (n+1)! = (n+2)!$$

as required.

To prove that $2^n \le n! + 2$, notice first that it's true for $n = 1$. If it's true for some $n \ge 1$, then by the inequality already proved,

$$2^{n+1} = 2 \cdot 2^n \le 2(n! + 2) = 2n! + 2 + 2 \le (n+1)! + 2$$

as required.

1.4.8. If $n = 1$ or $n = 2$, the result is trivial. If $n \ge 3$, then by the Binomial Formula,

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} > \binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

1.4.9. a) If $m = k^2$, then $\sqrt{m} = k$ by definition. On the other hand, if m is not a perfect square, then by Remark 1.28, $\sqrt{m}$ is irrational. In particular, it cannot be rational.

b) If $\sqrt{n+3}+\sqrt{n} \in \mathbf{Q}$ then $n+3+2\sqrt{n+3}\sqrt{n}+n = (\sqrt{n+3}+\sqrt{n})^2 \in \mathbf{Q}$. Since $\mathbf{Q}$ is closed under subtraction and division, it follows that $\sqrt{n^2+3n} \in \mathbf{Q}$. In particular, $n^2+3n = m^2$ for some $m \in \mathbf{N}$. Now n^2+3n is a perfect square when $n = 1$ but if $n > 1$ then

$$(n+1)^2 = n^2+2n+1 < n^2+2n+n = \boldsymbol{n^2+3n} =< n^2+4n+4 = (n+2)^2.$$

Therefore, the original expression is rational if and only if $n = 1$.

c) By repeating the steps in b), we see that the original expression is rational if and only if $n(n+7) = n^2+7n = m^2$ for some $m \in \mathbf{N}$. If $n > 9$ then

$$(n+3)^2 = n^2+6n+9 < \boldsymbol{n^2+7n} < n^2+8n+16 = (n+4)^2.$$

Thus the original expression cannot be rational when $n > 9$. On the other hand, it is easy to check that n^2+7n is not a perfect square for $n = 1, 2, \dots, 8$ but is a perfect square, namely $144 = 12^2$, when $n = 9$. Thus the original expression is rational if and only if $n = 9$.

1.4.10. The result holds for $n = 0$ since $c_0 - b_0 = 1$ and $a_0^2 + b_0^2 = c_0^2$. Suppose that $c_{n-1} - b_{n-1} = 1$ and $a_{n-1}^2 + b_{n-1}^2 = c_{n-1}^2$ hold for some $n \geq 0$. By definition, $c_n - b_n = c_{n-1} - b_{n-1} = 1$, so by induction, this difference is always 1. Moreover, by the Binomial Formula, the inductive hypothesis, and what we just proved,

$$\begin{aligned}
a_n^2 + b_n^2 &= (a_{n-1}+2)^2 + (2a_{n-1}+b_{n-1}+2)^2 \\
&= a_{n-1}^2 + 4a_{n-1} + 4 + (2a_{n-1}+2)^2 + 2b_{n-1}(2a_{n-1}+2) + b_{n-1}^2 \\
&= c_{n-1}^2 + 2(a_{n-1}+2) + (2a_{n-1}+2)^2 + 2(c_{n-1}-1)(2a_{n-1}+2) \\
&= c_{n-1}^2 + (2a_{n-1}+2)^2 + 2c_{n-1}(2a_{n-1}+2) \\
&= (2a_{n-1}+c_{n-1}+2)^2 \equiv c_n^2.
\end{aligned}$$

1.5 Inverse Functions and Images.

1.5.0. a) False. Since $(\sin x)' = \cos x$ is negative on $[\pi/2, 3\pi/2]$, f is 1–1 there, but the domain of $\arcsin x$ is $[-\pi/2, \pi/2]$. Thus here, $f^{-1}(x) = \arcsin(\pi - x)$.

b) True. By elementary set algebra and Theorem 1.37,

$$(f^{-1}(A) \cap f^{-1}(B)) \cup f^{-1}(C) = f^{-1}(A \cap B) \cup f^{-1}(C) \supset f^{-1}(A \cap B) \neq \emptyset.$$

c) False. If $X = [0,2]$, $A = [0,1]$ and $B = \{1\}$, then $B \setminus A = \emptyset$ but $(A \setminus B)^c = [0,1)^c = [1,2]$.

d) False. Let $f(x) = x+1$ for $-1 \leq x \leq 0$ and $f(x) = 2x-1$ for $0 < x \leq 1$. Then f takes $[-1,1]$ onto $[-1,1]$ and $f(0) = 1$, but $f^{-1}(f(0)) = f^{-1}(1) = \{0,1\}$.

1.5.1. α) f is 1–1 since $f'(x) = 3 > 0$ for $x \in \mathbf{R}$. If $y = 3x-7$ then $x = (y+7)/3$. Therefore $f^{-1}(x) = (x+7)/3$. By looking at the graph, we see that $f(E) = \mathbf{R}$.

β) f is 1–1 since $f'(x) = -e^{1/x}/x^2 > 0$ for $x \in (0,\infty)$. If $y = e^{1/x}$ then $\log y = 1/x$, i.e., $x = 1/\log y$. Therefore, $f^{-1}(x) = 1/\log x$. By looking at the graph, we see that $f(E) = (1,\infty)$.

γ) f is 1–1 on $(\pi/2, 3\pi/2)$ because $f'(x) = \sec^2 x > 0$ there. The inverse is $f^{-1}(x) = \arctan(x-\pi)$. By looking at the graph, we see that $f(E) = (-\infty,\infty)$.

δ) Since $f'(x) = 2x+2 < 0$ for $x < -6$, f is 1–1 on $[-\infty,-6]$. Since $y = x^2+2x-5$ is a quadratic in x, we have $x = (-2 \pm \sqrt{4+4(5+y)})/2 = -1 \pm \sqrt{6+y}$. But x is negative on $(-\infty,-6]$, so we must use the negative sign. Hence $f^{-1}(x) = -1-\sqrt{6+x}$. By looking at the graph, we see that $f(E) = [19,\infty)$.

ε) By definition,

$$f(x) = \begin{cases} 3x+2 & x \leq 0 \\ x+2 & 0 < x \leq 2 \\ 3x-2 & x > 2. \end{cases}$$

Thus f is strictly increasing, hence 1–1, and

$$f^{-1}(x) = \begin{cases} (x-2)/3 & x \leq 2 \\ x-2 & 2 < x \leq 4 \\ (x+2)/3 & x > 4, \end{cases}$$

i.e., $f^{-1}(x) = (x + |x-2| - |x-4|)/3$. By looking at the graph, we see that $f(E) = (-\infty, \infty)$.

ζ) Since $f'(x) = (1-x^2)/(x^2+1)^2$ is never zero on $(-1,1)$, f is 1–1 on $[-1,1]$. By the quadratic formula, $y = f(x)$ implies $x = (1 \pm \sqrt{1-4y^2})/2y$. Since $x \in [-1,1]$ we must take the minus sign. Hence

$$f^{-1}(x) = \begin{cases} (1-\sqrt{1-4x^2})/2x & x \neq 0 \\ 0 & x = 0. \end{cases}$$

By looking at the graph, we see that $f(E) = (-0.5, 0.5)$.

1.5.2. a) f decreases and $f(-1) = 5$, $f(2) = -4$. Therefore, $f(E) = (-4,5)$. Since $f(x) = -1$ implies $x = 1$ and $f(x) = 2$ implies $x = 0$, we also have $f^{-1}(E) = (0,1)$.

b) The graph of f is a parabola whose absolute minimum is 1 at $x = 0$ and whose maximum on $(-1,2]$ is 5 at $x = 2$. Therefore, $f(E) = [1,5]$. Since f takes ± 1 to 2, $f^{-1}(E) = [-1,1]$.

c) The graph of f is a parabola whose absolute maximum is 1 at $x = 1$. Since $f(-2) = -8$, it follows that $f(E) = [-8,1]$. Since $2x - x^2 = -2$ implies $x = 1 \pm \sqrt{3}$, we also have $f^{-1}(E) = [1-\sqrt{3}, 1+\sqrt{3}]$.

d) The graph of $x^2 - 2x + 2$ is a parabola whose minimum is 1 at $x = 1$. Since log increases on $(0,\infty)$, $f(1) = \log(1) = 0$, and $f(3) = \log(5)$, it follows that $f(E) = [0, \log(5)]$. Since $3 = \log(x^2 + x + 1)$ implies $x = 1 \pm \sqrt{e^3 - 1}$, we also have

$$f^{-1}(E) = [1 - \sqrt{e^3-1}, 1) \cup (1, 1 + \sqrt{e^3-1}].$$

e) Since $\cos x$ is periodic with maximum 1 and minimum -1, $f(E) = [-1,1]$. Since $\cos x$ is nonnegative when $(4k-1)\pi/2 \le x \le (4k+1)\pi/2$ for some $k \in \mathbf{Z}$, it follows that

$$f^{-1}(E) = \bigcup_{k \in \mathbf{Z}} [(4k-1)\pi/2, (4k+1)\pi/2].$$

1.5.3. a) The minimum of $x-2$ on $[0,1]$ is -2 and the maximum of $x+1$ on $[0,1]$ is 2. Thus $\cup_{x\in[0,1]}[x-2, x+1] = [-2,2]$.

b) The maximum of $x-1$ on $[0,1]$ is 0 and the minimum of $x+1$ on $[0,1]$ is 1. Thus $\cap_{x\in(0,1]}[x-1, x+1] = (0,1]$.

c) The minimum of $1/k$ for $k \in \mathbf{N}$ is 0 and $0 \in [-1/k, 1/k]$ for all $k \in \mathbf{N}$. Thus $\cap_{k\in\mathbf{N}}[-1/k, 1/k] = \{0\}$.

d) The maximum of $1/k$ for $k \in \mathbf{N}$ is 1. Thus $\cup_{k\in\mathbf{N}}[-1/k, 0] = [-1,0]$.

e) The maximum of $1/k$ for $k \in \mathbf{N}$ is 1 and the minimum of $-k$ for $k \in \mathbf{N}$ is $-\infty$. Thus $\cup_{k\in\mathbf{N}}[-k, 1/k) = (-\infty, 1)$.

f) The maximum of $(k-1)/k$ and the minimum of $(k+1)/k$ for $k \in \mathbf{N}$ is 1. Thus $\cap_{k\in\mathbf{N}}[(k-1)/k, (k+1)/k) = \{1\}$.

1.5.4. Suppose x belongs to the left side of (16), i.e., $x \in X$ and $x \notin \cap_{\alpha\in A} E_\alpha$. By definition, $x \in X$ and $x \notin E_\alpha$ for some $\alpha \in A$. Therefore, $x \in E_\alpha^c$ for some $\alpha \in A$, i.e., x belongs to the right side of (16). These steps are reversible.

1.5.5. a) By definition, $x \in f^{-1}(\cup_{\alpha\in A} E_\alpha)$ if and only if $f(x) \in E_\alpha$ for some $\alpha \in A$ if and only if $x \in \cup_{\alpha\in A} f^{-1}(E_\alpha)$.

b) By definition, $x \in f^{-1}(\cap_{\alpha\in A} E_\alpha)$ if and only if $f(x) \in E_\alpha$ for all $\alpha \in A$ if and only if $x \in \cap_{\alpha\in A} f^{-1}(E_\alpha)$.

c) To show $f(f^{-1}(E)) = E$, let $x \in E$. Since $E \subseteq f(X)$, choose $a \in X$ such that $x = f(a)$. By definition, $a \in f^{-1}(E)$ so $x \equiv f(a) \in f(f^{-1}(E))$. Conversely, if $x \in f(f^{-1}(E))$, then $x = f(a)$ for some $a \in f^{-1}(E)$. By definition, this means $x = f(a)$ and $f(a) \in E$. In particular, $x \in E$.

To show $E \subseteq f^{-1}(f(E))$, let $x \in E$. Then $f(x) \in f(E)$, so by definition, $x \in f^{-1}(f(E))$.

1.5.6. a) Let $C = [0,1]$ and $B = [-1,0]$. Then $C \setminus B = \{0\}$ and $f(C) = f(B) = [0,1]$. Thus $f(C \setminus B) = \{0\} \neq \emptyset = f(C) \setminus f(B)$.

b) Let $E = [0,1]$. Then $f(E) = [0,1]$ so $f^{-1}(f(E)) = [-1,1] \neq [0,1] = E$.

1.5.7. *a) implies b).* By definition, $f(A \setminus B) \supseteq f(A) \setminus f(B)$ holds whether f is 1–1 or not. To prove the reverse inequality, suppose f is 1–1 and $y \in f(A \setminus B)$. Then $y = f(a)$ for some $a \in A \setminus B$. Since f is 1–1, $a = f^{-1}(\{y\})$. Thus $y \neq f(b)$ for any $b \in B$. In particular, $y \in f(A) \setminus f(B)$.

b) implies c). By definition, $A \subseteq f^{-1}(f(A))$ holds whether f is 1–1 or not. Conversely, suppose $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$ so $f(x) = f(a)$ for some $a \in A$. If $x \notin A$, then it follows from b) that $f(A) = f(A \setminus \{x\}) = f(A) \setminus f(\{x\})$, i.e., $f(x) \notin f(A)$, a contradiction.

c) implies d). By Theorem 1.37, $f(A \cap B) \subseteq f(A) \cap f(B)$. Conversely, suppose $y \in f(A) \cap f(B)$. Then $y = f(a) = f(b)$ for some $a \in A$ and $b \in B$. If $y \notin f(A \cap B)$ then $a \notin B$ and $b \notin A$. Consequently, $f^{-1}(f(\{a\})) \supseteq \{a, b\} \supset \{a\}$, which contradicts c).

d) implies a). If f is not 1–1 then there exist $a, b \in X$ such that $a \neq b$ and $y := f(a) = f(b)$. Hence by d), $\{y\} = f(\{a\}) \cap f(\{b\}) = \emptyset$, a contradiction.

1.6 Countable and uncountable sets.

1.6.0. a) False. The function $f(x) = x$ for $x \in \mathbf{N}$ and $f(x) = 1$ for $x \in \mathbf{R} \setminus \mathbf{N}$ takes $\mathbf{R}$ onto $\mathbf{N}$, but $\mathbf{R}$ is not at most countable.

b) False. The sets $A_m := \{\frac{k}{n} : k \in \mathbf{N} \text{ and } -2^m \le k \le 2^m\}$ are finite, hence at most countable. Since the dyadic rationals are the union of the A_m's as m ranges over $\mathbf{N}$, they must be at most countable by Theorem 1.42ii.

c) True. If B were at most countable, then its subset $f(A)$ would be at most countable by Theorem 1.41, i.e., there is a function g which takes $f(A)$ onto $\mathbf{N}$. Hence by Exercise 1.6.5a, $g \circ f$ takes A onto $\mathbf{N}$. It follows from Lemma 1.40 that A is at most countable, a contradiction.

d) False, beguiling as it seems! Let $E_n = \{0, 1, \dots, 9\}$ and define f on $E_1 \times E_2 \times \cdots$ by taking each point $(x_1, x_2, \dots)$ onto the number with decimal expansion $0.x_1x_2\cdots$. Clearly (see the proof of Remark 1.39), f takes E onto $[0, 1]$. Since $[0, 1]$ is uncountable, it follows from 1.6.0c that $E_1 \times E_2 \times \cdots$ is uncountable.

1.6.1. The function $2x - 1$ is 1–1 and takes $\mathbf{N}$ onto $\{1, 3, 5, \dots\}$. Thus this set is countable by definition.

1.6.2. By two applications of Theorem 1.42i, $\mathbf{Q} \times \mathbf{Q}$ is countable, hence $\mathbf{Q}^3 := (\mathbf{Q} \times \mathbf{Q}) \times \mathbf{Q}$ is also countable.

1.6.3. Let g be a function that takes A onto B. If A is at most countable, then by Lemma 1.40 there is a function f which takes $\mathbf{N}$ onto A. It follows (see Exercise 1.6.5a) that $g \circ f$ takes $\mathbf{N}$ onto B. Hence by Lemma 1.40, B is at most countable, a contradiction.

1.6.4. By definition, there is an $n \in \mathbf{N}$ and a 1–1 function ϕ which takes $Z := \{1, 2, \dots, n\}$ onto A. Let $\psi(x) := f(\phi(x))$ for $x \in Z$. Since f and ϕ are 1–1, $\psi(x) = \psi(y)$ implies $\phi(x) = \phi(y)$ implies $x = y$. Moreover, since f and ϕ are onto, given $b \in B$ there is an $a \in A$ such that $f(a) = b$, and an $x \in Z$ such that $\phi(x) = a$, hence $\psi(x) \equiv f(\phi(x)) = f(a) = b$. Thus ψ is 1–1 from Z onto B. By definition, then, B is finite.

1.6.5. a) Repeat the proof in Exercise 1.6.4 without referring to $\mathbf{N}$ and Z.

b) By the definition of B_0, it is clear that f takes A onto B_0. Suppose $f^{-1}(x) = f^{-1}(y)$ for some $x, y \in B_0$. Since f is 1–1 from A onto B_0, it follows from Theorem 1.30 that $x = f(f^{-1}(x)) = f(f^{-1}(y)) = y$. Thus f^{-1} is 1–1 on B_0.

c) If f is 1–1 (respectively, onto), then it follows from part a) that $g \circ f$ is 1–1 (respectively, onto).

Conversely, if $g \circ f$ is 1–1 (respectively, onto), then by parts a) and b), $f \equiv g^{-1} \circ g \circ f$ is 1–1 (respectively, onto).

1.6.6. a) We prove this result by induction on n.

Suppose $n = 1$. Since $\phi : \{1\} \to \{1\}$, it must satisfy $\phi(1) = 1$. In particular, in this case ϕ is both 1–1 and onto and there is nothing to prove.

Suppose that the result holds for some integer $n \ge 1$ and let $\phi : \{1, 2, \dots, n+1\} \to \{1, 2, \dots, n+1\}$. Set $k_0 = \phi(n+1)$ and define ψ by

$$\psi(\ell) = \begin{cases} \ell & \ell < k_0 \\ \ell - 1 & \ell > k_0. \end{cases}$$

The ψ is 1–1 from $\{1, 2, \dots, k_0 - 1, k_0 + 1, \dots, n+1\}$ onto $\{1, 2, \dots, n\}$.

Suppose ϕ is 1–1 on $\{1, 2, \dots, n+1\}$. Then ϕ is 1–1 on $\{1, 2, \dots, n\}$, hence $\psi \circ \phi$ is 1–1 from $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, n\}$. It follows from the inductive hypothesis that $\psi \circ \phi$ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, n\}$. By Exercise 1.6.5, ϕ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, k_0 - 1, k_0 + 1, \dots, n+1\}$. Since $\phi(n+1) = k_0$, we conclude that ϕ takes $\{1, 2, \dots, n+1\}$ onto $\{1, 2, \dots, n+1\}$.

Conversely, if ϕ takes $\{1, 2, \dots, n+1\}$ onto $\{1, 2, \dots, n+1\}$, then ϕ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, k_0 - 1, k_0 + 1, \dots, n+1\}$, so $\psi \circ \phi$ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, n\}$. It follows from the inductive hypothesis that $\psi \circ \phi$ is 1–1 on $\{1, 2, \dots, n\}$. Hence by Exercise 1.6.5 and construction, ϕ is 1–1 on $\{1, 2, \dots, n+1\}$.

b) We may suppose that E is nonempty. Hence by hypothesis, there is an $n \in \mathbf{N}$ and a 1–1 function ϕ from E onto $\{1, 2, \dots, n\}$. Moreover, by Exercise 1.6.5b, the function ϕ^{-1} is 1–1 from $\{1, 2, \dots, n\}$ onto E.

Consider the function $\phi^{-1} \circ f \circ \phi$. Clearly, it takes $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, n\}$. Hence by part a), $\phi^{-1} \circ f \circ \phi$ is 1–1 if and only if it is onto. In particular, it follows from Exercise 1.6.5c that f is 1–1 if and only if f is onto.

1.6.7. a) Let $q = k/j$. If $k = 0$ then $n^q = 1$ is a root of the polynomial $x - 1$. If $k > 0$ then n^q is a root of the polynomial $x^j - n^k$. If $k < 0$ then n^q is a root of the polynomial $n^{-k}x^j - 1$. Thus n^q is algebraic.

b) By Theorem 1.42, there are countably many polynomials with integer coefficients. Each polynomial of degree n has at most n roots. Hence the class of algebraic numbers of degree n is a countable union of finite sets, hence countable.

c) Since any number is either algebraic or transcendental, **R** is the union of the set of algebraic numbers and the set of transcendental numbers. By b), the former set is countable. Therefore, the latter must be uncountable by the argument of Remark 1.43.

CHAPTER 2

2.1 Limits of Sequences.

2.1.0. a) True. If x_n converges, then there is an $M > 0$ such that $|x_n| \le M$. Choose by Archimedes an $N \in \mathbf{N}$ such that $N > M/\varepsilon$. Then $n \ge N$ implies $|x_n/n| \le M/n \le M/N < \varepsilon$.

b) False. $x_n = \sqrt{n}$ does not converge, but $x_n/n = 1/\sqrt{n} \to 0$ as $n \to \infty$.

c) False. $x_n = 1$ converges and $y_n = (-1)^n$ is bounded, but $x_n y_n = (-1)^n$ does not converge.

d) False. $x_n = 1/n$ converges to 0 and $y_n = n^2 > 0$, but $x_n y_n = n$ does not converge.

2.1.1. a) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > 1/\varepsilon$. Thus $n \ge N$ implies

$$|(2 - 1/n) - 2| \equiv |1/n| \le 1/N < \varepsilon.$$

b) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > \pi^2/\varepsilon^2$. Thus $n \ge N$ implies

$$|1 + \pi/\sqrt{n} - 1| \equiv |\pi/\sqrt{n}| \le \pi/\sqrt{N} < \varepsilon.$$

c) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > 3/\varepsilon$. Thus $n \ge N$ implies

$$|3(1 + 1/n) - 3| \equiv |3/n| \le 3/N < \varepsilon.$$

d) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > 1/\sqrt{3\varepsilon}$. Thus $n \ge N$ implies

$$|(2n^2 + 1)/(3n^2) - 2/3| \equiv |1/(3n^2)| \le 1/(3N^2) < \varepsilon.$$

2.1.2. a) By hypothesis, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $|x_n - 1| < \varepsilon/2$. Thus $n \ge N$ implies

$$|1 + 2x_n - 3| \equiv 2\,|x_n - 1| < \varepsilon.$$

b) By hypothesis, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n > 1/2$ and $|x_n - 1| < \varepsilon/4$. In particular, $1/x_n < 2$. Thus $n \ge N$ implies

$$|(\pi x_n - 2)/x_n - (\pi - 2)| \equiv 2\,|(x_n - 1)/x_n| < 4\,|x_n - 1| < \varepsilon.$$

c) By hypothesis, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n > 1/2$ and $|x_n - 1| < \varepsilon/(1 + 2e)$. Thus $n \ge N$ and the triangle inequality imply

$$|(x_n^2 - e)/x_n - (1 - e)| \equiv |x_n - 1|\left|1 + \frac{e}{x_n}\right| \le |x_n - 1|\left(1 + \frac{e}{|x_n|}\right) < |x_n - 1|(1 + 2e) < \varepsilon.$$

2.1.3. a) If $n_k = 2k$, then $3 - (-1)^{n_k} \equiv 2$ converges to 2; if $n_k = 2k + 1$, then $3 - (-1)^{n_k} \equiv 4$ converges to 4.

b) If $n_k = 2k$, then $(-1)^{3n_k} + 2 \equiv (-1)^{6k} + 2 = 1 + 2 = 3$ converges to 3; if $n_k = 2k + 1$, then $(-1)^{3n_k} + 2 \equiv (-1)^{6k+3} + 2 = -1 + 2 = 1$ converges to 1.

c) If $n_k = 2k$, then $(n_k - (-1)^{n_k} n_k - 1)/n_k \equiv -1/(2k)$ converges to 0; if $n_k = 2k+1$, then $(n_k - (-1)^{n_k} n_k - 1)/n_k \equiv (2n_k - 1)/n_k = (4k + 1)/(2k + 1)$ converges to 2.

2.1.4. Suppose x_n is bounded. By Definition 2.7, there are numbers M and m such that $m \le x_n \le M$ for all $n \in \mathbf{N}$. Set $C := \max\{1, |M|, |m|\}$. Then $C > 0$, $M \le C$, and $m \ge -C$. Therefore, $-C \le x_n \le C$, i.e., $|x_n| < C$ for all $n \in \mathbf{N}$.

Conversely, if $|x_n| < C$ for all $n \in \mathbf{N}$, then x_n is bounded above by C and below by $-C$.

2.1.5. If $C = 0$, there is nothing to prove. Otherwise, given $\varepsilon > 0$ use Definition 2.1 to choose an $N \in \mathbf{N}$ such that $n \ge N$ implies $|b_n| \equiv b_n < \varepsilon/|C|$. Hence by hypothesis, $n \ge N$ implies

$$|x_n - a| \le |C| b_n < \varepsilon.$$

By definition, $x_n \to a$ as $n \to \infty$.

2.1.6. If $x_n = a$ for all n, then $|x_n - a| = 0$ is less than any positive ε for all $n \in \mathbf{N}$. Thus, by definition, $x_n \to a$ as $n \to \infty$.

2.1.7. a) Let a be the common limit point. Given $\varepsilon > 0$, choose $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - a|$ and $|y_n - a|$ are both $< \varepsilon/2$. By the Triangle Inequality, $n \geq N$ implies

$$|x_n - y_n| \leq |x_n - a| + |y_n - a| < \varepsilon.$$

By definition, $x_n - y_n \to 0$ as $n \to \infty$.

b) If n converges to some a, then given $\varepsilon = 1/2$, $1 = |(n+1) - n| < |(n+1) - a| + |n - a| < 1$ for n sufficiently large, a contradiction.

c) Let $x_n = n$ and $y_n = n + 1/n$. Then $|x_n - y_n| = 1/n \to 0$ as $n \to \infty$, but neither x_n nor y_n converges.

2.1.8. By Theorem 2.6, if $x_n \to a$ then $x_{n_k} \to a$. Conversely, if $x_{n_k} \to a$ for every subsequence, then it converges for the "subsequence" x_n.

2.2 Limit Theorems.

2.2.0. a) False. Let $x_n = n^2$ and $y_n = -n$ and note by Exercise 2.2.2a that $x_n + y_n \to \infty$ as $n \to \infty$.

b) True. Let $\varepsilon > 0$. If $x_n \to -\infty$ as $n \to \infty$, then choose $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n < -1/\varepsilon$. Then $x_n < 0$ so $|x_n| = -x_n > 0$. Multiply $x_n < -1/\varepsilon$ by $\varepsilon/(-x_n)$ which is positive. We obtain $-\varepsilon < 1/x_n$, i.e., $|1/x_n| = -1/x_n < \varepsilon$.

c) False. Let $x_n = (-1)^n/n$. Then $1/x_n = (-1)^n n$ has no limit as $n \to \infty$.

d) True. Since $(2^x - x)' = 2^x \log 2 - 1 > 1$ for all $x \geq 2$, i.e., $2^x - x$ is increasing on $[2, \infty)$. In particular, $2^x - x \geq 2^2 - 2 > 0$, i.e., $2^x > x$ for $x \geq 2$. Thus, since $x_n \to \infty$ as $n \to \infty$, we have $2^{x_n} > x_n$ for n large, hence

$$2^{-x_n} < \frac{1}{x_n} \to 0$$

as $n \to \infty$.

2.2.1. a) $|x_n| \leq 1/n \to 0$ as $n \to \infty$ and we can apply the Squeeze Theorem.

b) $2n/(n^2 + \pi) = (2/n)/(1 + \pi/n^2) \to 0/(1+0) = 0$ by Theorem 2.12.

c) $(\sqrt{2n} + 1)/(n + \sqrt{2}) = ((\sqrt{2}/\sqrt{n}) + (1/n))/(1 + (\sqrt{2}/n)) \to 0/(1+0) = 0$ by Exercise 2.2.5 and Theorem 2.12.

d) An easy induction argument shows that $2n + 1 < 2^n$ for $n = 3, 4, \dots$. We will use this to prove that $n^2 \leq 2^n$ for $n = 4, 5, \dots$. It's surely true for $n = 4$. If it's true for some $n \geq 4$, then the inductive hypothesis and the fact that $2n + 1 < 2^n$ imply

$$(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 < 2^n + 2^n = 2^{n+1}$$

so the second inequality has been proved.

Now the second inequality implies $n/2^n < 1/n$ for $n \geq 4$. Hence by the Squeeze Theorem, $n/2^n \to 0$ as $n \to \infty$.

2.2.2. a) Let $M \in \mathbf{R}$ and choose by Archimedes an $N \in \mathbf{N}$ such that $N > \max\{M, 2\}$. Then $n \geq N$ implies $n^2 - n = n(n-1) \geq N(N-1) > M(2-1) = M$.

b) Let $M \in \mathbf{R}$ and choose by Archimedes an $N \in \mathbf{N}$ such that $N > -M/2$. Notice that $n \geq 1$ implies $-3n \leq -3$ so $1 - 3n \leq -2$. Thus $n \geq N$ implies $n - 3n^2 = n(1 - 3n) \leq -2n \leq -2N < M$.

c) Let $M \in \mathbf{R}$ and choose by Archimedes an $N \in \mathbf{N}$ such that $N > M$. Then $n \geq N$ implies $(n^2 + 1)/n = n + 1/n > N + 0 > M$.

d) Let $M \in \mathbf{R}$ satisfy $M \leq 0$. Then $2 + \sin\theta \geq 2 - 1 = 1$ implies $n^2(2 + \sin(n^3 + n + 1)) \geq n^2 \cdot 1 > 0 \geq M$ for all $n \in \mathbf{N}$. On the other hand, if $M > 0$, then choose by Archimedes an $N \in \mathbf{N}$ such that $N > \sqrt{M}$. Then $n \geq N$ implies $n^2(2 + \sin(n^3 + n + 1)) \geq n^2 \cdot 1 \geq N^2 > M$.

2.2.3. a) Following Example 2.13,

$$\frac{2 + 3n - 4n^2}{1 - 2n + 3n^2} = \frac{(2/n^2) + (3/n) - 4}{(1/n^2) - (2/n) + 3} \to \frac{-4}{3}$$

as $n \to \infty$.

b) Following Example 2.13,

$$\frac{n^3 + n - 2}{2n^3 + n - 2} = \frac{1 + (1/n^2) - (2/n^3)}{2 + (1/n^2) - (2/n^3)} \to \frac{1}{2}$$

as $n \to \infty$.

c) Rationalizing the expression, we obtain

$$\sqrt{3n+2}-\sqrt{n}=\frac{(\sqrt{3n+2}-\sqrt{n})(\sqrt{3n+2}+\sqrt{n})}{\sqrt{3n+2}+\sqrt{n}}=\frac{2n+2}{\sqrt{3n+2}+\sqrt{n}}\to\infty$$

as $n\to\infty$ by the method of Example 2.13. (Multiply top and bottom by $1/\sqrt{n}$.)

d) Multiply top and bottom by $1/\sqrt{n}$ to obtain

$$\frac{\sqrt{4n+1}-\sqrt{n}}{\sqrt{9n+1}-\sqrt{n+2}}=\frac{\sqrt{4+1/n}-\sqrt{1-1/n}}{\sqrt{9+1/n}-\sqrt{1+2/n}}\to\frac{2-1}{3-1}=\frac{1}{2}.$$

2.2.4. a) Clearly,

$$\frac{x_n}{y_n}-\frac{x}{y}=\frac{x_ny-xy_n}{yy_n}=\frac{x_ny-xy+xy-xy_n}{yy_n}.$$

Thus

$$\left|\frac{x_n}{y_n}-\frac{x}{y}\right|\le\frac{1}{|y_n|}|x_n-x|+\frac{|x|}{|yy_n|}|y_n-y|.$$

Since $y\neq 0$, $|y_n|\ge|y|/2$ for large n. Thus

$$\left|\frac{x_n}{y_n}-\frac{x}{y}\right|\le\frac{2}{|y|}|x_n-x|+\frac{2|x|}{|y|^2}|y_n-y|\to 0$$

as $n\to\infty$ by Theorem 2.12i and ii. Hence by the Squeeze Theorem, $x_n/y_n\to x/y$ as $n\to\infty$.

b) By symmetry, we may suppose that $x=y=\infty$. Since $y_n\to\infty$ implies $y_n>0$ for n large, we can apply Theorem 2.15 directly to obtain the conclusions when $\alpha>0$. For the case $\alpha<0$, $x_n>M$ implies $\alpha x_n<\alpha M$. Since any $M_0\in\mathbf{R}$ can be written as αM for some $M\in\mathbf{R}$, we see by definition that $x_n\to-\infty$ as $n\to\infty$.

2.2.5. *Case 1.* $x=0$. Let $\epsilon>0$ and choose N so large that $n\ge N$ implies $|x_n|<\epsilon^2$. By (8) in 1.1, $\sqrt{x_n}<\epsilon$ for $n\ge N$, i.e., $\sqrt{x_n}\to 0$ as $n\to\infty$.

Case 2. $x>0$. Then

$$\sqrt{x_n}-\sqrt{x}=(\sqrt{x_n}-\sqrt{x})\left(\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}}\right)=\frac{x_n-x}{\sqrt{x_n}+\sqrt{x}}.$$

Since $\sqrt{x_n}\ge 0$, it follows that

$$|\sqrt{x_n}-\sqrt{x}|\le\frac{|x_n-x|}{\sqrt{x}}.$$

This last quotient converges to 0 by Theorem 2.12. Hence it follows from the Squeeze Theorem that $\sqrt{x_n}\to\sqrt{x}$ as $n\to\infty$.

2.2.6. By the Density of Rationals, there is an r_n between $x+1/n$ and x for each $n\in\mathbf{N}$. Since $|x-r_n|<1/n$, it follows from the Squeeze Theorem that $r_n\to x$ as $n\to\infty$.

2.2.7. a) By Theorem 2.9 we may suppose that $|x|=\infty$. By symmetry, we may suppose that $x=\infty$. By definition, given $M\in\mathbf{R}$, there is an $N\in\mathbf{N}$ such that $n\ge N$ implies $x_n>M$. Since $w_n\ge x_n$, it follows that $w_n>M$ for all $n\ge N$. By definition, then, $w_n\to\infty$ as $n\to\infty$.

b) If x and y are finite, then the result follows from Theorem 2.17. If $x=y=\pm\infty$ or $-x=y=\infty$, there is nothing to prove. It remains to consider the case $x=\infty$ and $y=-\infty$. But by Definition 2.14 (with $M=0$), $x_n>0>y_n$ for n sufficiently large, which contradicts the hypothesis $x_n\le y_n$.

2.2.8. a) Take the limit of $x_{n+1}=1-\sqrt{1-x_n}$, as $n\to\infty$. We obtain $x=1-\sqrt{1-x}$, i.e., $x^2-x=0$. Therefore, $x=0,1$.

b) Take the limit of $x_{n+1}=2+\sqrt{x_n-2}$ as $n\to\infty$. We obtain $x=2+\sqrt{x-2}$, i.e., $x^2-5x+6=0$. Therefore, $x=2,3$. But $x_1>3$ and induction shows that $x_{n+1}=2+\sqrt{x_n-2}>2+\sqrt{3-2}=3$, so the limit must be $x=3$.

c) Take the limit of $x_{n+1}=\sqrt{2+x_n}$ as $n\to\infty$. We obtain $x=\sqrt{2+x}$, i.e., $x^2-x-2=0$. Therefore, $x=2,-1$. But $x_{n+1}=\sqrt{2+x_n}\ge 0$ by definition (all square roots are nonnegative), so the limit must be $x=2$.

This proof doesn't change if $x_1>-2$, so the limit is again $x=2$.

2.2.9. a) Let $E = \{k \in \mathbf{Z} : k \geq 0 \text{ and } k \leq 10^{n+1}y\}$. Since $10^{n+1}y < 10$, $E \subseteq \{0, 1, \dots, 9\}$. Hence $w := \sup E \in E$. It follows that $w \leq 10^{n+1}y$, i.e., $w/10^{n+1} \leq y$. On the other hand, since $w + 1$ is not the supremum of E, $w + 1 > 10^{n+1}y$. Therefore, $y < w/10^{n+1} + 1/10^{n+1}$.

b) Apply a) for $n = 0$ to choose $x_1 = w$ such that $x_1/10 \leq x < x_1/10 + 1/10$. Suppose

$$s_n := \sum_{k=1}^{n} \frac{x_k}{10^k} \leq x < \sum_{k=1}^{n} \frac{x_k}{10^k} + \frac{1}{10^n}.$$

Then $0 < x - s_n < 1/10^n$, so by a) choose x_{n+1} such that $x_{n+1}/10^{n+1} \leq x - s_n < x_{n+1}/10^{n+1} + 1/10^{n+1}$, i.e.,

$$\sum_{k=1}^{n+1} \frac{x_k}{10^k} \leq x < \sum_{k=1}^{n+1} \frac{x_k}{10^k} + \frac{1}{10^{n+1}}.$$

c) Combine b) with the Squeeze Theorem.

d) Since an easy induction proves that $9^n > n$ for all $n \in \mathbf{N}$, we have $9^{-n} < 1/n$. Hence the Squeeze Theorem implies that $9^{-n} \to 0$ as $n \to \infty$. Hence, it follows from Exercise 1.4.4c and definition that

$$.4999 \cdots = \frac{4}{10} + \lim_{n\to\infty} \sum_{k=2}^{n} \frac{9}{10^k} = \frac{4}{10} + \lim_{n\to\infty} \frac{1}{10}\left(1 - \frac{1}{9^n}\right) = \frac{4}{10} + \frac{1}{10} = 0.5.$$

Similarly,

$$.999 \cdots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = \lim_{n\to\infty} \left(1 - \frac{1}{9^n}\right) = 1.$$

2.3 The Bolzano–Weierstrass Theorem.

2.3.0. a) False. $x_n = 1/4 + 1/(n + 4)$ is strictly decreasing and $|x_n| \leq 1/4 + 1/5 < 1/2$, but $x_n \to 1/4$ as $n \to \infty$.

b) True. Since $(n - 1)/(2n - 1) \to 1/2$ as $n \to \infty$, this factor is bounded. Since $|\cos(n^2 + n + 1)| \leq 1$, it follows that $\{x_n\}$ is bounded. Hence it has a convergent subsequence by the Bolzano–Weierstrass Theorem.

c) False. $x_n = 1/2 - 1/n$ is strictly increasing and $|x_n| \leq 1/2 < 1 + 1/n$, but $x_n \to 1/2$ as $n \to \infty$.

d) False. $x_n = (1 + (-1)^n)n$ satisfies $x_n = 0$ for n odd and $x_n = 2n$ for n even. Thus $x_{2k+1} \to 0$ as $k \to \infty$, but x_n is NOT bounded.

2.3.1. Suppose that $-1 < x_{n-1} < 0$ for some $n \geq 0$. Then $0 < x_{n-1} + 1 < 1$ so $0 < x_{n-1} + 1 < \sqrt{x_{n-1} + 1}$ and it follows that $x_{n-1} < \sqrt{x_{n-1} + 1} - 1 = x_n$. Moreover, $\sqrt{x_{n-1} + 1} - 1 \leq 1 - 1 = 0$. Hence by induction, x_n is increasing and bounded above by 0. It follows from the Monotone Convergence Theorem that $x_n \to a$ as $n \to \infty$. Taking the limit of $\sqrt{x_{n-1} + 1} - 1 = x_n$ we see that $a^2 + a = 0$, i.e., $a = -1, 0$. Since x_n increases from $x_0 > -1$, the limit is 0. If $x_0 = -1$, then $x_n = -1$ for all n. If $x_0 = 0$, then $x_n = 0$ for all n.

Finally, it is easy to verify that if $x_0 = \ell$ for $\ell = -1$ or 0, then $x_n = \ell$ for all n, hence $x_n \to \ell$ as $n \to \infty$.

2.3.2. If $x_1 = 0$ then $x_n = 0$ for all n, hence converges to 0. If $0 < x_1 < 1$, then by 1.4.1c, x_n is decreasing and bounded below. Thus the limit, a, exists by the Monotone Convergence Theorem. Taking the limit of $x_{n+1} = 1 - \sqrt{1 - x_n}$, as $n \to \infty$, we have $a = 1 - \sqrt{1 - a}$, i.e., $a = 0, 1$. Since $x_1 < 1$, the limit must be zero.

Finally,

$$\frac{x_{n+1}}{x_n} = \frac{1 - \sqrt{1 - x_n}}{x_n} = \frac{1 - (1 - x_n)}{x_n(1 + \sqrt{1 - x_n})} \to \frac{1}{1 + 1} = \frac{1}{2}.$$

2.3.3. *Case 1.* $x_0 = 2$. Then $x_n = 2$ for all n, so the limit is 2.

Case 2. $2 < x_0 < 3$. Suppose that $2 < x_{n-1} \leq 3$ for some $n \geq 1$. Then $0 < x_{n-1} - 2 \leq 1$ so $\sqrt{x_{n-1} - 2} \geq x_{n-1} - 2$, i.e., $x_n = 2 + \sqrt{x_{n-1} - 2} \geq x_{n-1}$. Moreover, $x_n = 2 + \sqrt{x_{n-1} - 2} \leq 2 + 1 = 3$. Hence by induction, x_n is increasing and bounded above by 3. It follows from the Monotone Convergence Theorem that $x_n \to a$ as $n \to \infty$. Taking the limit of $2 + \sqrt{x_{n-1} - 2} = x_n$ we see that $a^2 - 5a + 6 = 0$, i.e., $a = 2, 3$. Since x_n increases from $x_0 > 2$, the limit is 3.

Case 3. $x_0 \geq 3$. Suppose that $x_{n-1} \geq 3$ for some $n \geq 1$. Then $x_{n-1} - 2 \geq 1$ so $\sqrt{x_{n-1} - 2} \leq x_{n-1} - 2$, i.e., $x_n = 2 + \sqrt{x_{n-1} - 2} \leq x_{n-1}$. Moreover, $x_n = 2 + \sqrt{x_{n-1} - 2} \geq 2 + 1 = 3$. Hence by induction, x_n is decreasing and bounded above by 3. By repeating the steps in Case 2, we conclude that x_n decreases from $x_0 \geq 3$ to the limit 3.

2.3.4. *Case 1.* $x_0 < 1$. Suppose $x_{n-1} < 1$. Then

$$x_{n-1} = \frac{2x_{n-1}}{2} < \frac{1 + x_{n-1}}{2} = x_n < \frac{2}{2} = 1.$$

Thus $\{x_n\}$ is increasing and bounded above, so $x_n \to x$. Taking the limit of $x_n = (1 + x_{n-1})/2$ as $n \to \infty$, we see that $x = (1 + x)/2$, i.e., $x = 1$.

Case 2. $x_0 \geq 1$. If $x_{n-1} \geq 1$ then

$$1 = \frac{2}{2} \leq \frac{1 + x_{n-1}}{2} = x_n \leq \frac{2x_{n-1}}{2} = x_{n-1}.$$

Thus $\{x_n\}$ is decreasing and bounded below. Repeating the argument in Case 1, we conclude that $x_n \to 1$ as $n \to \infty$.

2.3.5. The result is obvious when $x = 0$. If $x > 0$ then by Example 2.2 and Theorem 2.6,

$$\lim_{n \to \infty} x^{1/(2n-1)} = \lim_{m \to \infty} x^{1/m} = 1.$$

If $x < 0$ then since $2n - 1$ is odd, we have by the previous case that $x^{1/(2n-1)} = -(-x)^{1/(2n-1)} \to -1$ as $n \to \infty$.

2.3.6. a) Suppose that $\{x_n\}$ is increasing. If $\{x_n\}$ is bounded above, then there is an $x \in \mathbf{R}$ such that $x_n \to x$ (by the Monotone Convergence Theorem). Otherwise, given any $M > 0$ there is an $N \in \mathbf{N}$ such that $x_N > M$. Since $\{x_n\}$ is increasing, $n \geq N$ implies $x_n \geq x_N > M$. Hence $x_n \to \infty$ as $n \to \infty$.

b) If $\{x_n\}$ is decreasing, then $-x_n$ is increasing, so part a) applies.

2.3.7. Choose by the Approximation Property an $x_1 \in E$ such that $\sup E - 1 < x_1 \leq \sup E$. Since $\sup E \notin E$, we also have $x_1 < \sup E$. Suppose $x_1 < x_2 < \cdots < x_n$ in E have been chosen so that $\sup E - 1/n < x_n < \sup E$. Choose by the Approximation Property an $x_{n+1} \in E$ such that $\max\{x_n, \sup E - 1/(n+1)\} < x_{n+1} \leq \sup E$. Then $\sup E - 1/(n+1) < x_{n+1} < \sup E$ and $x_n < x_{n+1}$. Thus by induction, $x_1 < x_2 < \ldots$ and by the Squeeze Theorem, $x_n \to \sup E$ as $n \to \infty$.

2.3.8. a) This follows immediately from Exercise 1.2.6.

b) By a), $x_{n+1} = (x_n + y_n)/2 < 2x_n/2 = x_n$. Thus $y_{n+1} < x_{n+1} < \cdots < x_1$. Similarly, $y_n = \sqrt{y_n^2} < \sqrt{x_n y_n} = y_{n+1}$ implies $x_{n+1} > y_{n+1} > y_n \cdots > y_1$. Thus $\{x_n\}$ is decreasing and bounded below by y_1 and $\{y_n\}$ is increasing and bounded above by x_1.

c) By b),

$$x_{n+1} - y_{n+1} = \frac{x_n + y_n}{2} - \sqrt{x_n y_n} < \frac{x_n + y_n}{2} - y_n = \frac{x_n - y_n}{2}.$$

Hence by induction and a), $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/2^n$.

d) By b), there exist $x, y \in \mathbf{R}$ such that $x_n \downarrow x$ and $y_n \uparrow y$ as $n \to \infty$. By c), $|x - y| \leq (x_1 - y_1) \cdot 0 = 0$. Hence $x = y$.

2.3.9. Since $x_0 = 1$ and $y_0 = 0$,

$$\begin{aligned} x_{n+1}^2 - 2y_{n+1}^2 &= (x_n + 2y_n)^2 - 2(x_n + y_n)^2 \\ &= -x_n^2 + 2y_n^2 = \cdots = (-1)^n(x_0 - 2y_0) = (-1)^n. \end{aligned}$$

Notice that $x_1 = 1 = y_1$. If $y_{n-1} \geq n - 1$ and $x_{n-1} \geq 1$ then $y_n = x_{n-1} + y_{n-1} \geq 1 + (n-1) = n$ and $x_n = x_{n-1} + 2y_{n-1} \geq 1$. Thus $1/y_n \to 0$ as $n \to \infty$ and $x_n \geq 1$ for all $n \in \mathbf{N}$. Since

$$\left| \frac{x_n^2}{y_n^2} - 2 \right| = \left| \frac{x_n^2 - 2y_n^2}{y_n^2} \right| = \frac{1}{y_n^2} \to 0$$

as $n \to \infty$, it follows that $x_n/y_n \to \pm\sqrt{2}$ as $n \to \infty$. Since $x_n, y_n > 0$, the limit must be $\sqrt{2}$.

2.3.10. a) Notice $x_0 > y_0 > 1$. If $x_{n-1} > y_{n-1} > 1$ then $y_{n-1}^2 - x_{n-1}y_{n-1} = y_{n-1}(y_{n-1} - x_{n-1}) > 0$ so $y_{n-1}(y_{n-1} + x_{n-1}) < 2x_{n-1}y_{n-1}$. In particular,

$$x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}} > y_{n-1}.$$

It follows that $\sqrt{x_n} > \sqrt{y_{n-1}} > 1$, so $x_n > \sqrt{x_n y_{n-1}} = y_n > 1 \cdot 1 = 1$. Hence by induction, $x_n > y_n > 1$ for all $n \in \mathbf{N}$.

Now $y_n < x_n$ implies $2y_n < x_n + y_n$. Thus

$$x_{n+1} = \frac{2x_n y_n}{x_n + y_n} < x_n.$$

Hence, $\{x_n\}$ is decreasing and bounded below (by 1). Thus by the Monotone Convergence Theorem, $x_n \to x$ for some $x \in \mathbf{R}$.

On the other hand, y_{n+1} is the geometric mean of x_{n+1} and y_n, so by Exercise 1.2.6, $y_{n+1} \geq y_n$. Since y_n is bounded above (by x_0), we conclude that $y_n \to y$ as $n \to \infty$ for some $y \in \mathbf{R}$.

b) Let $n \to \infty$ in the identity $y_{n+1} = \sqrt{x_{n+1}y_n}$. We obtain, from part a), $y = \sqrt{xy}$, i.e., $x = y$. A direct calculation yields $y_6 > 3.141557494$ and $x_7 < 3.14161012$.

2.4 Cauchy sequences.

2.4.0. a) False. $a_n = 1$ is Cauchy and $b_n = (-1)^n$ is bounded, but $a_n b_n = (-1)^n$ does not converge, hence cannot be Cauchy by Theorem 2.29.

b) False. $a_n = 1$ and $b_n = 1/n$ are Cauchy, but $a_n/b_n = n$ does not converge, hence cannot be Cauchy by Theorem 2.29.

c) True. If $(a_n + b_n)^{-1}$ converged to 0, then given any $M \in \mathbf{R}$, $M \neq 0$, there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $|a_n + b_n|^{-1} < 1/|M|$. It follows that $n \geq N$ implies $|a_n + b_n| > |M| > 0 > M$. In particular, $|a_n + b_n|$ diverges to ∞. But if a_n and b_n are Cauchy, then by Theorem 2.29, $a_n + b_n \to x$ where $x \in \mathbf{R}$. Thus $|a_n + b_n| \to |x|$, NOT ∞.

d) False. If $x_{2^k} = \log k$ and $x_n = 0$ for $n \neq 2^k$, then $x_{2^k} - x_{2^{k-1}} = \log(k/(k-1)) \to 0$ as $k \to \infty$, but x_k does not converge, hence cannot be Cauchy by Theorem 2.29.

2.4.1. Since $(2n^2 + 3)/(n^3 + 5n^2 + 3n + 1) \to 0$ as $n \to \infty$, it follows from the Squeeze Theorem that $x_n \to 0$ as $n \to \infty$. Hence by Theorem 2.29, x_n is Cauchy.

2.4.2. If x_n is Cauchy, then there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - x_N| < 1$. Since $x_n - x_N \in \mathbf{Z}$, it follows that $x_n = x_N$ for all $n \geq N$. Thus set $a := x_N$.

2.4.3. Suppose x_n and y_n are Cauchy and let $\varepsilon > 0$.

a) If $\alpha = 0$, then $\alpha x_n = 0$ for all $n \in \mathbf{N}$, hence is Cauchy. If $\alpha \neq 0$, then there is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x_m| < \varepsilon/|\alpha|$. Hence

$$|\alpha x_n - \alpha x_m| \leq |\alpha|\,|x_n - x_m| < \varepsilon$$

for $n, m \geq N$.

b) There is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x_m|$ and $|y_n - y_m|$ are $< \varepsilon/2$. Hence

$$|x_n + y_n - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \varepsilon$$

for $n, m \geq N$.

c) By repeating the proof of Theorem 2.8, we can show that every Cauchy sequence is bounded. Thus choose $M > 0$ such that $|x_n|$ and $|y_n|$ are both $\leq M$ for all $n \in \mathbf{N}$. There is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x_m|$ and $|y_n - y_m|$ are both $< \varepsilon/(2M)$. Hence

$$|x_n y_n - (x_m y_m)| \leq |x_n - x_m|\,|y_m| + |x_n|\,|y_n - y_m| < \varepsilon$$

for $n, m \geq N$.

2.4.4. Let $s_n = \sum_{k=1}^{n-1} x_k$ for $n = 2, 3, \ldots$. If $m > n$ then $s_{m+1} - s_n = \sum_{k=n}^{m} x_k$. Therefore, s_n is Cauchy by hypothesis. Hence s_n converges by Theorem 2.29.

2.4.5. Let $x_n = \sum_{k=1}^n (-1)^k/k$ for $n \in \mathbf{N}$. Suppose n and m are even and $m > n$. Then

$$S := \sum_{k=n}^{m} \frac{(-1)^k}{k} \equiv \frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \cdots - \left(\frac{1}{m-1} - \frac{1}{m}\right).$$

Each term in parentheses is positive, so the absolute value of S is dominated by $1/n$. Similar arguments prevail for all integers n and m. Since $1/n \to 0$ as $n \to \infty$, it follows that x_n satisfies the hypotheses of Exercise 2.4.4. Hence x_n must converge to a finite real number.

2.4.6. By Exercise 1.4.4c, if $m \geq n$ then

$$|x_{m+1} - x_n| = |\sum_{k=n}^{m}(x_{k+1} - x_k)| \leq \sum_{k=n}^{m} \frac{1}{a^k} = \left(1 - \frac{1}{a^m} - (1 - \frac{1}{a^n})\right) \frac{1}{a-1}.$$

Thus $|x_{m+1} - x_n| \leq (1/a^n - 1/a^m)/(a-1) \to 0$ as $n, m \to \infty$ since $a > 1$. Hence $\{x_n\}$ is Cauchy and must converge by Theorem 2.29.

2.4.7. a) Suppose a is a cluster point for some set E and let $r > 0$. Since $E \cap (a-r, a+r)$ contains infinitely many points, so does $E \cap (a-r, a+r) \setminus \{a\}$. Hence this set is nonempty. Conversely, if $E \cap (a-s, a+s) \setminus \{a\}$ is always nonempty for all $s > 0$ and $r > 0$ is given, choose $x_1 \in E \cap (a-r, a+r)$. If distinct points $x_1, \dots, x_k$ have been chosen so that $x_k \in E \cap (a-r, a+r)$ and $s := \min\{|x_1 - a|, \dots, |x_k - a|\}$, then by hypothesis there is an $x_{k+1} \in E \cap (a-s, a+s)$. By construction, x_{k+1} does not equal any x_j for $1 \leq j \leq k$. Hence $x_1, \dots, x_{k+1}$ are distinct points in $E \cap (a-r, a+r)$. By induction, there are infinitely many points in $E \cap (a-r, a+r)$.

b) If E is a bounded infinite set, then it contains distinct points $x_1, x_2, \dots$. Since $\{x_n\} \subseteq E$, it is bounded. It follows from the Bolzano–Weierstrass Theorem that x_n contains a convergent subsequence, i.e., there is an $a \in \mathbf{R}$ such that given $r > 0$ there is an $N \in \mathbf{N}$ such that $k \geq N$ implies $|x_{n_k} - a| < r$. Since there are infinitely many x_{n_k}'s and they all belong to E, a is by definition a cluster point of E.

2.4.8. a) To show $E := [a, b]$ is sequentially compact, let $x_n \in E$. By the Bolzano–Weierstrass Theorem, x_n has a convergent subsequence, i.e., there is an $x_0 \in \mathbf{R}$ and integers n_k such that $x_{n_k} \to x_0$ as $k \to \infty$. Moreover, by the Comparison Theorem, $x_n \in E$ implies $x_0 \in E$. Thus E is sequentially compact by definition.

b) $(0, 1)$ is bounded and $1/n \in (0, 1)$ has no convergent subsequence with limit in $(0, 1)$.

c) $[0, \infty)$ is closed and $n \in [0, \infty)$ is a sequence which has no convergent subsequence.

2.5 Limits supremum and infimum.

2.5.1. a) Since $3 - (-1)^n = 2$ when n is even and 4 when n is odd, $\limsup_{n\to\infty} x_n = 4$ and $\liminf_{n\to\infty} x_n = 2$.

b) Since $\cos(n\pi/2) = 0$ if n is odd, 1 if $n = 4m$ and -1 if $n = 4m + 2$, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

c) Since $(-1)^{n+1} + (-1)^n/n = -1 + 1/n$ when n is even and $1 - 1/n$ when n is odd, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

d) Since $x_n \to 1/2$ as $n \to \infty$, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 1/2$ by Theorem 2.36.

e) Since $|y_n| \leq M$, $|y_n/n| \leq M/n \to 0$ as $n \to \infty$. Therefore, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 0$ by Theorem 2.36.

f) Since $n(1 + (-1)^n) + n^{-1}((-1)^n - 1) = 2n$ when n is even and $-2/n$ when n is odd, $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 0$.

g) Clearly $x_n \to \infty$ as $n \to \infty$. Therefore, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \infty$ by Theorem 2.36.

2.5.2. By Theorem 1.20,

$$\liminf_{n\to\infty}(-x_n) := \lim_{n\to\infty}(\inf_{k\geq n}(-x_k)) = -\lim_{n\to\infty}(\sup_{k\geq n} x_k) = -\limsup_{n\to\infty} x_n.$$

A similar argument establishes the second identity.

2.5.3. a) Since $\lim_{n\to\infty}(\sup_{k\geq n} x_k) < r$, there is an $N \in \mathbf{N}$ such that $\sup_{k\geq N} x_k < r$, i.e., $x_k < r$ for all $k \geq N$.

b) Since $\lim_{n\to\infty}(\sup_{k\geq n} x_k) > r$, there is an $N \in \mathbf{N}$ such that $\sup_{k\geq N} x_k > r$, i.e., there is a $k_1 \in \mathbf{N}$ such that $x_{k_1} > r$. Suppose $k_\nu \in \mathbf{N}$ have been chosen so that $k_1 < k_2 < \cdots < k_j$ and $x_{k_\nu} > r$ for $\nu = 1, 2, \dots, j$. Choose $N > k_j$ such that $\sup_{k\geq N} x_k > r$. Then there is a $k_{j+1} > N > k_j$ such that $x_{k_{j+1}} > r$. Hence by induction, there are distinct natural numbers $k_1, k_2, \dots$ such that $x_{k_j} > r$ for all $j \in \mathbf{N}$.

2.5.4. a) Since $\inf_{k\ge n} x_k + \inf_{k\ge n} y_k$ is a lower bound of $x_j + y_j$ for any $j \ge n$, we have $\inf_{k\ge n} x_k + \inf_{k\ge n} y_k \le \inf_{j\ge n}(x_j + y_j)$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty}(x_n + y_n).$$

Note, we used Corollary 1.16 and the fact that the sum on the left is not of the form $\infty - \infty$. Similarly, for each $j \ge n$,

$$\inf_{k\ge n}(x_k + y_k) \le x_j + y_j \le \sup_{k\ge n} x_k \; + y_j.$$

Taking the infimum of this inequality over all $j \ge n$, we obtain $\inf_{k\ge n}(x_k+y_k) \le \sup_{k\ge n} x_k + \inf_{j\ge n} y_j$. Therefore,

$$\liminf_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

The remaining two inequalities follow from Exercise 2.5.2. For example,

$$\begin{aligned}\limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &= -\liminf_{n\to\infty}(-x_n) - \limsup_{n\to\infty}(-y_n)\\ &\le -\liminf_{n\to\infty}(-x_n - y_n) = \limsup_{n\to\infty}(x_n + y_n).\end{aligned}$$

b) It suffices to prove the first identity. By Theorem 2.36 and a),

$$\lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty}(x_n + y_n).$$

To obtain the reverse inequality, notice by the Approximation Property that for each $n \in \mathbf{N}$ there is a $j_n > n$ such that $\inf_{k\ge n}(x_k + y_k) > x_{j_n} - 1/n + y_{j_n}$. Hence

$$\inf_{k\ge n}(x_k + y_k) > x_{j_n} - \frac{1}{n} + \inf_{k\ge n} y_k$$

for all $n \in \mathbf{N}$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$\liminf_{n\to\infty}(x_n + y_n) \ge \lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

c) Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then the limits infimum are both -1, the limits supremum are both 1, but $x_n + y_n = 0 \to 0$ as $n \to \infty$. If $x_n = (-1)^n$ and $y_n = 0$ then

$$\liminf_{n\to\infty}(x_n + y_n) = -1 < 1 = \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

2.5.5. a) For any $j \ge n$, $x_j \le \sup_{k\ge n} x_k$ and $y_j \le \sup_{k\ge n} y_k$. Multiplying these inequalities, we have $x_j y_j \le (\sup_{k\ge n} x_k)(\sup_{k\ge n} y_k)$, i.e.,

$$\sup_{j\ge n} x_j y_j \le (\sup_{k\ge n} x_k)(\sup_{k\ge n} y_k).$$

Taking the limit of this inequality as $n \to \infty$ establishes a). The inequality can be strict because if

$$x_n = 1 - y_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

then $\limsup_{n\to\infty}(x_n y_n) = 0 < 1 = (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n)$.

b) By a),

$$\liminf_{n\to\infty}(x_n y_n) = -\limsup_{n\to\infty}(-x_n y_n) \ge -\limsup_{n\to\infty}(-x_n) \limsup_{n\to\infty} y_n = \liminf_{n\to\infty} x_n \limsup_{n\to\infty} y_n.$$

2.5.6. *Case 1.* $x = \infty$. By hypothesis, $C := \limsup_{n\to\infty} y_n > 0$. Let $M > 0$ and choose $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \ge 2M/C$ and $\sup_{n\ge N} y_n > C/2$. Then $\sup_{k\ge N}(x_k y_k) \ge x_n y_n \ge (2M/C) y_n$ for any $n \ge N$ and $\sup_{k\ge N}(x_k y_k) \ge (2M/C) \sup_{n\ge N} y_n > M$. Therefore, $\limsup_{n\to\infty}(x_n y_n) = \infty$.

Case 2. $0 \le x < \infty$. By Exercise 2.5.6a and Theorem 2.36,

$$\limsup_{n\to\infty}(x_n y_n) \le (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n) = x \limsup_{n\to\infty} y_n.$$

On the other hand, given $\epsilon > 0$ choose $n \in \mathbf{N}$ so that $x_k > x - \epsilon$ for $k \ge n$. Then $x_k y_k \ge (x-\epsilon) y_k$ for each $k \ge n$, i.e., $\sup_{k\ge n}(x_k y_k) \ge (x-\epsilon)\sup_{k\ge n} y_k$. Taking the limit of this inequality as $n \to \infty$ and as $\epsilon \to 0$, we obtain

$$\limsup_{n\to\infty}(x_n y_n) \ge x \limsup_{n\to\infty} y_n.$$

2.5.7. It suffices to prove the first identity. Let $s = \inf_{n\in\mathbf{N}}(\sup_{k\ge n} x_k)$.
Case 1. $s = \infty$. Then $\sup_{k\ge n} x_k = \infty$ for all $n \in \mathbf{N}$ so by definition,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}(\sup_{k\ge n} x_k) = \infty = s.$$

Case 2. $s = -\infty$. Let $M > 0$ and choose $N \in \mathbf{N}$ such that $\sup_{k\ge N} x_k \le -M$. Then $\sup_{k\ge n} x_k \le \sup_{k\ge N} x_k \le -M$ for all $n \ge N$, i.e., $\limsup_{n\to\infty} x_n = -\infty$.
Case 3. $-\infty < s < -\infty$. Let $\epsilon > 0$ and use the Approximation Property to choose $N \in \mathbf{N}$ such that $\sup_{k\ge N} x_k < s + \epsilon$. Since $\sup_{k\ge n} x_k \le \sup_{k\ge N} x_k < s+\epsilon$ for all $n \ge N$, it follows that

$$s - \epsilon < s \le \sup_{k\ge n} x_k < s + \epsilon$$

for $n \ge N$, i.e., $\limsup_{n\to\infty} x_n = s$.

2.5.8. It suffices to establish the first identity. Let $s = \liminf_{n\to\infty} x_n$.
Case 1. $s = 0$. Then by Theorem 2.35 there is a subsequence k_j such that $x_{k_j} \to 0$, i.e., $1/x_{k_j} \to \infty$ as $j \to \infty$. In particular, $\sup_{k\ge n}(1/x_k) = \infty$ for all $n \in \mathbf{N}$, i.e., $\limsup_{n\to\infty}(1/x_n) = \infty = 1/s$.
Case 2. $s = \infty$. Then $x_k \to \infty$, i.e., $1/x_k \to 0$, as $k \to \infty$. Thus by Theorem 2.36, $\limsup_{n\to\infty}(1/x_n) = 0 = 1/s$.
Case 3. $0 < s < \infty$. Fix $j \ge n$. Since $1/\inf_{k\ge n} x_k \ge 1/x_j$ implies $1/\inf_{k\ge n} x_k \ge \sup_{j\ge n}(1/x_j)$, it is clear that $1/s \ge \limsup_{n\to\infty}(1/x_n)$. On the other hand, given $\epsilon > 0$ and $n \in \mathbf{N}$, choose $j > N$ such that $\inf_{k\ge n} x_k + \epsilon > x_j$, i.e., $1/(\inf_{k\ge n} x_k + \epsilon) < 1/x_j \le \sup_{k\ge n}(1/x_k)$. Taking the limit of this inequality as $n \to \infty$ and as $\epsilon \to 0$, we conclude that $1/s \le \limsup_{n\to\infty}(1/x_n)$.

2.5.9. If $x_n \to 0$, then $|x_n| \to 0$. Thus by Theorem 2.36, $\limsup_{n\to\infty}|x_n| = 0$. Conversely, if $\limsup_{n\to\infty}|x_n| \le 0$, then

$$0 \le \liminf_{n\to\infty}|x_n| \le \limsup_{n\to\infty}|x_n| \le 0,$$

implies that the limits supremum and infimum of $|x_n|$ are equal (to zero). Hence by Theorem 2.36, the limit exists and equals zero.

CHAPTER 3

3.1 Two–Sided Limits.

3.1.0. a) True. Since $|x^n \sin(x^{-n})| \le |x|^n$ and $|x|^n \to 0$ as $x \to 0$ (by Theorem 3.8), it follows from the Squeeze Theorem that $x^n \sin(x^{-n}) \to 0$ as $x \to 0$.

b) False. See Example 3.7.

c) False. Let $a = 0$, $f(x) = x$, and $g(x) = 1/x^2$ for $x \ne 0$ and $g(0) = 0$. Then for $x \ne 0$ we have $f(x)g(x) = 1/x$ which has no limit as $x \to 0$.

d) False. Let $f(x) = \sin(1/x)$ and $g(x) = 1$. Then f has no limit as $x \to 0$, but the limit of g is 1.

3.1.1. a) Let $\varepsilon > 0$ and set $\delta := \min\{1, \varepsilon/7\}$. Since $\delta < 1$, $|x-2| < \delta$ implies $|x+1| < 7$. Thus

$$|f(x) - L| = |x^2 + 2x - 8| = |x+4|\,|x-2| < 7\delta \le \varepsilon$$

for every x which satisfies $0 < |x-2| < \delta$.

b) Let $\varepsilon > 0$ and set $\delta := \varepsilon$. Notice that $(x^2 + x - 2)/(x-1) = x + 2$ for every x which satisfies $|x-1| > 0$. Thus

$$|f(x) - L| = |x - 1| < \delta = \varepsilon$$

for every x which satisfies $0 < |x-1| < \delta$.

c) Let $\varepsilon > 0$ and set $\delta := \min\{1, \varepsilon/9\}$. Since $\delta < 1$, $|x-1| < \delta$ implies $|x^2 + x + 3| < 9$. Thus

$$|f(x) - L| = |x^3 + 2x - 3| = |x-1|\,|x^2 + x + 3| < 9\delta \le \varepsilon$$

for every x which satisfies $0 < |x-1| < \delta$.

d) Let $\varepsilon > 0$ and set $\delta := \sqrt[3]{\varepsilon}$. Since $|\sin\theta| \le 1$ no matter what θ is, we have

$$|f(x) - L| = |x^3 \sin(e^{x^2})| < \delta^3 \cdot 1 = \varepsilon$$

for every x which satisfies $0 < |x| < \delta$.

3.1.2. a) If $x_n = 4/((2n+1)\pi)$, then $x_n \to 0$ but $\tan(1/x_n) = (-1)^n$ has no limit. Thus $\lim_{x\to 0} \tan(1/x)$ does not exist.

b) Since $|x\cos((x^2+1)/x^3)| \le |x|$ for all $x \ne 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} x\cos((x^2 + 1)/x^3) = 0$.

c) If $x_n = 1 + 1/n$, then $x_n \to 1$ and $1/\log x_n \to +\infty$ as $n \to \infty$. On the other hand, if $x_n = 1 - 1/n$, then $x_n \to 1$ and $1/\log x_n \to -\infty$ as $n \to \infty$. Thus $\lim_{x\to 0} \log x$ does not exist.

3.1.3. a) By Remark 3.4 and Theorem 3.8,

$$\lim_{x\to 1} \frac{x^2 + 2x - 3}{x^3 - x} = \lim_{x\to 1} \frac{(x+3)(x-1)}{x(x-1)(x+1)} = \lim_{x\to 1} \frac{x+3}{x(x+1)} = \frac{4}{2} = 2.$$

b) By Remark 3.4 and Theorem 3.8,

$$\lim_{x\to 1} \frac{x^n - 1}{x - 1} = \lim_{x\to 1} x^{n-1} + \cdots + x + 1 = 1 + \cdots + 1 + 1 = n.$$

c) By Theorem 3.8, the limit is $0/\cos 0 = 0/1 = 0$.

d) Clearly, $2\sin^2 x + 2x - 2x\cos^2 x = 2(x+1)\sin^2 x = (x+1)(1 - \cos(2x))$. Since $1 - \cos^2(2x) = (1 - \cos(2x))(1 + \cos(2x))$, it follows that

$$\frac{2\sin^2 x + 2x - 2x\cos^2 x}{1 - \cos^2(2x)} = \frac{x+1}{1 + \cos(2x)} \to \frac{1}{1 + \cos 0} = \frac{1}{2}$$

as $x \to 0$.

e) Since $\sin(1/x^2)$ is dominated by 1 and $\tan x \to 0$ as $x \to 0$, it follows from Theorem 3.9 that this limit is zero.

3.1.4. a) Let $x_n \in I \setminus \{a\}$ converge to a. By Theorem 2.9i, $h(x_n) \to L$ as $n \to \infty$. Hence by the Sequential Characterization of Limits, $h(x) \to L$ as $x \to a$.

b) Similarly, by Theorem 2.9ii $f(x_n)g(x_n) \to 0$ for all $x_n \in I \setminus \{a\}$ which converge to a. Hence by the Sequential Characterization of Limits, $f(x)g(x) \to 0$ as $x \to a$.

3.1.5. Let $f(x) \to L$ and $g(x) \to M$ as $x \to a$, and $x_n \in I \setminus \{a\}$ converge to a. By the Sequential Characterization of Limits, $f(x_n) \to L$ and $g(x_n) \to M$ as $n \to \infty$. Hence by Theorem 2.17, $L \le M$.

3.1.6. a) By Theorem 1.16, $0 \le ||f(x)| - |L|| \le |f(x) - L|$. It follows from the Squeeze Theorem that $|f(x)| \to |L|$ as $x \to x_0$ through E.

b) If $f(x) = |x|/x$ then $|f(x)| = 1 \to 1$ as $x \to 0$, but $f(x)$ has no limit as $x \to 0$.

3.1.7. a) Since $f(x) \le |f(x)|$ it is clear that $f^+(x) \ge 0$ and $f^-(x) \ge 0$. Also $f^+ - f^- = 2f/2 = f$ and $f^+ + f^- = 2|f|/2 = |f|$.

b) By Exercise 3.1.6, $|f(x)| \to |L|$ as $x \to x_0$ through E. Hence by Theorem 3.8 and part a), $f^+(x) \to L^+$ and $f^-(x) \to L^-$ as $x \to x_0$ through E.

3.1.8. a) By symmetry, it suffices to show the first identity.

Case 1. $f(x) \ge g(x)$. Then $(f \vee g)(x) = f(x)$ and $|f(x) - g(x)| = f(x) - g(x)$ so

$$\frac{f(x) + g(x) + |f(x) + g(x)|}{2} = \frac{2f(x)}{2} = f(x) = (f \vee g)(x).$$

Case 2. $f(x) \le g(x)$. Then $(f \vee g)(x) = g(x)$ and $|f(x) - g(x)| = g(x) - f(x)$ so

$$\frac{f(x) + g(x) + |f(x) + g(x)|}{2} = \frac{2g(x)}{2} = g(x) = (f \vee g)(x).$$

b) If $f(x) \to L$ and $g(x) \to M$ as $x \to x_0$ through E then $f(x) + g(x) \to L + M$ by Theorem 3.8, and $|f(x) + g(x)| \to |L + M|$ by Exercise 3.1.6. Thus by part a), $(f \vee g)(x) \to (L + M + |L - M|)/2 = L \vee M$ as $x \to x_0$ through E. A similar argument works for $f \wedge g$.

3.1.9. Let

$$\varepsilon = \min\left\{\frac{M - f(a)}{2}, \frac{f(a) - m}{2}\right\}.$$

By Definition 3.1, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$. Observe that these inequalities also hold for $x = a$.

Fix x such that $|x - a| < \delta$. By the choice of δ, the definition of ε, and a little algebra that

$$f(x) < f(a) + \frac{M - f(a)}{2} = \frac{M + f(a)}{2} = M - \varepsilon.$$

Similarly, $f(x) > f(a) - \varepsilon = m + \varepsilon$.

3.2 One–Sided Limits and Limits at Infinity.

3.2.0. a) False. If $f(x) = x^2 + 1 = g(x)$, then $f(x) \to \infty$, $g(x) > 0$ for all x, but $f(x)/g(x) = 1$ does not converge to 0.

b) False. If $f(x) = -x^2$ and $g(x) = 1$, then $f(x) \to 0$ as $x \to 0+$ and $g(x) \ge 1$, but $g(x)/f(x) = -1/x^2 \to -\infty$ as $x \to 0+$.

c) True. Given $\varepsilon > 0$, choose $M \in \mathbf{R}$ such that $x > M$ implies $f(x) > 1/\varepsilon$. Then $x > M$ implies $|\sin(x^2 + x + 1)/f(x)| \le 1/f(x) < \varepsilon$.

d) True. Let $P(x) = a_n x^n + \cdots + a_0$ and $Q(x) = b_m x^m + \cdots + b_0$, where $m \ge n$. Dividing top and bottom by x^m, we have

$$\frac{P(x)}{Q(x)} = \frac{a_n x^{n-m} + a_{n-1} x^{n-m-1} + \cdots + a_0/x^m}{b_m + \cdots + b_0 x^{-m}}.$$

If $m = n$, then $x^{n-m} = 1$ for $x \ne 0$ and $x^{n-m-k} \to 0$ as $x \to \pm\infty$ for all $k > 0$. Thus $P(x)/Q(x)$ converges to a_n/b_m as $x \to \pm\infty$. On the other hand, if $m > n$, then $x^{n-m} \to 0$ as $x \to \pm\infty$ too. Hence, $P(x)/x^m \to 0$ as $x \to \pm\infty$. Since $Q(x)/x^m \to b_m$, we conclude that $P(x)/Q(x) \to 0$ as $x \to \pm\infty$.

3.2.1. a) Let $L = -1$ and notice that $\sqrt{x^2} = -x$ when $x < 0$. Let $\varepsilon > 0$ and set $\delta = 1$. If $-\delta < x < 0$, then $f(x) = -x/x = -1$ so $|f(x) - L| = |-1 + 1| = 0 < \varepsilon$.

b) Let $L = 0$ and suppose $\varepsilon > 0$. Set $M = 1/\sqrt{\varepsilon}$. If $x > M$, then $|f(x) - L| \le 1/x^2 < 1/M^2 = \varepsilon$.

c) Let $L = -\infty$ and suppose without loss of generality that $M < 0$. Set $\delta = \min\{-1/(2M), 1\}$. If $-1 < x < -1 + \delta$, then $-1 < x < 0$ since $\delta \le 1$. Hence, $0 < 1 - x < 2$ and $x^2 - 1 < 0$. Thus $0 < 1 - x^2 = (1 - x)(1 + x) < 2\delta \le -1/M$, i.e., $1/(1 - x^2) > -M$. We conclude that $f(x) = 1/(x^2 - 1) < M$.

d) Let $L = \infty$ and suppose without loss of generality that $M > 0$. Set $\delta = \min\{1/(7M), 1\}$. If $1 < x < 1+\delta$, then $1 < x \le 2$ since $\delta \le 1$. Thus $5 < 2x+3 < 7$ and $0 < x-1 < \delta$. It follows that $0 < 2x^2+x-3 = (x-1)(2x+3) < 7\delta \le 1/M$. Therefore, $1/(2x^2+x-3 \ge M$. $f(x) > |2M|/2 = M$. Since $3-x \ge 3-2 = 1$, we conclude that $(x-3)(3-x-2x^2) = (3-x)/(2x^2+x-3) \ge M$.

e) Let $L = 0$ and suppose that $\varepsilon > 0$. Set $M = -1-1/\varepsilon$. If $x < M$, then $x+1 < -1/\varepsilon < 0$ so $|x+1| = -(x+1) > 1/\varepsilon$. Since $|\cos(\theta))| \le 1$ for any θ, it follows that $|f(x)-L| = -|\cos(\tan x)|/(x+1) \le -1/(x+1) < \varepsilon$.

3.2.2. a) $x^3-x^2-4 = (x-2)(x^2+x+2)$ and $x^2-4 = (x+2)(x-2)$ so $(x^3-x^2-4)/(x^2-4) = (x^2+x+2)/(x+2) \to 8/4 = 2$ as $x \to 2-$.

b) Multiplying top and bottom by $1/x^2$ we have

$$\frac{5x^2+3x-2}{3x^2-2x+1} = \frac{5+3/x-2/x^2}{3-2/x+1/x^2} \to \frac{5}{3}$$

as $x \to \infty$.

c) $-1/x^2 \to 0$ as $x \to -\infty$ so $e^{-1/x^2} \to e^0 = 1$.

d) $x^2+2x-1 \to -1$ as $x \to 0+$ and $\sin x$ is positive as $x \to 0+$, so $e^{x^2+2x-1}/\sin x \to e^{-1}/0+ = \infty$ as $x \to 0+$.

e) $\sin(x+\pi/2) \to \sin(\pi/2) = 1$ as $x \to 0-$ and $\sqrt[3]{\cos x - 1}$ is negative as $x \to 0-$ so $\sin(x+\pi/2)/\sqrt[3]{\cos x-1} \to 1/0- = -\infty$ as $x \to 0-$.

f) Since $\sin^2 x = 1-\cos^2 x$, it follows from factoring that

$$\frac{\sqrt{1-\cos x}}{\sin x} = \frac{1}{\sqrt{1+\cos x}}$$

so the limit is $1/\sqrt{2} \equiv \sqrt{2}/2$.

3.2.3. a) The result holds for $n = 0, 1$ by Example 3.2. Hence by Theorem 3.8,

$$\lim_{x\to x_0} x^n = (\lim_{x\to x_0} x)^n = x_0^n$$

for all $n \in \mathbf{N}$.

b) By Theorem 3.8 and part a),

$$\lim_{x\to x_0} P(x) = \lim_{x\to x_0} (a_n x^n + \cdots + a_0) = a_n x_0^n + \cdots + a_0 = P(x_0).$$

3.2.4. a) If $g(x) \to \infty$ as $x \to a$, then given $M \in \mathbf{R}$, choose $\delta > 0$ such that $0 < |x-a| < \delta$ implies $g(x) > M$. Since $f(x) > g(x)$, we also have $f(x) > M$. By definition, $g(x) \to \infty$ as $x \to a$.

b) Let $\varepsilon > 0$ and choose $M \in \mathbf{R}$ such that $x > M$ implies $|f(x)-L| < \varepsilon$ and $|h(x)-L| < \varepsilon$. By hypothesis, this means

$$L-\varepsilon < f(x) \le g(x) \le h(x) < L+\varepsilon.$$

Therefore, $x > M$ implies $|g(x)-L| < \varepsilon$.

3.2.5. Suppose $f(x) \to L$ as $x \to \infty$. Let $\epsilon > 0$ and choose $M \in \mathbf{R}$ such that $x > M$ implies $|f(x)-L| < \epsilon$. If $x_n \to \infty$ as $n \to \infty$, choose $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n > M$. Then $|f(x_n)-L| < \epsilon$ for all $n \ge N$, i.e., $f(x_n) \to L$ as $n \to \infty$.

Conversely, suppose $f(x)$ does not converge to L as $x \to \infty$. Then there is an $\epsilon_0 > 0$ such that given $n > 0$ there is an $x_n > n$ satisfying $f(x_n) \ge L+\epsilon_0$ or $f(x_n) \le L-\epsilon_0$, i.e., $|f(x_n)-L| \ge \epsilon_0$. Thus $x_n \to \infty$ but $f(x_n)$ does not converge to L as $n \to \infty$.

3.2.6. Given $x_0 \in [0,1]$, choose $q_n \in \mathbf{Q} \cap [0,1]$ such that $q_n \to x_0$ as $n \to \infty$. By Theorem 3.6, $f(q_n) \to f(x_0)$. If $f(q) = 0$ for all $q \in \mathbf{Q} \cap [0,1]$, it follows that $f(x_0) = 0$. Thus $f(x) = 0$ for all $x \in [0,1]$. The converse is trivial.

3.2.7. By Exercise 3.2.3b and symmetry, it suffices to prove $P(x)/(x-x_0) \to \infty$ as $x \to x_0+$. Since $m_0 := P(x_0)/2 > 0$ use 3.2.3b again to choose $\delta_0 > 0$ such that $0 < m_0 < P(x)$ for $|x-x_0| < \delta_0$. Let $M > 0$ and set $\delta = \min\{\delta_0, m_0/M\}$. If $x_0 < x < x_0+\delta$ then $P(x)/(x-x_0) > m_0/\delta \ge m_0(M/m_0) = M$. Hence by definition, $P(x)/(x-x_0) \to \infty$ as $x \to x_0+$.

3.2.8. Let $\varepsilon > 0$. Set $\varepsilon_n := \sup_{k\ge n} |f(k+1)-f(k)-L|$ and notice by hypothesis that $\varepsilon_n \to 0$ as $n \to \infty$. Thus choose N_0 such that $\varepsilon_{N_0} < \varepsilon/2$.

Let $n > N_0$ and write $f(n) = (f(n) - f(n-1)) + \cdots + (f(N_0+1) - f(N_0)) + f(N_0)$. Thus

$$\begin{aligned}\left|\frac{f(n)}{n} - L\right| &= \left|\frac{(f(n) - f(n-1)) + \cdots + (f(N_0+1) - f(N_0)) - (n - N_0)L}{n} + \frac{N_0L + f(N_0)}{n}\right| \\ &\le \left(\frac{n - N_0}{n}\right)\varepsilon_0 + \frac{|N_0L + f(N_0)|}{n} \\ &\le \varepsilon_0 + \frac{|N_0L + f(N_0)|}{n}.\end{aligned}$$

Thus if N is so big that $n \ge N$ implies $|N_0L + f(N_0)|/n < \varepsilon/2$, then the estimate above can be continued as $|f(n)/n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

3.3 Continuity.

3.3.0. a) True. By the Extreme Value Theorem, there exist $x_M, x_m \in [a,b]$ such that $f(x_m) = \alpha := \inf\{f(x) : x \in [a,b]\}$ and $f(x_M) = \beta := \sup\{f(x) : x \in [a,b]\}$. Thus $\alpha, \beta \in J$. If $t \in (\alpha, \beta)$, then by the Intermediate Value Theorem, there is a $x \in [a,b]$ such that $f(x) = t$, i.e., $t \in J$. We have shown that $[\alpha, \beta] \subseteq J$. On the other hand, if $t \in J$ then $t = f(x)$ for some $x \in [a,b]$ so by the choice of α and β, $\alpha \le t \le \beta$, i.e., $J \subseteq [\alpha, \beta]$.

b) True. Let $h(x) = g(x) - f(x)$. By hypothesis, $h(a) > 0$ and $h(b) < 0$. Hence by the Intermediate Value Theorem, there is a $c \in [a,b]$ such that $h(c) = 0$, i.e., such that $f(c) = g(c)$.

c) True. If f is continuous at a, then by Theorem 3.22, fg is too. Conversely, if fg is continuous at a and f is continuous and nonzero at a, then $g = fg/f$ is continuous at a by Theorem 3.22.

d) False. Let $f(x) = 2 - x$ for $x \le 1$, $f(x) = 1/x$ for $x \ge 1$, $g(x) = 1 - x$ for $x \le 0$ and $g(x) = -x$ for $x > 0$. Since $f(x) > 0$ for all x and g is continuous on $(0, \infty)$, it is clear that f and $g \circ f$ are continuous but g is not.

3.3.1. a) By Theorem 3.24, e^{x^2} and $\sqrt{\sin x}$ are continuous on $\mathbf{R}$. Since $\cos x \ne 0$ for $x \in [0,1]$, it follows from Theorem 3.22 that $e^{x^2}\sqrt{\sin x}/\cos x$ is continuous on $[0,1]$.

b) $x - 1 \ne 0$ for $x \in [0,1)$, so $f(x) := (x^2 + x - 2)/(x-1)$ is continuous on $[0,1)$ by Theorem 3.22. Since $f(x) \to 3$ as $x \to 1$ and $f(1) := 3$, it follows from Remark 3.20 that $f(x)$ is continuous on $[0,1]$.

c) $x \ne 0$ for $x \in (0,1]$, so $f(x) := e^{-1/x}$ is continuous on $(0,1]$ by Theorem 3.22. Since $-1/x \to -\infty$ as $x \to 0+$ implies that $e^{-1/x} \to e^{-\infty} = 0 =: f(0)$, it follows that from Remark 3.20 that f is continuous on $[0,1]$.

d) $\sqrt{x}\sin(1/x)$ is continuous for $x > 0$ by Theorem 3.22. Since $0 \le \sqrt{x}|\sin(1/x)| \le \sqrt{x}$, it follows from the Squeeze Theorem that $f(x) \to 0 =: f(0)$ as $x \to 0+$. Thus f is continuous on $[0,1]$.

3.3.2. a) Consider $f(x) = e^x - x^3$. This function is continuous, and $f(-1) = 1/e - 1 < 0 < 1 = f(0)$. Hence by the Intermediate Value Theorem, there is an x (between -1 and 0) such that $f(x) = 0$.

b) Consider $f(x) = e^x - 2\cos x - 1$. This function is continuous, and $f(0) = -2 < 0 < e + 1 = f(1)$. Hence by the Intermediate Value Theorem, there is an x (between 0 and 1) such that $f(x) = 0$.

c) Consider $f(x) = 2^x + 3x - 2$. This function is continuous, and $f(0) = -1 < 0 < 3 = f(1)$. Hence by the Intermediate Value Theorem, there is an x (between 0 and 1) such that $f(x) = 0$.

3.3.3. By Exercise 3.1.6, $|f|$ is continuous on $[a,b]$. Hence it follows from the Extreme Value Theorem that $|f|$ is bounded on $[a,b]$, i.e., $\sup_{x \in [a,b]} |f(x)|$ is finite.

3.3.4. Let $g(x) = f(x) - x$. Since $f(x) \in [a,b]$ for all $x \in [a,b]$, it is clear that $f(a) \ge a$ and $f(b) \le b$. Therefore, $g(a) = f(a) - a \ge 0$ and $g(b) = f(b) - b \le 0$. Since g is continuous on $[a,b]$, it follows from the Intermediate Value Theorem that there is a $c \in [a,b]$ such that $g(c) = 0$, i.e., such that $f(c) = c$.

3.3.5. Since $M - f(x) > 0$, it follows from the Sign Preserving Property that there is an interval I centered at x_0 such that $M - f(x) > 0$, i.e., $f(x) < M$ for all $x \in I$.

3.3.6. Let $f(x) = 1$ if $x \in \mathbf{Q}$, $f(x) = 0$ if $x \notin \mathbf{Q}$, and $g(x) = 1 - f(x)$. Then $f(x) + g(x) = 1$ and $f(x)g(x) = 0$ for all $x \in \mathbf{R}$. Hence $f + g$ and fg are continuous on $\mathbf{R}$ even though f and g are nowhere continuous.

3.3.7. If g is continuous at a then so is $f + g$ by Theorem 3.8. Conversely, if f and $f + g$ are continuous at a then so is $g = (f + g) - f$.

3.3.8. a) $f(0) = f(0 + 0) = f(0) + f(0) = 2f(0)$ implies $f(0) = 0$. For each $x \in \mathbf{R}$, $f(2x) = f(x + x) = f(x) + f(x) = 2f(x)$. And $0 = f(0) = f(x - x) = f(x) + f(-x)$ implies $f(-x) = -f(x)$. Hence we see by induction that $f(nx) = f(x + \cdots + x) = nf(x)$ holds for all $n \in \mathbf{Z}$.

b) Fix $x \in \mathbf{R}$. By part a), $f(x) = f(mx/m) = mf(x/m)$. Thus $f(x/m) = f(x)/m$ for $m \in \mathbf{N}$. Combining these statements, if $q \in \mathbf{Q}$ then $q = n/m$ so

$$f\left(\frac{n}{m}x\right) = nf\left(\frac{x}{m}\right) = \frac{n}{m}f(x) = qf(x)$$

for $x \in \mathbf{R}$.

c) Suppose f is continuous at 0 and $x \in \mathbf{R}$. If $x_n \to x$ then $x_n - x \to 0$, i.e., $|f(x_n) - f(x)| = |f(x_n - x)| \to 0$ as $n \to \infty$. Thus f is continuous on $\mathbf{R}$. The converse is trivial.

d) Let $m = f(1)$ and fix $x \in \mathbf{R}$. Choose $q_n \in \mathbf{Q}$ such that $q_n \to x$ as $n \to \infty$. Then by b) and c),

$$f(x) = f(\lim_{n\to\infty} q_n \cdot 1) = \lim_{n\to\infty} f(q_n \cdot 1) = \lim_{n\to\infty} q_n f(1) = mx.$$

3.3.9. a) $f(0) = f(0+0) = f(0)f(0) = f^2(0)$ implies $f(0) = 0$ or 1. But f has range $(0,\infty)$ so $f(0) = 1$. For each $x \in \mathbf{R}$, $1 = f(0) = f(x-x) = f(x)f(-x)$. Therefore, $f(-x) = 1/f(x)$.

b) Fix $x \in \mathbf{R}$. By induction, $f(nx) = f(x + \cdots + x) = f^n(x)$ for $n \in \mathbf{N}$. Also, $f(x) = f(mx/m) = f^m(x/m)$ implies $f(x/m) = \sqrt[m]{f(x)}$ for $m \in \mathbf{N}$. Combining these statements, if $q \in \mathbf{Q}$ then $q = n/m$ so

$$f(qx) = f\left(\frac{n}{m}x\right) = f^n\left(\frac{x}{m}\right) = (f^n(x))^{1/m} = f^{n/m}(x) = f^q(x)$$

for $x \in \mathbf{R}$. In particular, $f(q) = f(q \cdot 1) = f^q(1)$ for all $q \in \mathbf{Q}$.

c) Suppose f is continuous at 0 and $x \in \mathbf{R}$. If $x_n \to x$ then $x_n - x \to 0$, i.e., $f(x_n - x) \to f(0) = 1$ as $n \to \infty$. Therefore,

$$|f(x_n) - f(x)| = |f(x)| \left| \frac{f(x_n)}{f(x)} - 1 \right| = |f(x)|\,|f(x_n)f(-x) - 1| = |f(x)|\,|f(x_n - x) - 1| \to 0$$

as $n \to \infty$. Thus f is continuous on $\mathbf{R}$. The converse is trivial.

d) Let $a = f(1)$ and fix $x \in \mathbf{R}$. Choose $q_n \in \mathbf{Q}$ such that $q_n \to x$ as $n \to \infty$. Then by b) and c),

$$f(x) = f(\lim_{n\to\infty} q_n \cdot 1) = \lim_{n\to\infty} f(q_n \cdot 1) = \lim_{n\to\infty} f^{q_n}(1) = \lim_{n\to\infty} a^{q_n} = a^x.$$

3.3.10. Let $N \in \mathbf{N}$ be so large that $|x| \ge N$ implies $f(x) > f(0)$. By the Extreme Value Theorem, f has an absolute minimum on $[-N, N]$, say $f(x_m) = m$. Since $m \le f(0) < f(x)$ for all $x \notin [-N, N]$, it follows that m is the absolute minimum of f on $\mathbf{R}$.

3.3.11. a) Fix $a > 1$ and for each $x \in \mathbf{R}$ consider the set

$$E_x := \{a^q : q \in \mathbf{Q} \quad \text{and} \quad q \le x\}.$$

By the density of rationals, there are many $q \in \mathbf{Q}$ such that $q < x$. Thus E_x is nonempty. Moreover, if $q_0 \in \mathbf{Q}$ satisfies $q_0 > x$ and $q \in E_x$, then $q_0 > q$, so by hypothesis $a^q < a^{q_0}$. Thus E_x is bounded above by a^{q_0}. It follows from the Completeness Axiom that $A(x) = \sup E_x$ exists for every $x \in \mathbf{R}$.

To compute the value of $A(x)$ when $x = p_0 \in \mathbf{Q}$. Notice by hypothesis that $q \le p_0$ implies $a^q \le a^{p_0}$. Thus it is clear that $A(p_0) \le a^{p_0}$. On the other hand, since $a^{p_0} \in E_{p_0}$, it is also the case that $A(p_0) \ge a^{p_0}$. Thus $a^{p_0} = A(p_0)$.

b) Suppose that $x < y$. Since $q < x$ and $x < y$ imply $q < y$, it is clear by definition that $a^x \le a^y$. On the other hand, choose $p < q$ in $\mathbf{Q}$ such that $x < p < q < y$. Since $a > 1$, it follows from definition and hypothesis that

$$a^x \le a^p < a^q \le a^y,$$

thus $a^x < a^y$.

c) Let $\varepsilon > 0$. Since $a^{1/n} \to 1$ as $n \to \infty$ (see Example 2.21), choose an $N \in \mathbf{N}$ such that $|a^{1/N} - 1| \le \varepsilon/a^{x_0}$. Suppose that $\delta = 1/(2N)$ and $|x_0 - x| < \delta$. If $x < x_0$, choose $r, q \in \mathbf{Q}$ such that $q < x < x_0 < r$ and $r - q < 2\delta = 1/N$. By part a),

$$|a^{x_0} - a^x| = a^{x_0} - a^x < a^r - a^q = a^q \cdot (a^{r-q} - 1) < a^{x_0}(a^{1/N} - 1) < \varepsilon.$$

Similarly, if $x > x_0$, choose $r < x_0 < x < q$ such that $q - r < 1/N$. Then

$$|a^{x_0} - a^x| = a^x - a^{x_0} < a^q - a^r = a^r(a^{q-r} - 1) < a^{x_0}(a^{1/N} - 1) \le \varepsilon.$$

Thus by definition, a^x is continuous on $\mathbf{R}$ when $a > 1$.

d) Let $x, y \in \mathbf{R}$. Use Density of Rationals to choose t_n and $q_n \in \mathbf{Q}$ such that $t_n \to x$ and $q_n \to y$. Thus by the continuity of a^x, Theorem 3.8, and the fact that the laws of exponents hold for rational powers, we conclude that

$$a^{x+y} = \lim_{n\to\infty} a^{t_n+q_n} = \lim_{n\to\infty} a^{t_n} a^{q_n} = a^x a^y.$$

Similarly, $(a^x)^y = a^{xy}$ and $a^{-x} = 1/a^x$.

e) Notice $0 < b < 1$ implies $1/b > 1$. Thus $b^x := (1/b)^{-x}$ defines b^x, and by parts a) and c), b^x is a continuous extension of b^q from $\mathbf{Q}$ to $\mathbf{R}$ which satisfies the exponential properties.

Finally, if $x < y$ and $0 < b < 1$, then $-x > -y$ and by b), $b^x = (1/b)^{-x} > (1/b)^{-y} = b^y$. Thus b^x is decreasing (not increasing) when $b \in (0,1)$.

3.4 Uniform continuity.

3.4.0 a) False. Let $f(x) = x$ and $g(x) = 1$ if $x < 2$ and 2 if $x \ge 2$. Then f is uniformly continuous on $(0,\infty)$ and g is positive and bounded, but $f(x)g(x)$ is not continuous at $x = 2$ so cannot be uniformly continuous.

b) True. By l'Hôpital's Rule, $x\log(1/x) \to 0$ as $x \to 0+$, so use Theorem 3.40.

c) False. Let $m = -b = 1$. Then $\cos x/(mx + b) \to \cos 1/0- = -\infty$ as $x \to 1-$, so by Theorem 3.40, this function cannot possibly be uniformly continuous on $(0,1)$.

d) True. Both f and g are bounded on $[a,b]$ by the Extreme Value Theorem. Since $g(x) \ne 0$, it follows from the Intermediate Value Theorem that either $g(x) > 0$ or $g(x) < 0$ for all $x \in [a,b]$. We may suppose $g(x) > 0$. By the Extreme Value Theorem, $g(x) \ge \epsilon_0 > 0$ for $x \in [a,b]$. Therefore, f/g is uniformly continuous on $[a,b]$ by Exercise 3.4.5d.

3.4.1. a) Let $\varepsilon > 0$ and let $\delta = \varepsilon/3$. If $x, a \in (0,1)$ and $|x - a| < \delta$, then

$$|f(x) - f(a)| = |x - a|\,|x + a + 1| \le 3|x - a| < 3\frac{\varepsilon}{3} = \varepsilon.$$

b) Let $\varepsilon > 0$ and let $\delta = \varepsilon/4$. If $x, a \in (0,1)$ and $|x - a| < \delta$, then

$$|f(x) - f(a)| = |x - a|\,|x^2 + xa + a^2 - 1| \le 4|x - a| < 4\frac{\varepsilon}{4} = \varepsilon.$$

c) Let $\varepsilon > 0$ and let $\delta = \varepsilon/3$. If $x, a \in (0,1)$ and $|x - a| < \delta$, then

$$\begin{aligned}|f(x) - f(a)| &\le |x(\sin 2x - \sin 2a)| + |(x - a)\sin 2a| \\ &\le 2|\sin(x - a)| + |x - a| \le 3|x - a| < 3\frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

3.4.2. a) By L'Hôpital's Rule, $\sin x/x \to 1$ as $x \to 0$. Therefore, f is uniformly continuous on $(0,1)$ by Theorem 3.39.

b) By the Squeeze Theorem, $x\cos(1/x^2) \to 0$ as $x \to 0$. Therefore, f is uniformly continuous on $(0,1)$ by Theorem 3.39.

c) By L'Hôpital's Rule, $x\log x \to 0$ as $x \to 0+$. Therefore, f is uniformly continuous on $(0,1)$ by Theorem 3.39.

d) By l'Hôpital's Rule, $(1 - x^2)^{1/x} \to 1$ as $x \to 0+$. Therefore, f is uniformly continuous on $(0,1)$ by Theorem 3.39.

3.4.3. If $\alpha > 0$ then $|x^\alpha \sin(1/x)| \le x^\alpha \to 0$ as $x \to 0+$. Thus $x^\alpha \sin(1/x)$ is uniformly continuous on $(0,1)$ for all $\alpha > 0$.

If $\alpha \le 0$ and $x_n = 2/((2n + 1)\pi)$ then $x_n^\alpha \sin(1/x_n) = (-1)^n x_n^\alpha$ does not converge as $n \to \infty$, i.e., $x^\alpha \sin(1/x)$ has no limit as $x \to 0+$. Therefore, $x^\alpha \sin(1/x)$ is not uniformly continuous on $(0,1)$ when $\alpha \le 0$.

3.4.4. a) Given $\epsilon > 0$ choose N so large that $x \ge N$ implies $|f(x) - L| < \epsilon/3$. By Theorem 3.40, f is uniformly continuous on $[0,N]$. Thus there is a $\delta > 0$ such that $|x - y| < \delta$ and $x, y \in [0,N]$ implies $|f(x) - f(y)| < \epsilon/3$.

Let $x, y \in [0, \infty)$ and suppose $|x - y| < \delta$. If $x, y \in [0, N]$, then $|f(x) - f(y)| < \epsilon$. If both $x, y \notin [0, N]$ then $|f(x) - f(y)| \leq |f(x) - L| + |f(y) - L| < 2\epsilon/3 < \epsilon$. If one of the pair x, y belongs to $[0, N]$ and the other does not, for example, if $x \in [0, N]$ and $y \notin [0, N]$, then $|x - N| \leq |x - y| < \delta$. Thus

$$|f(x) - f(y)| \leq |f(x) - f(N)| + |f(N) - L| + |f(y) - L| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

b) Since $f(x) = 1/(x^2 + 1)$ is continuous on $\mathbf{R}$ and $1/(x^2 + 1) \to 0$ as $x \to \infty$, f is uniformly continuous on $[0, \infty)$ by part a). But $f(-x) = f(x)$. Hence f is uniformly continuous on $\mathbf{R}$ by symmetry.

3.4.5. a) Given $\epsilon > 0$ choose $\delta > 0$ such that $x, y \in E$ and $|x-y| < \delta$ imply $|f(x)-f(y)|$ and $|g(x)-g(y)| < \epsilon/2$. If $x, y \in E$ and $|x - y| < \delta$ then

$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

A similar argument proves αf is uniformly continuous on E.

b) Let $M = \sup\{1+|f(x)|+|g(x)| : x \in E\}$. Then $M > 0$ and both $|f(x)|$ and $|g(x)|$ are less than M for $x \in E$. Given $\epsilon > 0$ choose $\delta > 0$ such that $x, y \in E$ and $|x - y| < \delta$ imply $|f(x) - f(y)|$ and $|g(x) - g(y)| < \epsilon/(2M)$. If $x, y \in E$ and $|x - y| < \delta$ then

$$|(fg)(x) - (fg)(y)| \leq |g(y)|\,|f(x) - f(y)| + |f(x)|\,|g(x) - g(y)| < M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon.$$

c) Let $f(x) = x = g(x)$. Then f and g are uniformly continuous on $\mathbf{R}$ but $(fg)(x) = x^2$ is not (see Example 3.36).

d) If $g(x) \geq \epsilon_0 > 0$, then $1/g$ is continuous on E and bounded by $1/\epsilon_0$. Moreover, since g is uniformly continuous on E and

$$\left|\ \frac{1}{g(x)} - \frac{1}{g(y)}\ \right| = \frac{|g(x) - g(y)|}{|g(x)|\,|g(y)|} \leq \epsilon_0^{-2}|g(x) - g(y)|,$$

$1/g$ is uniformly continuous on E. Hence by b), $f/g := f(1/g)$ is uniformly continuous on E.

e) Let $f(x) = x$ and $g(x) = x^2$. Then f and g are uniformly continuous on $(0, 1)$, but $(f/g)(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

3.4.6. a) Suppose I has endpoints a, b. By Theorem 3.39, there is a continuous function g on $[a, b]$ such that $f(x) = g(x)$ for all $x \in I$. By the Extreme Value Theorem, g is bounded on $[a, b]$. Therefore, f is bounded on $I \subseteq [a, b]$.

b) $f(x) = x$ is uniformly continuous on $[0, \infty)$ but not bounded there. On the other hand, $f(x) = 1/x$ is continuous on $(0, 1)$ but not bounded there either.

3.4.7. Since f is uniformly continuous on $[0, 1]$, given $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in [0, 1]$ and $|x-y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. Choose $N \in \mathbf{N}$ so large that $N > (b - a)/\delta$ and set $x_k = a + k(b - a)/N$. Then $a = x_0 < x_1 < \cdots < x_N = b$, and $|x_k - x_{k-1}| = (b - a)/N < \delta$ by the choice of N for all k.

Fix $k \in \{1, 2, \ldots, N\}$ and observe by the choice of δ that $x^*, y^* \in [x_{k-1}, x_k]$ imply that $|f(x^*) - f(y^*)| < \epsilon$. But by the Extreme Value Theorem, there exist $x^*, y^* \in [x_{k-1}, x_k]$ such that $f(x^*) = \sup E_k$ and $f(y^*) = \inf E_k$. Consequently,

$$\left|\ \sup E_k - \inf E_k\ \right| = |f(x^*) - f(y^*)| < \epsilon.$$

3.4.8. a) By symmetry, we need only show $f(b-)$ exists. Let $L = \sup\{f(x) : x \in (a, b)\}$. Since f is bounded, $L < \infty$ by the Completeness Axiom. Given $\epsilon > 0$ choose by the Approximation Property an $x_0 \in (a, b)$ such that $L - \epsilon < f(x_0)$. Since f is increasing, $L - \epsilon < f(x_0) \leq f(x) \leq L$ for all $x_0 < x < b$. Hence by definition, $f(b-)$ exists and is equal to L.

b) Suppose f is increasing and continuous on (a, b). By part a), $f(a+)$ and $f(b-)$ exist. Thus the function defined by $g(x) = f(x)$, $x \in (a, b)$, $g(a) = f(a+)$ and $g(b) = f(b-)$ is continuous on $[a, b]$. Thus f is uniformly continuous on (a, b) by Theorem 3.40. The converse is trivial.

c) $g(x) = -1/x$ is increasing and continuous on $(0, 1)$ but not uniformly continuous there.

3.4.9. Suppose P is a polynomial of degree 0 or 1, i.e., $P(x) = mx + b$. Let $\epsilon > 0$ and set $\delta = \epsilon/(|m| + 1)$. If $|x - y| < \delta$ then

$$|P(x) - P(y)| \leq |m||x - y| < |m|\frac{\epsilon}{|m| + 1} < \epsilon.$$

Thus P is uniformly continuous on $\mathbf{R}$.

Suppose P is a polynomial of degree $n > 1$, i.e., $P(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$. Suppose without loss of generality that $a_n > 0$. Then

$$\frac{P(x)}{x^{n-1}} = a_n x + a_{n-1} + \frac{a_{n-2}}{x} + \frac{a_0}{x^{n-1}} \to \infty$$

as $x \to \infty$, i.e., $P(x) \to \infty$ as $x \to \infty$. Now $P(x) - P(y) = (x - y)Q(x, y)$ where

$$Q(x, y) = a_n(x^{n-1} + \cdots + y^{n-1}) + \cdots + a_2(x + y) + a_1.$$

By the argument above, $Q(x, y) \to \infty$ as $x, y \to \infty$. If P were uniformly continuous on $\mathbf{R}$, then given $0 < \epsilon < 1$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|P(x) - P(y)| < \epsilon$. Let $x_n \to \infty$ and set $y_n = x_n + \delta/2$. Then $x_n, y_n \to \infty$ so choose N so large that $Q(x_N, y_N) > 2/\delta$. Since $|x_N - y_N| = \delta/2 < \delta$, we have

$$1 = \frac{2}{\delta}|x_N - y_N| < |x_N - y_N|\,|Q(x_N, y_N)| = |P(x_N) - P(y_N)| < \epsilon$$

a contradiction.

4.1 The Derivative.

4.1.0. a) False. $g(x) = |x|$ is not differentiable at $x = 0$, but $g^2(x) = x^2$ is.

b) True. If f is differentiable, then f is continuous on $[a, b]$. Since $[a, b]$ is a closed, bounded interval, it follows from Theorem 3.39 that f is uniformly continuous on $[a, b]$.

c) False. Let $f(x) = 1/x - 1$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $(0, 1)$ and $f(0) = f(1) = 0$, but f is not even continuous on $[0, 1]$.

d) True. By Theorem 3.40, it suffices to prove that f is continuously extendable from the right at $x = a$, i.e., that if $x_n \in (a, b]$ and $x_n \to a$, then $f(x_n) \to L$ for some $L \in \mathbf{R}$. We claim that $L = 0$. If not, then some x_n satisfies $|f(x_{n_k})| > \varepsilon_0$ for $k \in \mathbf{N}$, so $|f(x_{n_k})|/(x_{n_k} - a) > \varepsilon_0/(x_{n_k} - a)$. But this cannot be, since the later converges to ∞ as $k \to \infty$. Thus $f(x_n) \to 0$ as $n \to \infty$, and $f(a) := 0$ continuously extends f from $(a, b]$ to $[a, b]$.

4.1.1. a) $(f(a + h) - f(a))/h = (2ah + h^2 + h)/h = 2a + h + 1 \to 2a + 1$ as $h \to 0$.

b) By rationalizing the numerator, $(f(a + h) - f(a))/h = (\sqrt{a + h} - \sqrt{a})/h = 1/(\sqrt{a + h} + \sqrt{a}) \to 1/2\sqrt{a}$ as $h \to 0$.

c) $(f(a + h) - f(a))/h = (-h/a(a + h))/h = -1/a(a + h) \to -1/a^2$ as $h \to 0$.

4.1.2. a) Let $n \in \mathbf{N}$. If $n = 1$ then $f'(x_0) = 1$ for all $x_0 \in \mathbf{R}$. Suppose $n > 1$ and $x \neq x_0$. Since $(f(x) - f(x_0))/(x - x_0) = (x^n - x_0^n)/(x - x_0) = x^{n-1} + \cdots + x_0^{n-1}$, it is clear that $f'(x_0) = nx_0^{n-1}$.

b) It's clear for $n = 0$. Let $n \in -\mathbf{N}$ and $x_0 > 0$. Since $-n \in \mathbf{N}$, we have by algebra and part a that

$$\frac{x^n - x_0^n}{x - x_0} = \frac{x_0^{-n} - x^{-n}}{x - x_0} \cdot x^n x_0^n \to n x_0^{-n-1} \cdot x^{2n} = n x_0^{n-1}.$$

Thus the function x^n is differentiable at x_0 and $f'(x_0) = nx_0^{n-1}$.

4.1.3. Clearly, $|f_\alpha(x)| \le |x|^\alpha$. If $\alpha > 0$ then $|x|^\alpha \to 0$ as $x \to 0$. Thus by the Squeeze Theorem, $f_\alpha(x) \to 0 = f(0)$ as $x \to 0$, i.e., f_α is continuous at $x = 0$. If $\alpha > 1$ then

$$\left| \frac{f_\alpha(h) - f_\alpha(0)}{h} \right| \le |h|^{\alpha - 1} \to 0$$

as $h \to 0$. Thus $f'_\alpha(0) = 0$ by the Squeeze Theorem.

The graph of f_1 has no tangent at $x = 0$ because it oscillates between $y = x$ and $y = -x$. The graph of f_2 has a tangent at $x = 0$ because it is trapped between $y = x^2$ and $y = -x^2$, hence squeezed flat at $x = 0$.

4.1.4. Since $|f(x)| \le |x|^\alpha$ for all $x \in I$, $f(0) = 0$. Thus by hypothesis,

$$\left| \frac{f(0 + h) - f(0)}{h} \right| = \frac{|f(h)|}{|h|} \le |h|^{\alpha - 1}.$$

Since $\alpha > 1$, this last number converges to zero as $h \to 0$ in I. It follows from the Squeeze Theorem that f is differentiable at $0 \in I$, and its derivative is zero.

Set $f(x) = |x|$. Then f satisfies the hypotheses with $\alpha = 1$ and $I = [-1, 1]$, but f is not differentiable at $0 \in [-1, 1]$.

4.1.5. a) $1 = y' = 1 + \cos x$ implies $\cos x = 0$, i.e., $x = (2k + 1)\pi/2$ for $k \in \mathbf{Z}$. Thus the points are $(a, b) = ((2k + 1)\pi/2, (-1)^k + (2k + 1)\pi/2)$ for $k \in \mathbf{Z}$.

b) The tangent line at (a, b) is $y = b + 6a(x - a)$. If it passes through $(-1, -7)$, then $3a^2 + 6a - 9 = 0$, i.e., $a = 1, -3$. Thus the points are $(1, 5)$ and $(-3, 29)$.

4.1.6. For $x < 0$ we have $f^{(n)}(x) = 0$ for all $n \in \mathbf{N}$. Thus we need only check whether $f^{(n)}_{[0,\infty)}(0) = 0$. For $x > 0$ we have $f(x) = x^3$, so

$$f'_{[0,\infty)}(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h^2 = 0.$$

Thus $f'(0) = 0$ exists. For $x > 0$ we have $f'(x) = 3x^2$, so

$$f''_{[0,\infty)}(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} 3h = 0.$$

Thus $f''(0) = 0$ exists. For $x > 0$ we have $f''(x) = 6x$, so

$$f'''_{[0,\infty)}(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} 6 \neq 0.$$

Thus $f'''(0)$ does not exist. Hence, $n = 1, 2$. It won't work for $n \geq 4$ either because f''' is not defined at $x = 0$ so no higher derivative exists by definition.

4.1.7. a) Let $y_n \to x_0 \in (0, \infty)$. If f is continuous at $x = 1$ then $|f(x_0) - f(y_n)| = |f(x_0/y_n)| \to |f(1)| = 0$ as $n \to \infty$, i.e., f is continuous at x_0. The converse is trivial.

b),c) If f is differentiable at $x = 1$ then for any $x \in (0, \infty)$,

$$\frac{f(x+h) - f(x)}{h} = \frac{f((x+h)/x)}{h} = \frac{1}{x}\left(\frac{f(1 + (h/x))}{h/x}\right) \to \frac{f'(1)}{x}$$

as $h \to 0$. Thus $f'(x)$ exists. The converse is trivial.

4.1.8. a) If f has a local maximum at x_0 then $f(x_0 + h) - f(x_0) \leq 0$ for h small. Hence

$$\frac{f(x_0+h) - f(x_0)}{h} \leq 0 \text{ when } h > 0 \text{ and} \frac{f(x_0+h) - f(x_0)}{h} \geq 0 \text{ when } h < 0.$$

b) If f is differentiable at x_0 then taking the limit of both inequalities in part a), we obtain estimates of the derivative of f at x_0. Indeed, $0 \leq f'(x_0) \leq 0$, i.e., $f'(x_0) = 0$.

c) Since f has a local maximum at x_0 if and only if $-f$ has a local minimum, it follows from b) that if f is differentiable at x_0 and f has a local minimum at x_0 then $f'(x_0) = 0$.

d) If $f(x) = x^3$ then $f'(x) = 3x^2$ is zero at $x = 0$ but $f(0)$ is neither a local maximum nor a local minimum.

4.1.9. a) Suppose that f is odd and differentiable on I and $x \in I$. For h so small that $x \pm h \in I$, we have $f(-x+h) = -f(x-h)$ and $-f(-x) = f(x)$. Therefore,

$$\lim_{h\to 0+} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0+} \frac{f(x-h) - f(x)}{-h} = \lim_{h\to 0-} \frac{f(x+h) - f(x)}{h} = f'(x),$$

i.e., the right derivative of f at $-x$ equals $f'(x)$. A similar argument proves that the left derivative of f at x is even. b) Repeat the argument in part a), but this time, $f(-x+h) = f(x-h)$ and $f(-x) = f(x)$.

4.2 Differentiability Theorems.

4.2.0. a) True. Apply the Product Rule twice: $(fgh)' = (fg)'h + (fg)h' = f'gh + fg'h + fgh'$.

b) True. Apply the Chain Rule twice and the Product Rule once: $(g \circ f)''(a) = (g'(f(a)) \cdot f'(a))' = (g''(f(a)) \cdot f'(a)) \cdot f'(a) + g'(f(a)) \cdot f''(a) = g'(f(a))f''(a) + g''(f(a))(f'(a))^2$.

c) True. It's true for $n = 1$. If it's true for some $n \geq 1$, then by the Inductive Hypothesis, definition, and the Sum Rule,

$$(f+g)^{(n+1)} = (f^{(n)} + g^{(n)})' = f^{(n+1)} + g^{(n+1)}.$$

d) False. Let $a \neq 0$, $f(x) = g(x) = x^2$, and $n = 2$. Then $(f/g)'' = 0$ but

$$\frac{g(a)f''(a) + f(a)g''(a)}{g^3(a)} = \frac{4a^2}{a^6}$$

is not zero.

4.2.1. a) By the Product Rule,

$$(fg)'(2) = f'(2)g(2) + f(2)g'(2) = 3a + c.$$

b) By the Quotient Rule,

$$\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{g^2(3)} = \frac{2b - d}{8}.$$

c) By the Chain Rule,

$$(g \circ f)'(3) = g'(f(3))f'(3) = bc.$$

d) By the Chain Rule,

$$(f \circ g)'(2) = f'(g(2))g'(2) = bc.$$

4.2.2. a) By the Product and Chain Rules, $g'(x) = 2x^2 f'(x^2) + f(x^2)$, so $g'(2) = 8f'(4) + f(4) = 8e + 3$.
b) By the Power and Chain Rules, $g'(x) = 2f(\sqrt{x}) \cdot f'(\sqrt{x})/(2\sqrt{x})$, so $g'(4) = 2f(2) \cdot f'(2)/(2\sqrt{4}) = \pi$.
c) By the Quotient and Chain Rules,

$$g'(x) = \frac{f(x^3) \cdot 1 - x \cdot 3x^2 f'(x^3)}{f^2(x^3)},$$

so $g'(\sqrt[3]{2}) = (f(2) - 6f'(2))/f^2(2) = (1 - 3\pi)/2$.

4.2.3 $(x^\alpha)' = (e^{\alpha \log x})' = \alpha e^{\alpha \log x}/x = \alpha x^{\alpha - 1}$ for all $x > 0$.

4.2.4. By Exercise 4.1.2a, $(x^n)' = nx^{n-1}$ for each $n \in \mathbf{N}$. If $P(x) = a_n x^n + \cdots + a_0$, then it follows from Theorem 4.10 that $P'(x) = na_n x^{n-1} + \cdots + a_1$ exists and is a polynomial. Hence by induction, $P^{(k)}$ exists for all $k \in \mathbf{N}$ (and in fact is evidently zero for large k).

4.2.5. a) If $f(a) \neq 0$, $|f(a+h)| > |f(a)|/2 > 0$ for h small (like the proof of Lemma 3.28).
b) Choose $h \neq 0$ small enough so that $f(a+h) \neq 0$. Using $f(a)f(a+h)$ as a common denominator, we have

$$\frac{1}{f(a+h)} - \frac{1}{f(a)} = \frac{f(a) - f(a+h)}{f(a)f(a+h)}.$$

Dividing this by h and taking the limit as $h \to 0$, we find that $1/f$ is differentiable at a with derivative $-f'(a)/f^2(a)$.
c) Using part b) and the product rule,

$$(f/g)' = (f \cdot 1/g)' = f' \cdot \frac{1}{g} - f\frac{g'}{g^2} = \frac{gf' - fg'}{g^2}.$$

4.2.6. By the Product Rule, this formula holds for $n = 1$. Suppose it holds for some $n \geq 1$. Then

$$\begin{aligned}
(fg)^{(n+1)} = ((fg)^{(n)})' &= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) f^{(k)} g^{(n+1-k)} + f^{(0)} g^{(n+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)})
\end{aligned}$$

by Lemma 1.25.

4.2.7. a) It is well-known that if $A, B \in \mathbf{R}$ and $m \in \mathbf{N}$, then

$$A^m - B^m = (A - B)(A^{m-1} + A^{m-2}B + \cdots + AB^{m-2} = B^{m-1}).$$

Thus the desired identity follows from setting $A = x^q$ and $B = a^q$, where $q = n/m$.
b) Let $x, a \in (0, \infty)$ and $q = n/m$. By part a,

$$\frac{x^q - a^q}{x - a} = \frac{x^n - a^n}{x - a} \cdot (x^{q(m-1)} + \cdots + a^{q(m-1)})^{-1} =: y(x) \cdot z(x).$$

By Exercise 4.1.2, $y(x) \to na^{n-1}$ as $x \to a$. Since $z(x)$ contains m terms, it is easy to see that $z(x) \to ma^{q(m-1)}$ as $x \to a$. Therefore, x^q is differentiable at a and the value of its derivative there is

$$na^{n-1} \cdot (ma^{q(m-1)})^{-1} = qa^{n-1-qm+q} = qa^{q-1}.$$

4.2.8. Clearly, $f'(x)$ exists when $x \neq 0$. By definition,

$$f'(0) = \lim_{h\to 0+} \frac{h/(1+e^{1/h}) - 0}{h} = \lim_{h\to 0+} \frac{1}{1+e^{1/h}} = 0.$$

Since this limit is 1 as $h \to 0-$, f is not differentiable at $x = 0$.

4.2.9. a) By assumptions ii) and vi), $0 \le |\sin x| \le |x|$ for $x \in [-\pi/2, \pi/2]$. Thus by the Squeeze Theorem and assumption i), $\sin x \to 0 = \sin(0)$ as $x \to 0$. Hence by assumptions iii) and i), $\cos x = 1 - 2\sin^2(x/2) \to 1 - 2\sin^2(0) = 1 = \cos(0)$ as $x \to 0$.

b) Let $x_0 \in \mathbf{R}$. By assumption iv) and part a), $\sin x = \sin((x-x_0)+x_0) = \sin(x-x_0)\cos x_0 + \cos(x-x_0)\sin x_0 \to 0 + \sin x_0 = \sin x_0$ as $x \to x_0$. Hence by assumption iii), $\cos x = 1 - 2\sin^2(x/2) \to 1 - 2\sin^2(x_0/2) = \cos x_0$ as $x \to x_0$.

c) Let $x \in (0, \pi/2]$. By assumption vi), $0 \le \cos x \le \sin x/x \le 1$. Hence by part a) and the Squeeze Theorem, $\sin x/x \to 1$ as $x \to 0+$. In particular, it follows from assumption ii) that $\sin x/x \to 1$ as $x \to 0$.

Let $x \in (0, \pi/2]$. By assumption i), $\cos^2 x \le \cos x$. Hence by assumptions iii) and vi),

$$0 \le 1 - \cos x \le 1 - \cos^2 x = \sin^2 x \le x^2,$$

i.e., $0 \le (1-\cos x)/x \le x \to 0$ as $x \to 0+$. In particular, it follows from assumption ii) that $(1 - \cos x)/x \to 0$ as $x \to 0$.

d) Let $x \in \mathbf{R}$. By assumption iv) and part c),

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right) \to \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

as $h \to 0$.

e) By assumption v), part d), and the Chain Rule, $(\cos x)' = -\cos(\pi/2 - x) = -\sin x$. Hence it follows from the Quotient Rule and assumption iii) that

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

4.3 The Mean Value Theorem.

4.3.0. a) True. If $x < y$ belong to $[a, b]$, then $f(x) \le f(y)$ and $g(x) \le g(y)$. Adding these inequalities, we obtain $f(x) + g(x) \le f(y) + g(y)$.

b) False. $f(x) = g(x) = x$ are increasing on $[-1, 0]$ but $f(x)g(x) = x^2$ is decreasing on $[-1, 0]$.

c) True. The function $g(x) = f(x)$ for $x \in (a, b)$ and $g(a) = f(a+)$ is continuous on $[a, x]$ for every $x \in (a, b)$. Thus by the Mean Value Theorem, there is a $c \in (a, b)$ such that

$$f(x) - f(a+) = g(x) - g(a) = g'(c)(x-a) = f'(c)(x-a).$$

d) True. For any $x \in (a, b)$, by the Mean Value Theorem, there are points c, d between a and x such that

$$|f(x) - f(a)| = (x-a)|f'(c)| \le (x-a)|g'(d)| = |g(x) - g(a)|.$$

4.3.1. a) Let $f(x) = e^x - 2x - 0.7$. Since $x \ge 1$, $f'(x) = e^x - 2 > 0$. Hence by Theorem 4.17i, f increases on $[1, \infty)$. In particular,

$$e^x - 2x - 0.7 \ge f(1) = e - 2.7 > 0.$$

b) Let $f(x) = \sqrt{x} - \log x - 0.6$. Since $x \ge 4$,

$$f'(x) = 1/(2\sqrt{x}) - 1/x = \frac{x - 2\sqrt{x}}{2x\sqrt{x}} \ge 0.$$

Hence by Theorem 4.17i, f increases on $[4,\infty)$. In particular,

$$\sqrt{x} - \log x - 0.6 \geq f(4) = 2 - \log 4 > 0.$$

c) Let $f(x) = 2|x| - \sin^2 x$ and suppose first that $x \geq 0$. Then $f'(x) = 2 - 2\sin x \cos x \geq 0$. Hence by Theorem 4.17i, f increases on $[0,\infty)$. In particular,

$$2x - \sin^2 x \geq f(0) = 0,$$

i.e., $\sin^2 x \leq 2|x|$ when $x \geq 0$. On the other hand, if $x < 0$, then by what we just showed, $\sin^2 x = \sin^2(-x) \leq 2(-x) = 2|x|$.

d) Let $f(x) = e^x - 1 + \sin x$. Since $x \geq 0$, $f'(x) = e^x + \cos x \geq 1 + \cos x \geq 0$. Hence by Theorem 4.17i, f increases on $[0,\infty)$. In particular,

$$e^x - 1 + \sin x \geq f(0) = 0.$$

4.3.2. By the Mean Value Theorem, $2 = f(2) - f(0) = 2f'(c)$ for some $c \in (0,2)$. Thus $f'(c) = 1$, i.e., $1 \in f'(0,2)$.

4.3.3. If a and b are roots of f, then by the Mean Value Theorem, $0 = f(b) - f(a) = (b-a)f'(c)$ for some $c \in (a,b)$. Hence c is a root of f'.

4.3.4. Suppose $M > 0$ satisfies $|f'(x)| \leq M$ for all $x \in (a,b)$. Let $\varepsilon > 0$ and set $\delta = \varepsilon/M$. By the Mean Value Theorem, if $x, y \in (a,b)$ and $|x-y| < \delta$, then

$$|f(x) - f(y)| = |f'(c)|\,|x-y| \leq M\,|x-y| < \varepsilon.$$

4.3.5. By the Mean Value Theorem, $|f(x) - 1| = |f(x) - f(0)| = |(x-0)f'(c)| = |x| \cdot |f'(c)|$ for some $c \in (0,x)$. Since $|f'(c)| \leq 1$, it follows that $|f(x)| \leq |f(x) - 1| + 1 \leq |x| + 1$ for all $x \in \mathbf{R}$.

4.3.6. By the Mean Value Theorem, $f(c) = f(c) - f(a) = (c-a)f'(x_1)$ and $f(c) = f(c) - f(b) = (c-b)f'(x_2)$ for some $x_1, x_2 \in (a,b)$. If $f(c) > 0$, then $c - a > 0$ and $c - b < 0$ imply that $f'(x_1) > 0 > f'(x_2)$. If $f(c) < 0$, then $c - a > 0$ and $c - b < 0$ imply that $f'(x_1) < 0 < f'(x_2)$.

4.3.7. By the Monotone Property for Suprema, F is increasing on $[a,b]$. Hence by Theorem 4.18, f has one-sided limits at each point in $[a,b]$. By symmetry, it suffices to prove that f is left continuous at each $c \in (a,b]$. Suppose not, i.e., that $F(c) - \lim_{x\to c-} F(x) =: \varepsilon_0 > 0$. Then there is a $\delta_0 > 0$ such that $0 < c - x < \delta_0$ implies $F(c) - F(x) > \varepsilon_0/2$.

By the Extreme Value Theorem, there is a $t_0 \in [a,c]$ such that $f(t_0) = F(c)$. Since f is continuous, we can choose a $\delta \in (0,\delta_0)$ such that $|t - t_0| < \delta$ implies $|f(t) - f(t_0)| < \varepsilon_0/2$. Fix x_0 with $0 < c - x_0 < \delta$. By the choice of δ_0, we have $F(x_0) < F(c)$. Since $f(t) \leq F(x_0)$ for all $t \in [a, x_0]$, it follows that $t_0 \in (x_0, c]$. In particular, $|x_0 - t_0| \leq c - x_0 < \delta$. By the choice of δ, we conclude that $F(x_0) \geq f(x_0) \geq f(t_0) - \varepsilon_0/2 = F(c) - \varepsilon_0/2$, i.e., $F(c) - F(x_0) \leq \varepsilon_0/2$ contrary to the choice of x_0.

4.3.8. By the Mean Value Theorem, $0 < f(x_1) - f(x_2) = (x_1 - x_2)f'(c_1)$ for some $x_1 < c_1 < x_2$ and $0 < f(x_3) - f(x_2) = (x_3 - x_2)f'(c_2)$ for some $x_2 < c_2 < x_3$. Thus $f'(c_1) < 0 < f'(c_2)$. Applying the Mean Value Theorem to f', there is a $c \in (c_1, c_2)$ such that $0 < f'(c_2) - f'(c_1) = (c_2 - c_1)f''(c)$. Since $c_2 > c_1$, it follows that $f''(c) > 0$.

4.3.9. Let A represent the limit of $\{f(n)\}$. By the Mean Value Theorem, $f(n+1) - f(n) = f'(c_n)$ for some $c_n \in (n, n+1)$, $n \in \mathbf{N}$. Since $c_n \to \infty$ as $n \to \infty$, it follows that

$$0 = A - A = \lim_{n\to\infty}(f(n+1) - f(n)) = \lim_{n\to\infty} f'(c_n) = L.$$

4.3.10. a) By Exercise 4.1.8, $f'(x_0) = 0$. Let $\delta > 0$. Since $f(x_0)$ is a proper local maximum, there is a $c \in (x_0 - \delta, x_0)$ such that $f(c) < f(x_0)$. Hence by the Mean Value Theorem there is an $x_1 \in (c, x_0)$ such that $0 < f(x_0) - f(c) = f'(x_1)$. A similar argument shows there is an $x_2 > x_0$ such that $f'(x_2) < 0$.

b) The statement is: If f is differentiable on (a,b) and has a proper local minimum at x_0, then $f'(x_0) = 0$ and given $\delta > 0$ there exist $x_1 < x_0 < x_2$ such that $f'(x_1) < 0$, $f'(x_2) > 0$, and $|x_j - x_0| < \delta$ for $j = 1, 2$.

Proof. The function $g := -f$ has a proper local maximum at x_0, hence by part a), such x_1, x_2 exist which satisfy $g'(x_1) > 0$ and $g'(x_2) < 0$. Since $f' = -g'$, it follows that $f'(x_1) < 0$ and $f'(x_2) > 0$.

4.3.11. Since E is a nonempty subset of $[a, b]$, $\sup E$ is a finite real number which belongs to $[a, b]$. Since f is increasing, $f(x) \le f(\sup E)$ for every $x \in E$. Therefore, $\sup f(E) \le f(\sup E)$. Let $x_k \in E$ such that $x_k \to \sup E$ as $k \to \infty$. Since f is continuous, $f(x_k) \to f(\sup E)$. Therefore, $\sup f(E) \ge f(x_k)$ for all k implies $\sup f(E) \ge f(\sup E)$.

4.3.12. Suppose $c \in (a, b)$ is a point of discontinuity of f'. Since f' is increasing, $f'(c-)$ and $f'(c+)$ exist by Theorem 4.18. Since c is a point of discontinuity, it follows that $f'(c-) < f'(c+)$. Thus by Darboux's Theorem, there is an x_0 between a and b such that $f'(c-) < f'(x_0) < f'(c+)$, a contradiction of the fact that f' is monotone on (a, b).

4.4 Taylor's Theorem and l'Hôpital's Rule.

4.4.0. a) False. $\log x$ is NOT real-valued for $x < 0$.

b) True. Since $|\sin(1/x)/x^n| \le 1/x$ for $x \ge 1$, it follows from the Squeeze Theorem that this limit exists and equals 0.

c) False, but it's not a l'Hôpital problem. $0^{-\infty} = \infty$ is not indeterminate.

d) False. Let $f(x) = x^3 + x$. Then $f'(x) = 3x^2 + 1 > 1$ for all $x \in (0, \infty)$, but $x^2/f(x) < 1/x \to 0$ as $x \to \infty$.

4.4.1. a) If $f(x) = \cos x$, then $f^{(2n)}(x) = (-1)^n \cos x$ and $f^{(2n-1)}(x) = (-1)^n \sin x$. Thus

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}.$$

b) By Taylor's Formula, there is a c between x and 0 such that $|\cos x - P_{2n}(x)| = |(-1)^{n+1}(\sin c) \cdot x^{2n+1}|/(2n+1)!$. Thus $|\cos x - P_{2n}(x)| \le 1/(2n+1)!$ for $x \in [-1, 1]$.

c) $1/(2n+1)! \le 0.00000005$ implies $(2n+1)! \ge 20,000,000$. Since $9! = 362,880$ and $11! = 39,916,800$, it follows that $n \ge 5$.

4.4.2. a) If $f(x) = \log x$, then $f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n$. Thus

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^{k-1}(x-1)^k}{k}.$$

b) By Taylor's Formula, there is a c between x and 1 such that $|\log x - P_n(x)| = |(-1)^n (x-1)^{n+1}|/(c^{n+1}(n+1))$. Since $x \in [1, 2]$, we have $|x - 1| = x - 1 \le 1$ and $c \ge 1$. Thus $|\log x - P_n(x)| \le 1/(n+1)$ for $x \in [1, 2]$.

c) $1/(n+1) \le 0.0005$ implies $(n+1) \ge 2000$, i.e., $n \ge 1999$.

4.4.3. By Taylor's Formula, there is a c between x and 0 such that

$$e^x - \left(1 + x + \cdots + \frac{x^n}{n!}\right) = \frac{e^c x^{n+1}}{(n+1)!}.$$

Since $x > 0$ implies $x^{n+1} > 0$ and $e^c > 1$, it follows that

$$e^x > 1 + x + \cdots + \frac{x^n}{n!}.$$

4.4.4. By Taylor's Formula, there is a c between x and 0 such that

$$\sin x - \left(x - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \frac{(-1)^{n+1}(\sin c) \cdot x^{2n+2}}{(2n+2)!} =: R.$$

Since $x \in (0, \pi)$ implies $x^{n+1} > 0$ and $\sin c > 0$, it follows that $R > 0$ when $n+1$ is even, i.e., when $n = 2m - 1$ and $R < 0$ when $n + 1$ is odd, i.e., when $n = 2m$. Therefore,

$$\left(x - \cdots + \frac{(-1)^n x^{4m-1}}{(4m-1)!}\right) < \sin x < \left(x - \cdots + \frac{(-1)^n x^{4m+1}}{(4m+1)!}\right).$$

4.4.5. a) $\lim_{x\to 0}\sin^2(5x)/x^2 = (\sin(5x)/x)^2 = (\lim_{x\to 0} 5\cos(5x)/1)^2 = 25$.

b) $\lim_{x\to 0+}(\cos x - e^x)/(\log(1+x^2)) = \lim_{x\to 0+}(-\sin x - e^x)/(2x/(1+x^2)) = -\infty$.

c) $\lim_{x\to 0}\log(x/\sin x)/x^2 = \lim_{x\to 0}(\sin x - x\cos x)/(2x^2\sin x) = \lim_{x\to 0} x\sin x/(2x^2\cos x + 4x\sin x) = \lim_{x\to 0}\sin x/(2x\cos x + 4\sin x) = \lim_{x\to 0}\cos x/(6\cos x - 2x\sin x) = 1/6$. Therefore, the original limit is $e^{1/6}$.

d) $\lim_{x\to 0+}\log(1-x^2)/x = \lim_{x\to 0+}(1-x^2)^{-1}(-2x)/1 = \lim_{x\to 0+} -2x/(1-x^2) = 0$. Thus the original limit is $e^0 = 1$.

e) $\lim_{x\to 1}\log x/(\sin(\pi x)) = \lim_{x\to 1}(1/x)/(\pi\cos(\pi x)) = -1/\pi$.

f) $\lim_{x\to 0+}\log(\log x)/(1/x) = \lim_{x\to 0+}(1/(x\log x)/(-1/x^2) = \lim_{x\to 0+} -x/\log x = \lim_{x\to 0+} -1/(1/x) = 0$. Thus the original limit is $e^0 = 1$.

g) Multiplying top and bottom by $(\sqrt{x^2+2}+\sqrt{x^2})(\sqrt{2x^2-1}+\sqrt{2x^2})$ we obtain

$$\frac{2}{-1}\frac{\sqrt{2x^2-1}+\sqrt{2x^2}}{\sqrt{x^2+2}+\sqrt{x^2}} = -2\frac{\sqrt{2-1/x^2}+\sqrt{2}}{\sqrt{1+2/x^2}+\sqrt{1}} \to -2\sqrt{2}$$

as $x\to\infty$.

h) Multiplying top and bottom by $(\sqrt{x+4}+\sqrt{x+1})(\sqrt{x+3}+\sqrt{x+1})$ we obtain

$$\frac{3}{2}\frac{\sqrt{x+3}+\sqrt{x+1}}{\sqrt{x+4}+\sqrt{x+1}} = \frac{3}{2}\frac{\sqrt{1+3/x}+\sqrt{1+1/x}}{\sqrt{1+4/x}+\sqrt{1+1/x}} \to \frac{3}{2}$$

as $x\to\infty$.

4.4.6. a) Let $f(x) = \log x/x^\alpha$. Since $f(x)\to 0$ as $x\to\infty$, $f(x) < 1$ for large x. Also, $0 = f'(x) = (1-\alpha\log x)/x^{\alpha+1}$ implies $\log x = 1/\alpha$, i.e., $x = e^{1/\alpha}$. Since $f'(x) > 0$ when $x < e^{1/\alpha}$ and $f'(x) < 0$ when $x > e^{1/\alpha}$, $C_\alpha := f(e^{1/\alpha}) = 1/(\alpha e)$ is the absolute maximum of f on $[1,\infty)$. Note that $C_\alpha\to\infty$ as $\alpha\to 0+$ and $C_\alpha \to 0$ as $\alpha\to\infty$.

b) The statements are: $x^\alpha \le e^x$ for x large, and there is a constant B_α such that $x^\alpha \le B_\alpha e^x$ for all $x\in(0,\infty)$, $B_\alpha\to 1$ as $\alpha\to 0+$, and $B_\alpha\to 0$ as $\alpha\to\infty$. Let $f(x) = x^\alpha/e^x$. Then $0 = f'(x) = x^{\alpha-1}(\alpha - x)/e^x$ implies $x=\alpha$. Since $f'(x) > 0$ when $x<\alpha$ and $f'(x) < 0$ when $x > \alpha$, $B_\alpha := f(\alpha) = (\alpha/e)^\alpha$ is the absolute maximum of f on $(0,\infty)$. By L'Hôpital's Rule, $B_\alpha\to 1$ as $\alpha\to 0+$ and $B_\alpha\to 0$ as $\alpha\to\infty$.

4.4.7. a) $f'(x) = 2e^{-1/x^2}/x^3$ is evidently continuous for $x\ne 0$. Also, by L'Hôpital's Rule, $\lim_{x\to 0} f'(x) = \lim_{x\to 0}(2/x^3)/e^{1/x^2} = \lim_{x\to 0}(6/x^4)/(2e^{1/x^2}/x^3) = \lim_{x\to 0}(3/x)/e^{1/x^2} = \lim_{x\to 0}(3/x^2)/(2e^{1/x^2}/x^3) = 0$. On the other hand, $f'(0) := \lim_{h\to 0}(e^{-1/h^2}-0)/h = \lim_{h\to 0}(1/h)/e^{1/h^2} = 0$. Thus f' exists and is continuous on $\mathbf{R}$.

b) We first prove that the functions $g(x) = e^{-1/x^2}/x^k$ satisfy $g(x)\to 0$ as $x\to 0$ for all integers $k\ge 0$. Indeed, this surely holds for $k=0$. Suppose it holds for some all $k\in[0,j]$ for some $j\ge 0$. Then

$$L := \lim_{x\to 0}\frac{e^{-1/x^2}}{x^{j+1}} = \lim_{x\to 0}\frac{1/x^{j+1}}{e^{1/x^2}} = \lim_{x\to 0}\frac{(j+1)/x^{j+2}}{2e^{1/x^2}/x^3} = \lim_{x\to 0}\frac{(j+1)/x^{j-1}}{2e^{1/x^2}}.$$

If $j-1\le 0$ then $1/x^{j-1} = x^{1-j}$ is bounded near 0, hence $L=0$. If $k := j-1 \ge 0$ then $0\le k<j$. Hence $L=0$ by the inductive hypothesis. Therefore, $g(x)\to 0$ as $x\to 0$ for all $k\ge 0$.

Next, notice by part a) that $f'(x) = (2/x^3)e^{-1/x^2}$ for $x\ne 0$ and $f'(0)=0$. Suppose

$$(*)\qquad f^{(n)}(x) = \begin{cases} \sum_{k=n+2}^N (a_k/x^k)e^{-1/x^2} & x\ne 0 \\ 0 & x = 0. \end{cases}$$

Then $f^{(n+1)}(x) = \sum_{k=n+2}^N(-ka_k/x^{k+1})e^{-1/x^2} + \sum_{k=n+2}^N(2a_k/x^{k+3})e^{-1/x^2}$. Moreover, by the claim and (*),

$$f^{(n+1)}(0) := \lim_{x\to 0}\frac{1}{x}\left(\sum_{k=n+2}^N (a_k/x^k)e^{-1/x^2} - 0\right) = 0.$$

Hence by induction, given $n\in\mathbf{N}$, there are integers $N = N(n)\in\mathbf{N}$, and $a_k = a_k^{(n)}\in\mathbf{Z}$ such that (*) holds. Moreover, by the claim,

$$\lim_{x\to 0} f^{(n+1)}(x) = \lim_{x\to 0}\left(\sum_{k=n+2}^N(-ka_k/x^{k+1})e^{-1/x^2} + \sum_{k=n+2}^N(2a_k/x^{k+3})e^{-1/x^2}\right) = 0 = f^{(n+1)}(0).$$

In particular, $f^{(n)}$ exists and is continuous on $\mathbf{R}$ for all $n \in \mathbf{N}$ and $f^{(n)}(0) = 0$.

4.4.8. The Taylor polynomials $P = P_{n-1}^{f,x_0}$ at $x_0 = a$ and $x_0 = b$ are zero. Thus by Taylor's Formula, there is an x_1 between a and c such that

$$f(c) = f^{(n)}(x_1)\frac{(c-a)^n}{n!}$$

and an x_2 between b and c such that

$$f(c) = f^{(n)}(x_2)\frac{(c-b)^n}{n!}.$$

Since n is odd, $(c-b)^n/n! < 0$ and $(c-a)^n/n! > 0$. Since $f(c)$ is the product of these factors and $f^{(n)}(x_j)$, it follows that the $f^{(n)}(x_j)$'s must have different signs.

4.4.9. a) Let $f(x) = \sin(x+\pi)$. Since $\sin(x+\pi) = -\sin x$, we have by Taylor's formula that

$$|\sin(x+\pi)+\delta| = |\delta - \sin\delta| = |-\cos c|\left|\frac{\delta^3}{3!}\right| \le \frac{\delta^3}{3!}.$$

b) Let $\delta_0 = |x-\pi|$. By a), $|(x-\pi)+\sin x| = |\delta_0 + \sin(\delta_0+\pi)| \le |x-\pi|^3/3! = \delta_0^3/3! < \delta^3/3!$.

4.4.10. We may suppose that $B = +\infty$. Consider the quotient g/f. By hypothesis, g and f are differentiable on $I \setminus \{a\}$, the limit of g/f is of the form $0/0$ or ∞/∞, and the limit of g'/f' is zero. Moreover, since $B = \infty$, we know that $f'(x)$ cannot be zero for large x. If $A = 0$, then $f(x) = f(x) - f(a) = f'(c)(x-a) \neq 0$, and if $A = \infty$, then $f(x)$ is large, hence also nonzero for large x. It follows that there is an interval $J \subset I$ which either contains a or has a as an endpoint on which both f and f' are never zero. Hence all the hypotheses of L'Hôpital's Rule are satisfied by g/f for the case when $B = 0$. Since that case has been proved in the text, it follows that

$$\lim_{\substack{x\to a\\ x\in I}} \frac{g(x)}{f(x)} = \lim_{\substack{x\to a\\ x\in I}} \frac{g'(x)}{f'(x)} = 0.$$

But $B = \infty$ implies that $f(x)/g(x)$ is positive for large x. In particular, we conclude that $f(x)/g(x) \to \infty = B$ as $x \to a$ through I.

4.4.11. Let $f(a) = g(a) = 0$ and $x \in I \setminus \{a\}$. By the Generalized Mean Value Theorem,

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$$

for some c between x and a. Since $c \in I$, we have by hypothesis that this last quotient is $\le M$. Therefore, $|f(x)|/|g(x)| \le M$.

If $f(x) = x + 100$ and $g(x) = x^2$, then $|f'/g'| = |1/x| \le 1$ for $x \in (1,\infty)$, but $(x+100)/x^2$ is not less than or equal to 1 when $x = 2$.

4.5 Inverse Functions.

4.5.0. a) False because I need not be an interval. For example, $f(x) = x$ for $x < 0$ and $= 1 - x$ for $x > 0$ is 1–1 on $[-1,0) \cup (0,1]$ but increases on the left half interval and decreases on the right half interval.

b) False. By the Inverse Function Theorem,

$$0 = (f^{-1})'(b) = \frac{1}{f'(a)}$$

implies $0 = 1$, a contradiction.

c) False. Let f and g be as in the solution to 3.3.0d. Then both f and g are strictly decreasing, hence 1–1 on $\mathbf{R}$.

d) True. Since $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$, combining the Inverse Function Theorem and the Chain Rule yields

$$((g\circ f)^{-1})'(g(b)) = \frac{1}{g'(g^{-1}(g(b)))\cdot f'(f^{-1}(g(g^{-1}(b))))} = \frac{1}{g'(b)\cdot f'(a)}.$$

4.5.1. a) By the Inverse Function Theorem,

$$(f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{\pi}.$$

b) By the Inverse Function Theorem,

$$(g^{-1})'(2) = \frac{1}{g'(1)} = \frac{1}{e}.$$

c) By the Product Rule and the Inverse Function Theorem,

$$\begin{aligned}(f^{-1}g^{-1})'(2) &= f^{-1}(2)(g^{-1})'(2) + g^{-1}(2)(f^{-1})'(2) \\ &= 0 \cdot \frac{1}{g'(1)} + 1 \cdot \frac{1}{f'(0)} = \frac{1}{\pi}.\end{aligned}$$

4.5.2. a) By the Intermediate Value Theorem, $f((0,\infty)) = (0,\infty)$. Also, $f'(x) = x^2(2xe^{x^2}) + 2xe^{x^2} = 2xe^{x^2}(x^2+1) > 0$ when $x > 0$, so f is strictly increasing, hence 1–1 on $(0,\infty)$. Since $f'(x)$ exists and is nonzero for all $x \in (0,\infty)$, it follows from Theorem 4.33 that f^{-1} is differentiable on $(0,\infty)$ and $(f^{-1})'(x) = 1/f'(f^{-1}(x))$.

b) $f^{-1}(e) = 1$. Thus by part a), $(f^{-1})'(e) = 1/f'(1) = 1/(4e)$.

4.5.3. Let $f(x) = \sin x$. Then $f'(x) = \cos x > 0$ for $x \in (-\pi/2, \pi/2)$ and $f(-\pi/2,\pi/2) = (-1,1)$. Hence by Theorem 4.33, $f^{-1}(x) := \arcsin x$ is differentiable on $(-1,1)$ with $(\arcsin x)' = 1/\cos y$ for $x = \sin y$. But by trigonometry, $\cos y = \sqrt{1-x^2}$. Hence $(\arcsin x)' = 1/\sqrt{1-x^2}$. Similarly, $(\arctan x)' = 1/\sec^2 y = 1/(1+x^2)$ for $x = \tan y \in (-\infty,\infty)$.

4.5.4. a) Since $f'(x) \neq 0$ and f' is continuous, it follows from the Intermediate Value Theorem that either $f' > 0$ on (a,b) or $f' < 0$ on (a,b). Thus either f is strictly increasing on (a,b) and takes (a,b) into $(f(a+), f(b-))$ or f is strictly decreasing and takes (a,b) into $(f(b-), f(a+))$. In both cases, f is onto by the Intermediate Value Theorem.

b) By Theorem 4.33, f^{-1} is differentiable on (c,d) and $(f^{-1})'(x) = 1/f'(f^{-1}(x))$. Let $x_n \to x_0 \in (c,d)$. Then $f^{-1}(x_n) \to f^{-1}(x_0)$ by Theorem 4.32. Thus $(f^{-1})'(x_n) = 1/f'(f^{-1}(x_n)) \to 1/f'(f^{-1}(x_0)) = (f^{-1})'(x_0)$ as $n \to \infty$. Hence by the Sequential Characterization of Continuity, $(f^{-1})'$ is continuous at x_0.

c) Let $a = -1$, $b = 1$, and $f(x) = x^3$. Then $f'(0) = 0$ and $f^{-1}(x) = \sqrt[3]{x}$ has no derivative at $x = 0$.

d) Since the range of $\tan x$ on $(-\pi/2, \pi/2)$ is $(-\infty,\infty)$, $c = -\infty$ and $d = \infty$.

4.5.5. a) Since $L(x)$ is the inverse of a^x and $(a^x)^y = a^{xy}$, it is easy to see that $L(x^y) = yL(x)$ for all $x \in (0,\infty)$ and $y \in \mathbf{R}$. Since Theorem 4.32 implies that L is continuous on $(0,\infty)$, it follows from hypothesis that

$$\lim_{t\to\infty} tL(1+1/t)) = \lim_{t\to\infty} L((1+1/t)^t) = L(a) = 1.$$

b) Fix $h > 0$ and let $y = a^h - 1$. Since $L(x)$ is the inverse function of a^x, we have $L(1+y) = h$. Therefore,

$$\frac{a^h - 1}{h} = \frac{y}{L(1+y)} = \frac{1}{tL(1+1/t)}$$

for $t = 1/y$. But $h \to 0+$ implies that $y \to 0+$ hence $t \to \infty$. We conclude by part a) that

$$\lim_{h\to 0+} \frac{a^h-1}{h} = \lim_{t\to\infty} \frac{1}{tL(1+1/t)} = 1.$$

A similar argument shows that $(a^h - 1)/h \to 1$ as $h \to 0-$.

c) By part b) and hypothesis, if $f(x) = a^x$, then

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \cdot 1 = a^x.$$

d) By the Inverse Function Theorem and part c), L is differentiable on $(0,\infty)$ and

$$L'(x) = \frac{1}{a^{L(x)}} = \frac{1}{x}.$$

4.5.6. a) Suppose that $x_1 \neq x_2$ belong to I. By the Mean Value Theorem, there is a c between x_1 and x_2 such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) \neq 0$, it follows that $f(x_2) \neq f(x_1)$, i.e., that f is 1–1 on I. Since differentiability implies continuity, it follows from the Inverse Function Theorem that f^{-1} is differentiable on $f(I)$, and $(f^{-1})'(x) = 1/f'(f^{-1}(x))$.

b) Since f' is continuous and nonzero on the closed, bounded interval I, we know from the Extreme Value Theorem that there is a $c \in I$ such that $|f'(x)| \geq |f'(c)| > 0$ for all $x \in I$. By part a, then, $|f^{-1}(x)| \leq 1/|f'(c)| < \infty$ for all $x \in I$.

4.5.7. By 4.5.6a, f^{-1} is differentiable on $[c, d]$. Let $x \in [c, d]$. Since $d - c \geq 2$, choose $x_2 \in [c, d]$ such that $x - x_2 = \pm 1$. Apply the Mean Value Theorem and the Inverse Function Theorem to f^{-1}. We find that there is an $x_0 \in (c, d)$ such that

$$f^{-1}(x) - f^{-1}(x_2) = (f^{-1})'(c) \cdot (x - x_2) = \frac{\pm 1}{f'(f^{-1}(x_0))}.$$

Thus set $x_1 = f^{-1}(x_0)$.

4.5.8. By 4.5.6a, f^{-1} is differentiable on $[a, b]$. Hence by the Generalized Mean Value Theorem, for each $x \in [a, b]$ there exists a c between a and x such that

$$f'(c)(f^{-1}(x) - f^{-1}(a)) = (f^{-1})'(c)(f(x) - f(a)).$$

Set $x_1 = c$, $x_2 = f^{-1}(c)$, and apply the Inverse Function Theorem. We conclude that

$$f'(x_1)(f^{-1}(x) - f^{-1}(a)) = \frac{f(x) - f(a)}{f'(x_2)}.$$

4.5.9. Since f is 1–1 and $f'(x) \neq 0$ for each $x \in (a, b)$, we have for each $y = f(x)$, $x \in (a, b)$, that $(f^{-1})'(y) = 1/f'(x)$ by Theorem 4.33. Hence by hypothesis, $f'(x)/\alpha = 1/f'(x)$, i.e., $(f'(x))^2 = \alpha$ for each $x \in (a, b)$. We conclude that $f'(x) = \pm\sqrt{\alpha}$, i.e., $f(x) = \pm\sqrt{\alpha}\, x + c$ for some $c \in \mathbf{R}$.

4.5.10. Suppose $f'(x_0) > 0$. Since f' is continuous, it follows from the sign preserving lemma that there is an interval $I \subset (a, b)$ containing x_0 such that $f'(x) > 0$ for all $x \in I$. In particular, f is 1–1 (actually monotone increasing) on I. By Exercise 4.5.4b, it follows that f takes I onto some interval J, and f^{-1} is continuously differentiable on J.

4.5.11. Suppose f' is not strictly monotone on $[a, b]$, in fact, not strictly increasing. Then there are numbers $x_1 < x_2 < x_3$ in $[a, b]$ such that $f'(x_0) < f'(x_3) < f'(x_2)$. Thus by Theorem 4.23, there is an $x_0 \in (x_1, x_2)$ such that $f'(x_0) = f'(x_3)$. This contradicts the fact that f' is 1–1 on $[a, b]$. A similar argument works if f is not strictly decreasing.

CHAPTER 5

5.1 The Riemann Integral.

5.1.0. a) False. See Example 5.12.

b) False. Let $f(x) = 1$ for $x \in \mathbf{Q}$ and $f(x) = -1$ for $x \notin \mathbf{Q}$. Repeating the argument in Example 5.12, we can prove that f is not integrable on $[0, 1]$. However, $|f| = 1$ is integrable on $[0, 1]$.

c) False. If f is NOT integrable, the symbol $\int_a^b f(x)\,dx$ makes no sense.

d) False. If $f(x) = 1/x$ for $x \in [-1, 0)$ and $f(x) = 0$ for $x \in [0, 1]$, then f is not integrable because it's not even bounded below, so the lower Riemann sums are not finite.

5.1.1. a) $L(f, P) = 0.5f(0) + 0.5f(0.5) + f(1) = 17/16$.
$U(f, P) = 0.5f(0.5) + 0.5f(1) + f(2) = 137/16$.

The lower one is closer because $y = x^3$ is concave up on $[0, 2]$, hence closer to the lower sum approximation than the upper sum approximation.

b) $L(f, P) = 0.5f(0.5) + 0.5f(1) + f(2) = 11/8$.
$U(f, P) = 0.5f(0) + 0.5f(0.5) + f(1) = 39/8$.

The upper one is closer because $y = 3 - x^2$ is concave down on $[0, 2]$, hence closer to the upper sum approximation than the lower sum approximation.

c) $L(f, P) = 0.5f(0) + 0.5f(0.5) + f(1) \approx 0.2485860$.
$U(f, P) = 0.5f(0.5) + 0.5f(1) + f(2) \approx 0.5386697$.

The upper one is closer because $y = \sin(x/5)$ is concave down on $[0, 2]$, hence closer to the upper sum approximation than the lower sum approximation.

5.1.2. a) The points are obviously increasing, beginning with $0/n = 0$ and ending with $n/n = 1$.

b) By definition,

$$L(f, P_n) \le (L)\int_0^1 f(x)\,dx \le (U)\int_0^1 f(x)\,dx \le U(f, P_n)$$

for each $n \in \mathbf{N}$. If we let $n \to \infty$, then

$$\lim_{n\to\infty} L(f, P_n) \le (L)\int_0^1 f(x)\,dx \le (U)\int_0^1 f(x)\,dx \le \lim_{n\to\infty} U(f, P_n).$$

Since these two limits are equal, it follows that

$$(L)\int_0^1 f(x)\,dx = (U)\int_0^1 f(x)\,dx = \lim_{n\to\infty} U(f, P_n) := I.$$

Thus f is integrable on $[0, 1]$ and its integral equals I.

α) Since $f(x) = x$ is increasing, $M_j = x_j := j/n$ and $m_j = x_{j-1} := (j-1)/n$. Thus

$$U(f, P_n) - L(f, P_n) = \sum_{j=1}^n (x_j - x_{j-1})(x_j - x_{j-1}) = \frac{1}{n^2}\sum_{j=1}^n 1 = \frac{1}{n} \to 0$$

as $n \to \infty$, so f is integrable by Definition 5.9. Since

$$U(f, P_n) = \frac{1}{n^2}\sum_{k=1}^n k = \frac{n(n+1)}{2n^2} \to \frac{1}{2}$$

as $n \to \infty$, $\int_0^1 x\,dx = 1/2$.

β) Since $f(x) = x^2$ is increasing on $[0, \infty)$, $M_j = x_j^2 := (j/n)^2$ and $m_j = x_{j-1}^2 := ((j-1)/n)^2$. Thus

$$U(f, P_n) - L(f, P_n) = \sum_{j=1}^n (x_j^2 - x_{j-1}^2)(x_j - x_{j-1}) = \frac{1}{n^3}\sum_{j=1}^n (2j-1) = \frac{n^2}{n^3} \to 0$$

as $n \to \infty$, so f is integrable by Definition 5.9. Since

$$U(f, P_n) = \frac{1}{n^3}\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6n^3} \to \frac{1}{3}$$

as $n \to \infty$, $\int_0^1 x^2\,dx = 1/3$.

γ) Since the cases n odd and n even are similar, we will suppose that n is even. In this case, $M_j = m_j = 0$ when $j < n/2$, $M_j = m_j = 1$ when $j > n/2+1$, and $M_j(f) = 1 = m_j(f) + 1$ otherwise. Thus

$$U(f,P_n) - L(f,P_n) = \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \to 0$$

as $n \to \infty$, so f is integrable by Definition 5.9. Since

$$U(f,P_n) = \frac{1}{n}\sum_{k=n/2}^{n} 1^k = \frac{2n-n}{2n} \to \frac{1}{2}$$

as $n \to \infty$, $\int_0^1 f(x)\,dx = 1/2$.

5.1.3. a) Let $\epsilon > 0$ and suppose that f is bounded on $[a,b]$, say by $M > 0$, and continuous on $[a,b]$ except at a finite set E. Choose a partition $\{x_0, x_1, \dots, x_{2n+1}\}$ of $[a,b]$ such that each $x \in E$ belongs to $[x_{2k}, x_{2k+1}]$ for some $0 \le k \le n$, and

$$\sum_{k=0}^{n} |x_{2k} - x_{2k+1}| = \frac{\epsilon}{4M}.$$

Let $E_0 := \cup_{k=1}^n [x_{2k-1}, x_{2k}]$ and observe that f is continuous (hence, uniformly continuous) on E_0. Thus choose $\delta > 0$ such that

$$|x-y| < \delta \text{ and } x, y \in E_0 \text{ imply } |f(x) - f(y)| < \frac{\epsilon}{2(b-a)}.$$

Let $P_0 = \{t_0, t_1, \dots, t_m\}$ be a refinement of P obtained by adding enough points so that $\|P_0\| < \delta$.

For each $j = 0, 1, \dots, m$, let $I_j := [t_j, t_{j+1}]$, $|I_j| = t_j - t_{j-1}$, $M_j = \sup f(I_j)$, and $m_j = \inf f(I_j)$. Since f is bounded by M and $\|P_0\| < \delta$, it is clear that

$$M_j - m_j \le \begin{cases} 2M & \text{when } I_j \subseteq [x_{2k}, x_{2k+1}] \text{ for some } k \\ \epsilon/(2b-2a) & \text{when } I_j \subseteq [x_{2k-1}, x_{2k}] \text{ for some } k\ . \end{cases}$$

Therefore,

$$\begin{aligned} U(f,P_0) - L(f,P_0) &= \sum_{I_j \subseteq [x_{2k}, x_{2k+1}]} (M_j - m_j)|I_j| + \sum_{I_j \subseteq [x_{2k-1}, x_{2k}]} (M_j - m_j)|I_j| \\ &< 2M \sum_{I_j \subseteq [x_{2k}, x_{2k+1}]} |I_j| + \frac{\epsilon}{2(b-a)} \sum_{j=1}^{n} |I_j| \\ &= 2M \cdot \frac{\epsilon}{4M} + \frac{\epsilon}{2(b-a)} \cdot (b-a) = \epsilon. \end{aligned}$$

b) Since $L(f,P) = 0$ for all partitions P, $(L)\int_0^1 f(x)\,dx = 0$. On the other hand, given $\epsilon > 0$, choose an integer $n > 1$ so large that $1/n < \epsilon/2$ and choose $\delta > 0$ so small that $2(n-1)\delta < \epsilon/2$. Define x_j's by

$$\begin{aligned} x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{1}{n-1} - \delta < x_3 = \frac{1}{n-1} + \delta \\ < x_4 = \frac{1}{n-2} - \delta < \cdots < x_{2n-3} = 1 - \delta < x_{2n-2} = 1. \end{aligned}$$

If $P = \{x_0, x_1, \dots, x_{2n-2}\}$ then $M_k(f) = 1$ for all odd k, $M_2(f) = 1$, and $M_k(f) = 0$ for all even $k > 2$. Hence

$$U(f,P) = \frac{2}{n} + \sum_{k=1}^{n-1} 2\delta = \frac{2}{n} + 2(n-1)\delta < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $(U)\int_0^1 f(x)\,dx < \epsilon$ and it follows that $(U)\int_0^1 f(x)\,dx = 0$. Hence f is integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

5.1.4. a) If $f(x_0) \neq 0$ then given $\epsilon > 0$ choose by the Sign Preserving Property a $\delta > 0$ such that $|f(x)| \geq \epsilon$ for $|x - x_0| \leq \delta$. Let P be any partition which satisfies $x_{j-1} = x_0 - \delta$ and $x_j = x_0 + \delta$ for some j. Then $m_j(|f|) \geq \epsilon$ and it follows that

$$(L)\int_a^b |f(x)|\,dx \geq L(|f|, P) \geq m_j(|f|)(x_j - x_{j-1}) \geq \epsilon(2\delta) > 0.$$

b) Suppose $\int_a^b |f(x)|\,dx = 0$. If $f(x_0) \neq 0$ for some $x_0 \in [a,b]$, then by part a), $\int_a^b |f(x)|\,dx > 0$, a contradiction. Thus $f(x) = 0$ for all $x \in [a,b]$. The converse is trivial.

c) No. If $f(x) = x$, then $\int_{-1}^1 f(x)\,dx = 0$ but $f(x) \neq 0$ for all $x \neq 0$.

5.1.5. By hypothesis, $\int_c^d f(x)\,dx = \int_a^d f(x)\,dx - \int_a^c f(x)\,dx = 0$ for all $c, d \in [a,b]$. If $f(x_0) > 0$ for some $x_0 \in [a,b]$ then by the Sign Preserving Property there is a nondegenerate interval $[c,d] \subset [a,b]$ such that $f(x) > \epsilon_0$ for $x \in [c,d]$. Therefore, $\int_c^d f(x)\,dx \geq \epsilon_0(d-c) > 0$, a contradiction. A similar argument shows $f(x_0) < 0$ is also impossible. Thus $f(x) = 0$ for all $x \in [a,b]$. The converse is trivial.

5.1.6. Let m be the number of points in E. Since f and g are bounded, choose $C > 0$ such that $|f(x)| \leq C$ and $|g(x)| \leq C$ for $x \in [a,b]$. Since f is integrable, there is a partition P of $[a,b]$ such that $\|P\| < \epsilon/(8mC)$,

$$U(f,P) < \int_a^b f(x)\,dx + \frac{\epsilon}{2} \quad \text{and} \quad L(f,P) > \int_a^b f(x)\,dx - \frac{\epsilon}{2}.$$

Let $P_0 = P \cup E := \{x_0, \dots, x_n\}$. Set $A = \{j : E \cap [x_{j-1}, x_j] \neq \emptyset\}$ and $B = \{1, 2, \dots, n\} \setminus A$ and observe that $M_j(g) \leq C$ for all j and $M_j(g) = M_j(f)$ for all $j \in B$. Also notice, since a point of E can belong to at most two intervals of the form $[x_{j-1}, x_j]$, that the number of points in A is at most $2m$. Finally, since P_0 is finer than P, $U(f, P_0) \leq U(f,P)$. Therefore,

$$\begin{aligned} U(g, P_0) &= \sum_{j\in A} M_j(g)\Delta x_j + \sum_{j \in B} M_j(g)\Delta x_j \\ &= \sum_{j\in A} (M_j(g) - M_j(f))\Delta x_j + U(f, P_0) \\ &\leq \sum_{j \in A} 2C\Delta x_j + U(f,P) \\ &< 4mC\frac{\epsilon}{8mC} + \int_a^b f(x)\,dx + \frac{\epsilon}{2} \\ &= \int_a^b f(x)\,dx + \epsilon. \end{aligned}$$

It follows that $(U)\int_a^b g(x)\,dx < \int_a^b f(x)\,dx + \epsilon$. Taking the limit of this inequality as $\epsilon \to 0$, we obtain $(U)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$. Repeating this argument using lower sums and lower integrals, we obtain $(L)\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$. We conclude that g is integrable and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

5.1.7. a) Let P_1 and P_2 be partitions of $[a,b]$, and let $P = P_1 \cup P_2$. Since $M_j(f+g) \leq M_j(f) + M_j(g)$, we have

$$(U)\int_a^b (f(x) + g(x))\,dx \leq U(f+g, P) \leq U(f,P) + U(g,P) \leq U(f,P_1) + U(g,P_2).$$

Taking the infimum of this inequality over all partitions P_1 and P_2, we obtain $(U)\int_a^b (f(x)+g(x))\,dx \leq (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx$. A similar argument establishes an analogous inequality for lower integrals.

b) Let P be a partition of $[a,b]$, $P_0 = P \cup \{c\}$, $P_1 = P_0 \cap [a,c]$, and $P_2 = P_0 \cap [c,b]$. Since P_0 is finer than P,

$$U(f,P) \geq U(f,P_0) = U(f,P_1) + U(f,P_2) \geq (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx.$$

Thus $(U)\int_a^b f(x)\,dx \geq (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx$. On the other hand, if P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$ then $P = P_1 \cup P_2$ is a partition of $[a,b]$ and

$$(U)\int_a^b f(x)\,dx \leq U(f,P) = U(f,P_1) + U(f,P_2).$$

Taking the infimum of this inequality first over all partitions P_1 of $[a,c]$ and then over all partitions P_2 of $[c,b]$, we obtain $(U)\int_a^b f(x)\,dx \le (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx$. Hence these quantities are equal. A similar argument establishes an analogous identity for lower integrals.

5.1.8. Given $\epsilon > 0$, let P be any partition of $[a,b]$ which satisfies $\|P\| < \epsilon/((f(b)-f(a))$. Since f is increasing, $M_j = f(x_j)$ and $m_j = f(x_{j-1})$ for all j. Thus by telescoping,

$$\sum_{j=1}^{n}(M_j(f)-m_j(f))(x_j-x_{j-1}) \le \sum_{j=1}^{n}(f(x_j)-f(x_{j-1}))\|P\| = (f(b)-f(a))\|P\| < \epsilon.$$

In particular, f is integrable on $[a,b]$.

5.1.9. Let $\varepsilon > 0$ and choose a partition $P = \{x_0,\dots,x_n\}$ of $[a,b]$ such that $U(f,P)-L(f,P) < 2\sqrt{c}\,\varepsilon$. Fix $x,y \in [x_{k-1},x_k]$. Since $|f(u)| \ge c > 0$ for all $u \in [a,b]$, we have by rationalizing the numerator that

$$|\sqrt{f(x)}-\sqrt{f(y)}| \le \frac{|(\sqrt{f(x)}-\sqrt{f(y)})(\sqrt{f(x)}+\sqrt{f(y)})|}{2\sqrt{c}} = \frac{|f(x)-f(y)|}{2\sqrt{c}}.$$

It follows that $\sqrt{f(x)} \le (M_k(f)-m_k(f))/(2\sqrt{c}) + \sqrt{f(y)}$ for all $x \in [x_{k-1},x_k]$. Taking the supremum over $x \in [x_{k-1},x_k]$ and then the infimum over $y \in [x_{k-1},x_k]$, we have proved that $M_k(\sqrt{f}) - m_k(\sqrt{f}) \le (M_k(f) - m_k(f))/(2\sqrt{c})$ for $k = 1,2,\dots n$. We conclude by the choice of P that

$$U(\sqrt{f},P) - L(\sqrt{f},P) \le \frac{U(f,P)-L(f,P)}{2\sqrt{c}} < \varepsilon.$$

5.1.10. Suppose f is integrable on $[a,b]$. By definition, there is a partition P_ϵ of $[a,b]$ such that $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$. By Remark 5.8, if P is finer than P_ϵ then

$$U(f,P)-L(f,P) \le U(f,P_\epsilon)-L(f,P_\epsilon) < \epsilon.$$

The converse is trivial.

5.2 Riemann Sums.

5.2.0. a) True. Combine (6) and (7).

b) True. If f is integrable on $[a,b]$, then by Corollary 5.23, so are f^n for all $n \in \mathbf{N}$. Since $P(f)$ is a linear combination of f^n's and constants, it too is integrable on $[a,b]$ by Theorem 5.19.

c) True. Let M be the maximum of f on $[a,b]$. By the Extreme Value Theorem, there is an $x_0 \in [a,b]$ such that $f(x_0) = M$. Hence, the cited result follows immediately from the Second Mean Value Theorem for integrals.

d) False, even if f were positive on $[a,b]$. Indeed, let $f(x) = x+2$, $g(x) = x$, and $[a,b] = [-1,1]$. Then for all x_0,

$$\int_{-1}^{1}(x^2+2x)\,dx = \frac{2}{3} \ne 0 = (x_0+2)\int_{-1}^{1} x\,dx.$$

5.2.1. a) $|x+1| = x+1$ if $x \ge -1$ and $|x+1| = -x-1$ if $x \le -1$. Thus the graph of $y = |x+1|$ for $x \in [-2,2]$ consists of two triangles, the left one with base 1 and altitude 1, and the right one with base 3 and altitude 3. Therefore, $\int_{-2}^{2}|x+1|\,dx = 1/2(1) + 1/2(9) = 5$.

b) Since

$$|x+1|+|x| = \begin{cases} -2x-1 & x \le -1 \\ 1 & -1 \le x \le 0 \\ 2x+1 & x \ge 0, \end{cases}$$

the graph of the integrand consists of two trapezoids on either side of a square. Hence

$$\int_{-2}^{2}(|x+1|+|x|)\,dx = 1\cdot(3+1)/2 + 1 + 2\cdot(5+1)/2 = 9.$$

c) $y = \sqrt{a^2 - x^2}$ implies $x^2 + y^2 = a^2$. Thus the graph of $y = \sqrt{a^2 - x^2}$ is a semicircle centered at the origin of radius a. Hence the integral represents the area of that semicircle, i.e.,

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\pi a^2.$$

d) By completing the square, $y = \sqrt{2x + x^2}$ implies $(x-1)^2 + y^2 = 1$. Thus $y = \sqrt{2x + x^2}$ is a semicircle centered at $(1,0)$ of radius 1. Hence $y = 5 + \sqrt{2x + x^2}$ is a semicircle mounted on a rectangle with base 2 and altitude 5. It follows that

$$\int_{0}^{2} (5 + \sqrt{2x + x^2})\, dx = 10 + \frac{\pi}{2}.$$

5.2.2. a) By the First Mean Value Theorem for Integrals, there is a $c \in [a, b]$ such that

$$0 = \int_a^b f(x) x^n\, dx = f(c) \int_a^b x^n dx =: f(c) \cdot I.$$

Since n is even, $x^n \geq 0$. Since $a < b$, I is not zero. Thus $f(c) = 0$.
b) Let $f = 1$, $n = 1$, and $[a, b] = [-1, 1]$.
c) If $a + b \neq 0$, then $I \neq 0$ for all $n \in \mathbf{N}$ whether odd or even.

5.2.3. a) By Exercise 4.4.4, $x^2 - x^6/3! < \sin(x^2) < x^2 - x^6/3! + x^{10}/5!$. Thus

$$0.3095 \approx \int_0^1 (x^2 - x^6/3!)\, dx < \int_0^1 \sin(x^2)\, dx < \int_0^1 (x^2 - x^6/3! + x^{10}/5!)\, dx \approx 0.3103.$$

b) By Taylor's Formula, $e^{x^2} = 1 + x^2 + x^4/2 + x^6/6 + e^c x^8/24$ for some c between x and 0. But

$$0 \leq e^c \frac{x^8}{24} \leq \frac{e}{24} \approx 0.1132617 \quad \text{and} \quad \int_0^1 (1 + x^2 + x^4/2 + x^6/6)\, dx \approx 1.4571429.$$

Thus

$$1.4571429 < \int_0^1 e^{x^2}\, dx < 1.5704046.$$

5.2.4. $0 \leq e^{-y^2} \leq 1$ for all $y \in [0, x]$ and $f \geq 0$ on $[0, x]$, so by the Second Mean Value Theorem for Integrals, there is an $g(x) := x_0$ such that

$$\int_0^x e^{-y^2} f(y)\, dy = \int_{g(x)}^x f(y)\, dy.$$

Note: Yes, there may be more than one x_0 given x, but just pick one and call it $g(x)$.

5.2.5. Let $M = \sup_{x \in [0,1]} f(x)$. By the Comparison Theorem,

$$\left| n^\alpha \int_0^{1/n^\beta} f(x)\, dx \right| \leq \frac{n^\alpha M}{n^\beta} = \frac{M}{n^{\beta - \alpha}}.$$

Since $\beta > \alpha$, $n^{\beta - \alpha} \to \infty$ as $n \to \infty$. Hence it follows from the Squeeze Theorem that $n^\alpha \int_0^{1/n^\beta} f(x)\, dx \to 0$ as $n \to \infty$.

5.2.6. a) Since f is bounded, choose m, M such that $m \leq f(x) \leq M$ for $x \in [a, b]$. Then by the Comparison Theorem,

$$m \int_a^b g_n(x)\, dx \leq \int_a^b f(x) g_n(x)\, dx \leq M \int_a^b g_n(x)\, dx$$

for all $n \in \mathbf{N}$. By hypothesis, the two outer sequences converge to 0 as $n \to \infty$. Hence by the Squeeze Theorem, the sequence in the middle converges to 0 as $n \to \infty$.

b) This follows immediately from part a) since

$$\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \to 0$$

as $n \to \infty$.

5.2.7. By Theorem 5.20, $\sum_{k=0}^n \int_{x_k}^{x_{k+1}} f(x)\,dx = \int_a^{x_{n+1}} f(x)\,dx$. Hence by Theorem 5.26,

$$\lim_{n\to\infty} \sum_{k=0}^n \int_{x_k}^{x_{k+1}} f(x)\,dx = \lim_{n\to\infty} \int_a^{x_{n+1}} f(x)\,dx = \int_a^b f(x)\,dx.$$

5.2.8. a) Since f is continuous on $[a,b]$ and $M > 0$, choose by the Approximation Property and the Sign Preserving Property a nondegenerate interval $I \subset [a,b]$ such that $f(x) \ge M - \epsilon$ for $x \in I$. Then $(M-\epsilon)^n \le |f(x)|^n$ for $x \in I$ and $|f(x)|^n \le M^n$ for $x \in [a,b]$. It follows from the Comparison Theorem that

$$(M-\epsilon)^n |I| \le \int_I |f(x)|^n\,dx \le \int_a^b |f(x)|^n\,dx \le M^n |b-a|.$$

b) If $M = 0$ then $|f| = 0$ and the result is trivial. If $M > 0$ then choose $\epsilon > 0$ so small that $M - \epsilon/2 > 0$. By part a), choose I so that

$$\left(M - \frac{\epsilon}{2}\right) |I|^{1/n} \le \left(\int_a^b |f(x)|^n\,dx\right)^{1/n} \le M|b-a|^{1/n}.$$

holds. By Example 2.21, $|I|^{1/n}$ and $(b-a)^{1/n} \to 1$ as $n \to \infty$. Since $\xi := (M-\epsilon)/(M-\epsilon/2) < 1$, choose n so large that $|I|^{1/n} > \xi$ and $(b-a)^{1/n} < (M+\epsilon)/M$. It follows that

$$M - \epsilon = \xi\left(M - \frac{\epsilon}{2}\right) < \left(M - \frac{\epsilon}{2}\right)|I|^{1/n} \le \left(\int_a^b |f(x)|^n\,dx\right)^{1/n} < M\left(\frac{M+\epsilon}{M}\right) = M + \epsilon.$$

Hence by definition, $(\int_a^b |f(x)|^n\,dx)^{1/n} \to M$ as $n \to \infty$.

5.2.9. By Theorem 5.20, it suffices to show f is integrable on each $I := [x_{k-1}, x_k]$. Let g be the continuous extension of f to I. By Theorem 3.40, g is uniformly continuous on I. Thus given $\epsilon > 0$ there is a $\delta > 0$ such that

(*) $$|x-y| < \delta \quad \text{and} \quad x, y \in I \quad \text{imply} \quad |g(x) - g(y)| < \frac{\epsilon}{2|I|}.$$

Now f is bounded on I by $M := \sup_{x\in I} |g(x)| + |f(x_{k-1})| + |f(x_k)|$. Let $P = \{t_0, \dots, t_N\}$ be a partition of I which satisfies

$$\|P\| < \min\{\delta, \frac{\epsilon}{8M}\}.$$

Since $f = g$ on $[t_1, t_{N-1}]$, it follows from (*),

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{j=1}^N (M_j(f) - m_j(f))(t_j - t_{j-1}) \\ &\le 2M(t_1 - t_0) + \frac{\epsilon}{2|I|}\sum_{j=2}^{N-1}(t_j - t_{j-1}) + 2M(t_N - t_{N-1}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is integrable on I.

5.2.10. By Theorems 5.19 and 5.22, $f+g$ and $|f-g|$ are integrable on $[a,b]$, hence so are $f \vee g := ((f+g) + |f-g|)/2$ and $f \wedge g := ((f+g) - |f-g|)/2$.

5.2.11. a) If f is not bounded above on $[a,b]$, then choose s_0 and s_k in $[a,b]$ such that $s_k \to s_0$ and $f(s_k) \to \infty$ as $k \to \infty$. By symmetry, we may suppose that $s_0 \ne a$ and that $s_k < s_0$.

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. Choose j_0 such that $s_0 \in (x_{j_0-1}, x_{j_0}]$. Fix $t_j \in [x_{j-1}, x_j]$ for $j \ne j_0$. Since $f(s_k)\Delta t_{j_0} \to \infty$, we can choose $t_{j_0} = s_k$, k large, so that $\mathcal{S}(f, P, t_j) > M$.

b) If the Riemann sums converge to $I(f)$, then there is a partition P such that $|\mathcal{S}(f,P,t_j)| < |I(f)| + 1$ for all choices of $t_j \in [x_{j-1}, x_j]$. But by part a) and symmetry, if f is unbounded on $[a,b]$, then there are $t_j \in [x_{j-1}, x_j]$ such that $|\mathcal{S}(f,P,t_j)| > |I(f)| + 1$, a contradiction.

5.3 The Fundamental Theorem of Calculus.

5.3.0. a) True. Since g is increasing on $[a, b]$, $g'(x) \geq 0$ for all $x \in [a, b]$. Thus, by the Fundamental Theorem of Calculus and the Chain Rule,

$$F'(x) = f(g(x)) \cdot g'(x) \geq 0 \cdot 0 = 0.$$

b) True. First, since g is continuous and nonzero on $[a, b]$, $1/g$ is continuous, hence integrable on $[a, b]$. In particular, all the functions which appear in this problem are products of integrable and/or continuous functions, hence integrable. Thus the expressions on the right side of statement b) are both well defined.

To prove that they are equal, notice by the Quotient Rule that

$$g(x)\left(\frac{f(x)}{g(x)}\right)' = f'(x) - \frac{f(x)g'(x)}{g(x)}.$$

It follows that the sum of integrals on the right side of statement b) is just the indefinite integral of f'. But by the Fundamental Theorem of Calculus and the fact that $f(a) = 0$, we have

$$\int_a^x f'(t)\,dt = f(x) - f(a) = f(x)$$

as required.

c) True. By the Product Rule and the Fundamental Theorem of Calculus,

$$\int_a^b (f'(x)g(x) + f(x)g'(x))\,dx = \int_a^b (f(x)g(x))', dx = f(b)g(b) - f(a)g(a).$$

Thus the left-most integral equals zero if and only if $f(a)g(a) = f(b)g(b)$.

d) False. The function f might take $[a, b]$ to something outside the domain of g. For example, if $a = 0$, $b = 1$, $g(x) = \sqrt{x}$ and $f(x) = -x$, then $g(f(x))$ does not exist, so the integral on the right side of part d) is not defined.

5.3.1. a) By the Chain Rule

$$F'(x) = -\frac{d}{dx}\int_1^{x^2} f(t)\,dt = -f(x^2) \cdot 2x.$$

b) By the Chain Rule

$$F'(x) = \frac{d}{dx}\left(\int_0^{x^3} f(t)\,dt - \int_0^{x^2} f(t)\,dt\right) = 3x^2 f(x^3) - 2xf(x^2).$$

c) By the Chain and Product Rules

$$F'(x) = x\cos x f(x\cos x)\frac{d}{dx}(x\cos x) = x\cos x f(x\cos x)(\cos x - x\sin x).$$

d) Let $u = x - t$ so $du = dx$. Then by the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^t f(t-x)\,dt = \frac{d}{dx}\int_{-x}^0 f(u)\,du = f(-x).$$

5.3.2. a) $1 \leq x \leq 4$ implies $1 \leq \sqrt{x} \leq 2$, i.e., $1 \leq 2/\sqrt{x}$. Thus by the Comparison Theorem and u-substitution,

$$\int_1^4 f(\sqrt{x})\,dx \leq 2\int_1^4 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx = 4\int_1^2 f(u)\,du = 20.$$

b) $1/\sqrt{2} \leq x \leq 1$ implies $1/2\sqrt{2} \leq x^3 \leq 1$, i.e., $1 \leq 1/x^3 \leq 2\sqrt{2}$. Thus by the Comparison Theorem and u-substitution,

$$\int_{1/\sqrt{2}}^1 f(1/x^2)\,dx \leq \int_{1/\sqrt{2}}^1 \frac{f(1/x^2)}{x^3}\,dx = \frac{1}{2}\int_1^2 f(u)\,du = \frac{5}{2}.$$

c) Use u-substitution. If $u = x+1$ then $du = dx$ and $x^2 = u^2 - 2u + 1$. Thus

$$\int_0^1 x^2 f(x+1)\,dx = \int_1^2 (u^2 f(u) - 2uf(u) + f(u))\,du = 9 - 2\cdot 6 + 5 = 2.$$

5.3.3. a) If $x = \tan\theta$ then $dx = \sec^2\theta\,d\theta$ so

$$\int_0^1 x^3 f(x^2+1)\,dx = \int_0^{\pi/4} \tan^3\theta \sec^2\theta f(\sec^2\theta)\,d\theta.$$

If $u = \sec^2\theta$ then $du = 2\sec\theta\cdot\sec\theta\tan\theta\,d\theta$. Since $\tan^3\theta = \tan\theta(\sec^2\theta - 1)$, it follows that

$$\int_0^{\pi/4} \tan^3\theta\sec^2\theta f(\sec^2\theta)\,d\theta = \frac{1}{2}\int_1^2 (u-1)f(u)\,du = \frac{1}{2}(2-3) = -\frac{1}{2}.$$

b) If $x = \sin\theta$ then $dx = \cos\theta\,d\theta$ so

$$\int_0^{\sqrt{3}/2} \frac{x^3 f(\sqrt{1-x^2})}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/3} \frac{\sin^3\theta\cos\theta f(\cos\theta)}{\cos\theta}\,d\theta.$$

If $u = \cos\theta$ then $du = -\sin\theta\,d\theta$. Since $\sin^3\theta = \sin\theta(1-\cos^2\theta)$, it follows that

$$\int_0^{\pi/3} \sin^3\theta f(\cos\theta)\,d\theta = -\int_1^{1/2} (1-u^2)f(u)\,du = 3 - 7 = -4.$$

5.3.4. a) Let $u = \log x$ and $dv = f'(x)\,dx$. By parts,

$$\int_1^e f'(x)\log x\,dx = \log x f(x)\,\Big|_1^e - \int_1^e \frac{f(x)}{x}\,dx > f(e) - f(e) = 0.$$

b) Let $u = e^x$ and $dv = f'(x)\,dx$. By parts,

$$\int_0^1 e^x f'(x)\,dx = e^x f(x)\,\Big|_0^1 - \int_0^1 e^x f(x)\,dx = -\int_0^1 e^x f(x)\,dx.$$

c) Let $u = f(x)$ and $dv = g'(x)\,dx$. By parts,

$$\int_0^e f(x)g'(x)\,dx = f(x)g(x)\,\Big|_0^e - \int_0^e g(x)f'(x)\,dx.$$

By hypothesis, either $f(0) = 0$ or $g(0) = 0$, and either $f(e) = 0$ or $g(e) = 0$. Thus $f(x)g(x)\,\Big|_0^e = 0 - 0 = 0$.

5.3.5. By the Fundamental Theorem of Calculus and the First Mean Value Theorem for Integrals,

$$f(b) - f(a) = \int_a^b f'(t)\,dt = f'(x_0)\int_a^b dt = f'(x_0)(b-a)$$

for some x_0 between a and b.

5.3.6. Take the derivative of $0 = \alpha\int_a^c f(x)\,dx + \beta\int_c^b f(x)\,dx$ with respect to c. By the Fundamental Theorem of Calculus we obtain $0 = \alpha f(c) - \beta f(c) = (\alpha - \beta)f(c)$ for all $c \in [a,b]$. Since $\alpha \neq \beta$, it follows that $f(c) = 0$ for all $c \in [a,b]$.

5.3.7. a) Since $1/t$ is continuous on $(0,\infty)$, it follows from the Fundamental Theorem of Calculus that $L(x)$ is differentiable at each point $x \in (0,\infty)$ with $L'(x) = 1/x$. If $0 < x_1 < x_2$ then

$$L(x_2) - L(x_1) = \int_{x_1}^{x_2} \frac{dt}{t} > \frac{x_2 - x_1}{x_2} > 0.$$

Thus L is strictly increasing on $(0,\infty)$.

b) Clearly, $\int_1^{2^n} (1/t)\,dt > \sum_{k=1}^{2^n-1} 1/(k+1) > n/2$ for $n \ge 1$. Thus $L(2^n) \to \infty$ as $n \to \infty$. Since L is strictly increasing, it follows that $L(x) \to \infty$ as $x \to \infty$. A similar argument shows that $L(x) \to -\infty$ as $x \to 0+$.

c) Using the substitution $t = u^q$, we obtain

$$L(x^q) = \int_1^{x^q} \frac{dt}{t} = q \int_1^x \frac{du}{u} = qL(x).$$

d) Using the substitution $u = t/x$, we obtain

$$L(xy) = \int_1^x \frac{dt}{t} + \int_1^{xy} \frac{dt}{t} = L(x) + \int_1^y \frac{du}{u} = L(x) + L(y).$$

e) $L(e) = \lim_{n\to\infty} L((1+1/n)^n) = \lim_{n\to\infty} L(1+1/n)/(1/n)$. Since $L(1) = 0$ by definition and $L(x)$ is continuous at $x = 1$, this limit is of the form $0/0$. Hence by L'Hôpital's Rule, $L(e) = \lim_{n\to\infty} 1/(1+1/n) = 1$.

5.3.8. a) By Exercise 5.3.7, L is differentiable and strictly increasing, hence 1–1, on $(0,\infty)$, and takes $(0,\infty)$ onto $\mathbf{R}$. Hence by Theorems 4.32 and 4.33, $E(x) := L^{-1}(x)$ is differentiable and strictly increasing on $\mathbf{R}$ with $E'(x) = 1/L'(y)$ for $y = E(x)$. Since $L'(y) = 1/y$, it follows that $E'(x) = E(x)$. Since $L(1) = 0$ and $L(e) = 1$, we also have $E(0) = 1$ and $E(1) = e$.

b) By Exercise 5.3.7b, $E(x) \to \infty$ as $x \to \infty$, and $E(x) \to 0$ as $x \to -\infty$.

c) Let $y = E(xq)$ and $t = E(x)$. By definition and Exercise 5.3.7c, $L(y) = xq = qL(t) = L(t^q)$. Since L is 1–1, we have $y = t^q$, i.e., $E(xq) = (E(x))^q$. Hence by part a), $E(q) = E(1\cdot q) = (E(1))^q = e^q$ for all $q \in \mathbf{Q}$.

d) Let $s = E(x)$, $t = E(y)$, and $w = E(x+y)$. By definition and Exercise 5.3.7d, $L(w) = x + y = L(s) + L(t) = L(st)$. Hence, $w = st$, i.e., $E(x+y) = E(x)E(y)$.

e) Suppose $\alpha > 0$ and $x < y$. Then $L(x) < L(y)$ and $\alpha L(x) < \alpha L(y)$. Since E is increasing and $x^\alpha = E(\alpha L(x))$, it follows that $x^\alpha < y^\alpha$. A similar argument proves that $x^\alpha > y^\alpha$ when $\alpha < 0$ and $x < y$. By part d),

$$x^{\alpha+\beta} = E((\alpha+\beta)L(x)) = E(\alpha L(x))E(\beta L(x)) = x^\alpha x^\beta$$

and $x^\alpha \cdot x^{-\alpha} = E(\alpha - \alpha) = E(0) = 1$, i.e., $1/x^\alpha = x^{-\alpha}$. Finally, by the Chain Rule,

$$(x^\alpha)' = (E(\alpha L(x)))' = E(\alpha L(x)) \cdot \alpha L'(x) = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

5.3.9. Using the substitution $y = f(x)$ and integrating by parts, we have

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = \int_a^b x f'(x)\,dx = xf(x)\,\Big|_a^b - \int_a^b f(x)\,dx.$$

Therefore, $\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = xf(x)\,\big|_a^b = bf(b) - af(a)$.

5.3.10. By Theorem 5.34, $f\circ\phi\cdot|\phi'|$ is integrable on $[a,b]$. But ϕ' is never zero, so $|\phi'| = \pm\phi'$. Moreover, since ϕ' is nonzero, its reciprocal is continuous on $[a,b]$, hence integrable there. Therefore, by the product theorem (Corollary 5.23),

$$f\circ\phi = f\circ\phi\cdot\phi'\frac{1}{\phi'}$$

is integrable on $[a,b]$.

5.3.11. By Corollary 5.23 and the Fundamental Theorem of Calculus, it suffices to prove that f^{-1} and $f^{1/m}$ are integrable for all $m \in \mathbf{N}$. Let $\epsilon > 0$ and choose a partition $P = \{x_0, x_1, \dots, x_n\}$ such that

$$U(f,P) - L(f,P) < \min\{C_m\epsilon, c^{-2}\epsilon\},$$

where $C_m := (m+1)c^{(m-1)/m}$. Since $f(x) \ge c > 0$ for all $x \in [a,b]$, it is easy to see that if $x, y \in [x_{j-1}, x_j]$,then

$$f^{1/m}(x) - f^{1/m}(y) \le \frac{M_j(f) - m_j(f)}{(m+1)c^{(m-1)/m}} = \frac{M_j(f) - m_j(f)}{C_m}$$

(see Exercise 4.2.7) and that

$$\frac{1}{f(x)} - \frac{1}{f(y)} \le \frac{M_j(f) - m_j(f)}{c^2}.$$

It follows that $M_j(f^{1/m}) - m_j(f^{1/m}) \le (M_j(f) - m_j(f))/C_m$ and $M_j(1/f) - m_j(1/f) \le (M_j(f) - m_j(f))/c^2$. In particular, $U(f^{1/m}, P) - L(f^{1/m}, P) < \epsilon$ by the choice of P.

5.3.12. Since

$$\frac{(2n)!}{n!n^n} = \frac{(n+1)(n+2)\cdots(2n)}{n^n}$$

and

$$1 = \frac{n}{n}, \ \frac{n+1}{n}, \ \frac{n+2}{n}, \dots, \frac{2n}{n} = 2$$

is a partition of $[1,2]$ with norm $1/n$, it is easy to see that $\log(a_n)$ is a Riemann sum of $\int_1^2 \log x\, dx$. Thus $\log(a_n)$ converges to $\int_1^2 \log x\, dx$. Hence

$$\lim_{n\to\infty} \log(a_n) = \int_1^2 \log x\, dx = (x \log x - x \,\Big|_1^2 = \log\left(\frac{4}{e}\right).$$

Since e^x is continuous, we conclude that $a_n \to 4/e$ as $n \to \infty$.

5.4 Improper Riemann Integration.

5.4.0. a) False. Let $a = 0$, $b = 1$. Define f and g on $[0,1]$ by: $f(x) = 1$ when $x \in \mathbf{Q}$ and $f(x) = 0$ when $x \notin \mathbf{Q}$, and $g(x) = 1 + (\sqrt{x})^{-1}$. Then $|f| \le g$ and g is absolutely integrable on $[0,1]$ (the integral has value 3), but f is NOT even locally integrable on $(0,1)$ much less improperly integrable.

b) True. By the Extreme Value Theorem, there is an $\varepsilon > 0$ such that $|g(x)| \ge \varepsilon$ for all $x \in [a,b]$. Thus f/g is locally integrable and $|f|/|g| \le h/\varepsilon$ holds everywhere on (a,b). Hence this quotient is absolutely integrable by the Comparison Test.

c) True. Since $\sqrt{f}$ is continuous, it is locally integrable on (a,b). If $f(x) < 1$, then $\sqrt{f(x)} < 1 \le 1 + f(x)$. If $f(x) \ge 1$, then $\sqrt{f(x)} \le f(x) < 1 + f(x)$. Thus $\sqrt{f(x)} < 1 + f(x)$ for all $x \in (a,b)$. The function $1 + f$ is absolutely integrable on (a,b) by hypothesis and the fact that $b - a < \infty$. Thus $\sqrt{f}$ is absolutely integrable on (a,b) by the Comparison Theorem.

d) True. Since f and g are absolutely integrable on (a,b), it follows from Theorem 5.42 and the Comparison Theorem that any finite linear combination of f, g, and $|f - g|$ is absolutely integrable on (a,b). Hence by Exercise 3.18, $f \vee g$ and $f \wedge g$ are absolutely integrable on (a,b).

5.4.1. a) $\int_1^\infty (1+x)/x^3\, dx = \int_1^\infty x^{-3}\, dx + \int_1^\infty x^{-2}\, dx = 1/2 + 1 = 3/2$.

b) Using the substitution $u = x^3$, $dx = 3x^2\, dx$, we have

$$\int_{-\infty}^0 x^2 e^{x^3}\, dx/(1+x^2) = \frac{1}{3}\int_{-\infty}^0 e^u\, du = \frac{1}{3}.$$

c) Using the substitution $u = \sin x$, $du = \cos x\, dx$, we have $\int_0^{\pi/2} \cos x/\sqrt[3]{\sin x}\, dx = \int_0^1 u^{-1/3}\, du = 3/2$.

d) Integrating by parts and using l'Hôpital's Rule, we obtain

$$\int_0^1 \log x\, dx = x \log x - x \,\Big|_0^1 = -1.$$

5.4.2. a) If $p \ne 1$ then $\int_1^\infty dx/x^p = x^{1-p}/(1-p) \,\Big|_1^\infty$. Now x^{1-p} has a finite limit as $x \to \infty$ if and only if $1 - p > 0$, i.e., $p > 1$. When $p = 1$, the integral is $\log x \,\Big|_1^\infty$ which diverges. Thus the integral converges if and only if $p > 1$.

b) If $p \ne 1$ then $\int_0^1 dx/x^p = x^{1-p}/(1-p) \,\Big|_0^1$. Now x^{1-p} has a finite limit as $x \to 0+$ if and only if $1 - p > 0$, i.e., $p < 1$. When $p = 1$, the integral is $\log x \,\Big|_0^1$ which diverges. Thus the integral converges if and only if $p < 1$.

c) Using the substitution $u = \log x$, $du = dx/x$, we have $\int_e^\infty dx/(x \log^p x) = \int_1^\infty du/u^p$. Thus by part a), this integral converges if and only if $p > 1$.

d) $0 \le \int_0^\infty dx/(1+x^p) \le \int_0^1 dx/(1+x^p) + \int_1^\infty dx/x^p$. Thus the integral converges for all $p > 1$ by part a). If $p = 1$ then $\int_0^\infty dx/(1+x) = \log(1+x)\,\big|_0^\infty$ which diverges. Finally, if $p < 1$ then

$$\int_0^\infty \frac{dx}{1+x^p} \ge \int_1^\infty \frac{dx}{1+x^p} \ge \frac{1}{2}\int_1^\infty \frac{dx}{x^p}$$

since $2x^p \ge 1 + x^p$ for $x \ge 1$. Thus the integral diverges for $p < 1$ by part a).

e) Since $a > 0$ implies $\log^a x \ge 1$ for $x \ge 0$, $f(x) \ge x^p$ for all $x \in [1,\infty)$. Hence $f(x)$ cannot be improperly integrable if $p \le 1$. On the other hand, if $p > 1$, choose $q > 0$ such that $p - 1 > q$ and a constant C such that $\log x \le x^{q/p}$ for all $x \ge C$. Then $x \ge C$ implies

$$\frac{\log^c}{x^p} \le \frac{x^q}{x^p} = \frac{1}{x^{p-q}}.$$

Since $p - q > 1$, it follows from the comparison test that $f(x)$ is improperly integrable on $[1,\infty)$.

5.4.3. Since $p > 0$, integration by parts yields

$$\int_1^\infty \frac{\sin x}{x^p}\,dx = -\frac{\cos x}{x^p}\,\Big|_1^\infty - p\int_1^\infty \frac{\cos x}{x^{p+1}}\,dx = \cos(1) - p\int_1^\infty \frac{\cos x}{x^{p+1}}\,dx.$$

By Remark 5.46 and Exercise 5.4.2a, this last integral is absolutely integrable since $p + 1 > 1$. Hence $\sin x/x^p$ is improperly integrable on $[1,\infty)$ for all $p > 0$.

Similarly, $\int_e^\infty \cos x/\log^p x\,dx = -\sin(e) + p\int_e^\infty \sin x/(x\log^{p+1} x)\,dx$. By Remark 5.46 and Exercise 5.4.2c, this last integral converges absolutely since $p + 1 > 1$. Thus $\cos x/\log^p x$ is improperly integrable on $[e,\infty)$ for all $p > 0$.

5.4.4. a) $\int_0^\infty \sin x\,dx = -\cos x\,\big|_0^\infty$. Since $\cos x$ has no limit as $x \to \infty$, this integral diverges.

b) Since $\int_{-1}^1 dx/x^2 = \int_{-1}^0 dx/x^2 + \int_0^1 dx/x^2$, this integral diverges by Exercise 5.4.2b.

c) For $x \ge 1$, $\sin(1/x) = |\sin(1/x)| \le 1/x$. Hence by the Comparison Theorem and Exercise 5.4.2a the integral converges: $\int_1^\infty (1/x)\sin(1/x)\,dx \le \int_1^\infty (1/x^2)\,dx < \infty$.

d) Substitute $u = \sin x$, $du = \cos x\,dx$, i.e., $dx = du/\sqrt{1-u^2}$, to obtain

$$I := \int_0^1 \log(\sin x)\,dx = \int_0^{\sin 1} \frac{\log u}{\sqrt{1-u^2}}\,du.$$

Since $\sin 1 < 1$, this last integral is improper only at $u = 0$. But by l'Hôpital's Rule,

$$\lim_{u\to 0+} \frac{-\log u}{\sqrt{(1-u^2)/u}} = \lim_{u\to 0+} \frac{2\sqrt{u-u^3}}{1+u^2} = 0.$$

Thus $|\log u|\sqrt{u}/\sqrt{1-u^2} = -\log u/\sqrt{1-u^2}$ is bounded on $[0,\sin 1]$, i.e., there is an $M > 0$ such that $|\log u|/\sqrt{1-u^2} \le 1/\sqrt{u}$. Since the later is integrable on $[0,\sin 1]$, it follows from the Comparison Test that I converges absolutely.

e) Since $|1-\cos x|/x^2 \le 2/x^2$, the integral $\int_1^\infty (1-\cos x)/x^2\,dx$ converges. On the other hand, by L'Hôpital's Rule, $(1-\cos x)/x^2 \to 1/2$ as $x \to 0$. Therefore, $\int_0^1 (1-\cos x)/x^2\,dx$ is a Riemann integral. Therefore, the original integral converges.

5.4.5. By Exercise 5.4.2b, $1/\sqrt{x}$ is integrable on $(0,1)$, but $1/x = (1/\sqrt{x})(1/\sqrt{x})$ is not.

5.4.6. a) Choose $a < b_0 < b$ such that $f(x)/g(x) < 2L + 1$ for all $b_0 < x < b$. Since g is nonnegative, we have $f(x) < (2L+1)g(x)$ for $x \in (b_0, b)$. Hence by the Comparison Theorem, f is improperly integrable on $[a,b)$.

b) Choose $a < b_0 < b$ such that $f(x)/g(x) > M := \min\{L/2, 1\}$ for all $b_0 < x < b$. Notice that $M > 0$. Since g is nonnegative, we have $f(x) > Mg(x)$ for $x \in (b_0, b)$. Hence by the Comparison Theorem, f is not improperly integrable on $[a,b)$.

5.4.7. a) Suppose $L > 0$. If $f(x) \to L$ as $x \to \infty$, then choose $N \in \mathbf{N}$ such that $f(x) > L/2$ for $x \ge N$. By the Comparison Theorem, $\int_N^\infty f(x)\,dx \ge \int_N^\infty L/2\,dx = \infty$, a contradiction. A similar argument handles the case $L < 0$.

b) Clearly, $\int_0^N f(x)\,dx = 1/2 + 1/4 + \cdots + 1/2^{N-1} = 1 - 1/2^{N-1} \to 1$ as $N \to \infty$. The limit does not exist because $f(n) \to 1$ but $f(n+1/2) \to 0$ as $n \to \infty$.

5.4.8. Let $b \in [1,\infty)$ and $n > 1$. By Exercise 5.3.10, $f(x^n)$ is locally integrable on $[1,\infty)$. Since $x^{n-1} \geq 1$ when $x \geq 1$, we have

$$\int_1^b |f(x^n)|\,dx \leq \int_1^b x^{n-1}|f(x^n)|\,dx = \frac{1}{n}\int_1^{b^n} |f(u)|\,du.$$

Taking the limit of this last inequality as $b \to \infty$, we see that $f(x^n)$ is absolutely integrable on $[1,\infty)$ for each $n > 1$. Thus by Theorem 5.48,

$$0 \leq \left|\int_1^\infty f(x^n)\,dx\right| \leq \int_1^\infty x^{n-1}|f(x^n)|\,dx = \frac{1}{n}\int_1^\infty |f(u)|\,du.$$

Since f is absolutely integrable on $[1,\infty)$, this last integral converges to 0 as $n \to \infty$. Hence by the Squeeze Theorem, $\int_1^\infty f(x^n)\,dx \to 0$ as $n \to \infty$.

5.4.9. Integrating by parts and applying L'Hôpital's Rule, we obtain

$$\begin{aligned}\int_1^\infty x^n e^{-x}\,dx &= e^{-1} + \int_1^\infty n x^{n-1} e^{-x}\,dx = \dots \\ &= e^{-1}(1 + n + n(n-1) + \cdots + n!) + \int_1^\infty n! e^{-x}\,dx \\ &= e^{-1}(1 + n + n(n-1) + \cdots + 2n!).\end{aligned}$$

Therefore,

$$\frac{1}{n!}\int_1^\infty x^n e^{-x}\,dx = e^{-1}\left(\frac{1}{n!} + \frac{1}{(n-1)!} + \cdots + 1 + 1\right) = e^{-1}\left(\sum_{k=0}^n \frac{1}{k!}\right).$$

In particular, $(1/n!)\int_1^\infty x^n e^{-x}\,dx \to e^{-1}e = 1$ as $n \to \infty$.

5.4.10. a) Clearly, $\sin x \geq \sqrt{2}/2$ for $x \in [\pi/4, \pi/2]$. Since $f(x) := \sin x - 2x/\pi$ has no local minima in $[0,\pi/4]$ and $f(0) = 0 < f(\pi/4)$, we also have $\sin x \geq 2x/\pi$ for $x \in [0,\pi/4]$. Hence

$$\left|\int_0^{\pi/2} e^{-a\sin x}\,dx\right| \leq \int_0^{\pi/4} e^{-a\sin x}\,dx + \int_{\pi/4}^{\pi/2} e^{-a\sin x}\,dx \leq \frac{\pi}{4}e^{-a\sqrt{2}/2} + \int_0^{\pi/4} e^{-2ax/\pi}\,dx.$$

Using the substitution $t = 2ax/\pi$, $dt = 2a/\pi dx$, this last integral can be estimated by

$$\int_0^{\pi/4} e^{-2ax/\pi}\,dx = \frac{\pi}{2a}\int_0^{a/2} e^{-t}\,dt \leq \frac{\pi}{2a}\int_0^\infty e^{-t}\,dt \leq \frac{\pi}{2a}.$$

We claim that $e^{-a\sqrt{2}/2} \leq \sqrt{2}/(ae)$. If this claim holds, then the estimates above yield

$$\left|\int_0^{\pi/2} e^{-a\sin x}\,dx\right| \leq \frac{\pi}{4}e^{-a\sqrt{2}/2} + \frac{\pi}{2a} \leq \frac{\pi\sqrt{2}}{4ae} + \frac{\pi}{2a} = \frac{\pi}{2a}\left(\frac{1}{e\sqrt{2}} + 1\right) \approx \frac{1.9794076}{a}.$$

It remains to prove the claim.

Let $\phi(x) = xe^{-x}$. Then $0 = \phi'(x) = x(-e^{-x}) + e^{-x}$ implies $x = 1$. By the first derivative test, $\phi(1) = 1/e$ is a local maximum. Since $\phi(0) = 0$ and $\phi(x) \to 0$ as $x \to \infty$, it follows that $1/e$ is the maximum of ϕ on $[0,\infty)$. Hence, $e^{-x} \leq 1/(xe)$ for all $x > 0$. In particular, $e^{-a\sqrt{2}/2} \leq \sqrt{2}/(ae)$.

Note: If we replace ϕ by $g(x) = \sin x - 2\sqrt{2}x/\pi$, then the same argument shows $g(x) \geq 0$ for $x \in [0,\pi/4]$, and we obtain

$$\left|\int_0^{\pi/2} e^{-a\sin x}\,dx\right| \leq \frac{\pi}{4}e^{-a\sqrt{2}/2} + \frac{\pi}{2a\sqrt{2}} \leq \frac{\pi}{2a\sqrt{2}}\left(\frac{1}{e} + 1\right),$$

an improvement over the estimate we already obtained.

b) The same estimates can be obtained. Indeed, using the substitution $y = \pi/2 - x$, $dy = -dx$, we have

$$\int_0^{\pi/2} e^{-a\cos x}\,dx = \int_0^{\pi/2} e^{-a\sin y}\,dy.$$

5.5 Functions of Bounded Variation.

5.5.1. a) Fix $k \in \mathbf{N}$. The inequality holds if $4k^2 > 4k^2 - 1$, i.e., if $1 > 0$. Thus the inequality holds for all $k \in \mathbf{N}$.

b) By the Comparison Theorem,

$$\int_1^{2^n} \frac{1}{x}\,dx = \sum_{k=1}^{2^n-1} \int_k^{k+1} \frac{1}{x}\,dx < \sum_{k=1}^{2^n-1} \frac{1}{k} < \sum_{k=1}^{2^n} \frac{1}{k}$$

for all $n \in \mathbf{N}$.

c) Let $x_0 = 0$, $x_{2^n} = 1$, and $x_k = y_{2^n-k}$ for $0 < k < 2^n$, where $y_k^{-1} = \sqrt{(2k+1)\pi/2}$. Then $\{x_0, x_1, \dots, x_{2^n}\}$ is a partition of $[0,1]$. Since $\sin(1/y_k^2) = (-1)^k$, we have

$$\mathrm{Var}\,(\phi) \ge \sum_{k=1}^{2^n-1} |\phi(x_k) - \phi(x_{k-1})| = \sum_{k=1}^{2^n-1} |x_k^2 + x_{k-1}^2| = \frac{2}{\pi} \sum_{k=1}^{2^n-1} \frac{4k}{4k^2-1}.$$

Hence it follows from parts a) and b) that $\sum_{k=1}^{2^n-1} |\phi(x_k) - \phi(x_{k-1})| > 2\log(2^n)/\pi$ for all $n \in \mathbf{N}$. Since $n/\pi \to \infty$ as $n \to \infty$, we conclude that ϕ is not of bounded variation on $[0,1]$.

5.5.2. a) Fix $k \in \mathbf{N}$. The inequality holds if $8k^4 + 2k^2 < 16k^4 - 8k^2 + 1$, i.e., if $0 < 8k^4 - 10k^2 + 1$. By calculus, this function increases when $k > \sqrt{5/8}$. Hence, $8k^4 - 10k^2 + 1 > 0$ for all $k \ge 2$.

b) By the Comparison Theorem,

$$1 + \int_1^n \frac{1}{x^2}\,dx = 1 + \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x^2}\,dx \ge 1 + \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} = \sum_{k=1}^{n} \frac{1}{k^2}$$

for all $n \in \mathbf{N}$.

c) Let $x_k^{-1} = (2k+1)\pi/2$ for $k \in \mathbf{N}$ and notice that $\{x_n, x_{n-1}, \dots, x_1, x_0\}$ forms a partition of $[2/(2n+1)\pi, 2/\pi]$. (These x_k's have been chosen so that ϕ achieves its maximum variation.) Since $\sin(1/x_k) = (-1)^k$, parts a) and b) imply that

$$\sum_{k=1}^{n} |\phi(x_k) - \phi(x_{k-1})| = \sum_{k=1}^{n} |x_k^2 + x_{k-1}^2| = \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{8k^2+2}{(4k^2-1)^2} \le \frac{8}{\pi^2} < 1$$

for all $n \in \mathbf{N}$.

Let $\{y_0, \dots, y_N\}$ be any partition of $[0,1]$. Choose $n \in \mathbf{N}$ so large that $x_n < y_1$. Temporarily set $x_{n+1} = y_0 = 0$ and $x_{-1} = y_N = 1$. Since adding points only increases variation, e.g., $|f(y_{j+1}) - f(y_j)| \le |f(y_{j+1}) - f(x_k)| + |f(x_k) - f(y_j)|$, we may suppose that $\{x_0, \dots, x_n\} \subseteq \{y_0, \dots, y_N\}$.

Fix $k \in [-1, n-1]$ and choose μ, ν such that $x_{k+1} = y_\mu < y_{\mu+1} < \cdots < y_{\mu+\nu} = x_k$. Since ϕ is monotone between the x_k's, we can telescope to obtain

$$\sum_{j=\mu}^{\mu+\nu} |\phi(y_j) - \phi(y_{j+1})| = |\phi(x_{k+1}) - \phi(x_k)|.$$

It is also clear that $|\phi(x_n) - \phi(0)|$ and $|\phi(1) - \phi(x_0)|$ are both ≤ 1. Hence,

$$\sum_{j=0}^{N-1} |\phi(y_{j+1}) - \phi(y_j)| = |\phi(x_n) - \phi(0)| + \sum_{k=1}^{n} |\phi(x_{k+1}) - \phi(x_k)| + |\phi(1) - \phi(x_0)| \le 3.$$

We conclude that $\mathrm{Var}\,(\phi) \le 3$, i.e., ϕ is of bounded variation on $[0,1]$.

5.5.3. a) $\sum_{k=1}^n |\alpha\phi(x_k) - \alpha\phi(x_{k-1})| = |\alpha| \sum_{k=1}^n |\phi(x_k) - \phi(x_{k-1})|$ so $\mathrm{Var}(\alpha\phi) \le |\alpha|\mathrm{Var}(\phi)$.

b) By Remark 5.53,

$$\sum_{k=1}^{n}|\phi(x_k)\psi(x_k)-\phi(x_{k-1})\psi(x_{k-1})| \le \sum_{k=1}^{n}|\psi(x_k)|\,|\phi(x_k)-\phi(x_{k-1})| + \sum_{k=1}^{n}|\phi(x_{k-1})|\,|\psi(x_k)-\psi(x_{k-1})| \le M\sum_{k=1}^{n}|\phi(x_k)-\phi(x_{k-1})| + \widetilde{M}\sum_{k=1}^{n}|\psi(x_k)-\psi(x_{k-1})|.$$

Therefore, $\text{Var}(\phi\psi) \le M\text{Var}(\phi) + \widetilde{M}\text{Var}(\psi)$.

c) By hypothesis,

$$\sum_{k=1}^{n}\left|\frac{1}{\phi(x_k)} - \frac{1}{\phi(x_{k-1})}\right| = \sum_{k=1}^{n}\left|\frac{\phi(x_k)-\phi(x_{k-1})}{\phi(x_k)\phi(x_{k-1})}\right| \le \epsilon_0^2\sum_{k=1}^{n}|\phi(x_k)-\phi(x_{k-1})|.$$

Therefore, $\text{Var}(1/\phi) \le \epsilon_0^2\text{Var}(\phi)$.

5.5.4. If ϕ is uniformly continuous then ϕ is continuous. Conversely, suppose that ϕ is continuous on (a,b) and of bounded variation on $[a,b]$. Since ϕ is the difference of two increasing functions, it follows from Theorem 4.18 that ϕ has left and right limits at each point in $[a,b]$. Thus ϕ is continuously extendable to $[a,b]$. Hence by Theorem 3.40, ϕ is uniformly continuous on (a,b).

5.5.5. a) Repeat the proof of Remark 5.51.

b) Since $\phi'(x) = x^{-2/3}/3$, $\phi'(x)$ is unbounded at $x=0$ and ϕ is strictly increasing on $[-1,1]$. Hence by Remark 5.52, ϕ is of bounded variation on $[-1,1]$.

5.5.6. a) Since polynomials are $\mathcal{C}^1$ on $\mathbf{R}$, they are of bounded variation on any $[a,b]$ by Remark 5.51.

b) By the Mean Value Theorem,

$$\sum_{k=1}^{n}|P(x_k)-P(x_{k-1})| = \sum_{k=1}^{n}|x_k-x_{k-1}||P'(c_k)| \le (b-a)\cdot\sup_{x\in[a,b]}|P'(x)|.$$

However, the supremum of $P'(x)$ on $[a,b]$ occurs at $x=a$, $x=b$, or $x=r_j$, where $r_1,\dots,r_m$ are the roots of P'' which lie in $[a,b]$. Since P'' is of degree $N-2$, $m \le N-2$. Therefore,

$$\text{Var}\,(P) \le (b-a)\cdot\sup_{x\in[a,b]}|P'(x)| \le (b-a)\cdot\max\{|P'(r_1)|,\dots,|P'(r_m)|,|P'(a)|,|P'(b)|\}$$

is an estimate which involves no more than N points.

5.5.7. By Theorem 5.56, $|\phi(x)-\phi(x_0)| \le \Phi(x)-\Phi(x_0)$. If Φ is continuous at x_0 then $\Phi(x)-\Phi(x_0)\to 0$ as $x\to x_0$. Thus by the Squeeze Theorem, $\phi(x)-\phi(x_0)\to 0$ as $x\to x_0$, i.e., ϕ is continuous at x_0.

5.5.8. By the Comparison Theorem,

$$\sum_{k=1}^{n}|F(x_k)-F(x_{k-1})| \le \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}|f(t)|\,dt \le \sup_{x\in[a,b]}|f(x)|\,(b-a).$$

5.5.9. Let $\{x_0,\dots,x_n\}$ be a partition of $[a,b]$. Then by the Fundamental Theorem of Calculus,

$$\sum_{j=1}^{n}|f(x_j)-f(x_{j-1})| = \sum_{j=1}^{n}\left|\int_{x_{j-1}}^{x_j}f'(x)\,dx\right| \le \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}|f'(x)|\,dx = \int_a^b|f'(x)|\,dx.$$

Taking the supremum of this last inequality over all partitions of $[a,b]$, we obtain

$$\operatorname{Var} f \leq \int_a^b |f'(x)|\,dx.$$

On the other hand, by the Mean Value Theorem,

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| = \sum_{j=1}^{n} |f'(c_j)|(x_j - x_{j-1}) \geq \sum_{j=1}^{n} m_j(f')(x_j - x_{j-1}).$$

Taking the supremum of this inequality over all partitions of $[a,b]$, we conclude that

$$\operatorname{Var} f \geq (L)\int_a^b |f'(x)|\,dx = \int_a^b |f'(x)|\,dx.$$

Note: If f' is bounded, we obtain

$$(L)\int_a^b |f'(x)|\,dx \leq \operatorname{Var} f \leq (U)\int_a^b |f'(x)|\,dx.$$

For the first inequality, repeat the argument above. For the second inequality, observe that for any partitions P and Q, we can use the Mean Value Theorem to obtain

$$V(f,P) \leq V(f, P \cup Q) \leq U(|f|, P \cup Q) \leq U(|f|, Q).$$

5.6 Convex Functions.

5.6.1. If f and g are convex on I and $x, y \in I$, then $f(\alpha x + (1-\alpha)y) + g(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) + \alpha g(x) + (1-\alpha)g(y) = \alpha(f(x) + g(x)) + (1-\alpha)(f(y) + g(y))$. Thus $f+g$ is convex on I. If $c \geq 0$ then $cf(\alpha x + (1-\alpha)y) \leq c\alpha f(x) + c(1-\alpha)f(y)$. Thus cf is convex on I.

5.6.2. Let f_n be convex on I. For each $n \in \mathbf{N}$ and $x, y \in I$, $f_n(\alpha x + (1-\alpha)y) \leq \alpha f_n(x) + (1-\alpha)f_n(y)$. Taking the limit of this inequality as $n \to \infty$, we obtain $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. Thus f is convex on I.

5.6.3. By Remark 5.59, if f is both convex and concave on I then f and the chord from $(c, f(c))$ to $(d, f(d))$ coincide for every $c < d$ in I. Hence the graph $y = f(x)$, $x \in I$, is a straight line, i.e., $f(x) = mx + b$ for some $m, b \in \mathbf{R}$ and all $x \in I$. The converse is trivial.

5.6.4. Fix $x \in (0, \infty)$. Since $f''(x) = p(p-1)x^{p-2}$, it is clear that $f''(x) \geq 0$ when $p \geq 1$ and $f''(x) \leq 0$ when $0 < p \leq 1$. Therefore, the result follows immediately from Theorem 5.61.

5.6.5. Let $a < c < x < d < b$. Then

$$\frac{F(x) - F(c)}{x - c} = \frac{1}{x-c}\int_c^x f(t)\,dt.$$

Let R_1 be the rectangle with base $[c, x]$ and height $f(x-)$, and R_2 be the rectangle with base $[x, d]$ and height $f(x+)$. Then

$$\frac{\operatorname{Area}(R_1)}{x-c} = f(x-) \leq f(x+) = \frac{\operatorname{Area}(R_2)}{d-x}.$$

Since f increases on $[a,b]$, it follows that

$$\frac{1}{x-c}\int_c^x f(t)\,dt \leq \frac{\operatorname{Area}(R_1)}{x-c} \leq \frac{\operatorname{Area}(R_2)}{d-x} \leq \frac{1}{d-x}\int_x^d f(t)\,dt = \frac{F(d) - F(x)}{d-x}.$$

We conclude by Remark 5.60 that F is convex on $[a,b]$.

5.6.6. Let $x=(b-a)u+a$ and $\phi(u)=f((b-a)u+a)$. By a change of variables formula and Jensen's Inequality,

$$\begin{aligned}\left(\int_a^b |f(x)|\,dx\right)^2 &= \left((b-a)\int_0^1 |\phi(u)|\,du\right)^2\\ &\le (b-a)^2\int_0^1 |\phi(u)|^2\,du\\ &= (b-a)\int_a^b |f(x)|^2\,dx.\end{aligned}$$

Taking the square root of this inequality, we conclude that

$$\int_a^b |f(x)|\,dx \le (b-a)^{1/2}\left(\int_a^b f^2(x)\,dx\right)^{1/2}.$$

5.6.7. These are all straight forward applications of Jensen's Inequality.

a) Since $(e^x)''=e^x>0$ for all $x\in\mathbf{R}$, e^x is convex on $\mathbf{R}$. Apply Jensen for $\phi(x)=e^x$.

On the other hand, if $r\le 1$, then (by Exercise 5.6.4) $\phi(x)=x^{1/r}$ is convex on $[0,\infty)$. Hence by Jensen's inequality,

$$\left(\int_0^1 |f(x)|^r\,dx\right)^{1/r}=\phi\left(\int_0^1 |f(x)|^r\,dx\right)\le\int_0^1 \phi(|f(x)|^r)\,dx=\int_0^1 |f(x)|\,dx.$$

b) Let $p\le q$. By Exercise 5.6.4, $\phi(x)=x^{q/p}$ is convex on $[0,\infty)$. Hence by Jensen's inequality,

$$\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}=\left(\phi\left(\int_0^1 |f(x)|^p\,dx\right)\right)^{1/q}\le\left(\int_0^1 \phi(|f(x)|^p)\,dx\right)^{1/q}=\left(\int_0^1 |f(x)|^q\,dx\right)^{1/q}.$$

c) If $p<q$, if f is locally integrable on $(0,1)$, and if the improper integral

$$\|f\|_q:=\left(\int_0^1 |f(x)|^q\,dx\right)^{1/q}$$

is finite, then the improper integral $\|f\|_p$ is also finite. To prove this, combine the inequality in part b) above with the Comparison Theorem for improper integrals.

5.6.8. a) Let $E:=\{x\in[a,b]:f(x)>y_0\}$. Since $a\in E$ and E is bounded above by b, $x_0:=\sup E$ is a finite real number. By the Approximation Property, choose $x_n\in E$ such that $x_n\to x_0$. Since f is continuous and $f(x_n)>y_0$ for all $n\in\mathbf{N}$, we have $f(x_0)\ge y_0$. On the other hand, if $f(x_0)>y_0$ then choose $h_0>0$ such that $x_0+h_0<b$ and $f(x_0+h_0)>y_0$. Then $x_0+h_0\in E$ so x_0 cannot be the supremum of E. This contradiction proves that $f(x_0)=y_0$. Finally, since $x_0=\sup E$ we have $f(x_0+h)\le y_0=f(x_0)$ for any $h>0$. Hence $D_Rf(x_0)=\lim_{h\to 0+}(f(x_0+h)-f(x_0))/h\le 0$.

b) If $f(b)<f(a)$ then by part a), given $y_0\in(f(b),f(a))$ there is an $x_0\in(a,b)$ such that $y_0=f(x_0)$ and $D_Rf(x_0)\le 0$. In particular, there are uncountably many $x_0\in(a,b)$ which satisfy $D_Rf(x_0)\le 0$.

c) If f is not increasing on (a,b) then there are points $c<d$ in (a,b) such that $f(d)<f(c)$. Hence by part b), $D_Rf(x)\le 0$ for uncountably many $x\in(c,d)\subset(a,b)$, a contradiction.

d) $D_Rg(x)=D_Rf(x)+1/n>D_Rf(x)\ge 0$ for all but countably many $x\in(a,b)$.

e) By parts c) and d), $g_n(x):=f(x)+x/n$ is increasing on (a,b). Thus given $x_1<x_2$,

$$f(x_1)=\lim_{n\to\infty}g_n(x_1)\le\lim_{n\to\infty}g_n(x_2)=f(x_2).$$

CHAPTER 6

6.1 Introduction.

6.1.0. a) False. $a_k = 1/k$ is strictly decreasing to 0 but $\sum_{k=1}^{\infty} 1/k$ diverges.

b) False. The series associated with $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$ both diverge, but $\sum_{k=1}^{\infty}(a_k+b_k) = \sum_{k=1}^{\infty} 0 = 0$.

c) True. For example, if $\sum_{k=1}^{\infty}(a_k + b_k)$ and $\sum_{k=1}^{\infty} a_k$ converge, then $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}(a_k + b_k - a_k) = \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} a_k$ converges by Theorem 6.10.

d) True. By algebra and telescoping

$$\sum_{k=1}^{\infty}(a_k - a_{k+2}) = \sum_{k=1}^{\infty}(a_k - a_{k+1}) + \sum_{k=1}^{\infty}(a_{k+1} - a_{k+2}) = (a_1 - a) + (a_2 - a).$$

6.1.1. a) $\sum_{k=1}^{\infty}(-1)^{k+1}/e^{k-1} = \sum_{k=0}^{\infty}(-1/e)^k = 1/(1+1/e) = e/(1+e)$.

b) $\sum_{k=0}^{\infty}(-1)^{k-1}/\pi^{2k} = -\sum_{k=0}^{\infty}(-1/\pi^2)^k = -1/(1+1/\pi^2) = -\pi^2/(\pi^2+1)$.

c) $\sum_{k=2}^{\infty} 4^{k+1}/9^{k-1} = 36\sum_{k=2}^{\infty}(4/9)^k = 36(4/9)^2/(1-4/9) = 64/5$.

d) $\sum_{k=0}^{\infty}(5^{k+1} + (-3)^k)/7^{k+2} = (5\sum_{k=0}^{\infty}(5/7)^k + \sum_{k=0}^{\infty}(-3/7)^k)/7^2 = (5/2 + 1/10)/7 = 13/35$.

6.1.2. a) $\sum_{k=1}^{\infty} 1/(k(k+1)) = \sum_{k=1}^{\infty}(1/k \,-\, 1/(k+1)) = 1 - \lim_{k\to\infty} 1/k = 1$.

b) $\sum_{k=1}^{\infty} 12/(k+2)(k+3) = -3\sum_{k=1}^{\infty}(2k/(k+2) - (2k+2)/(k+3)) = -3(2/3 - 2) = 4$.

c) $\log(k(k+2)/(k+1)^2) = \log(k/(k+1)) - \log((k+1)/(k+2))$. Therefore, by telescoping we obtain

$$\sum_{k=2}^{\infty}\log(k(k+2)/(k+1)^2) = \log(2/3) - \lim_{k\to\infty}\log(k/(k+1)) = \log(2/3).$$

d) Since $2\sin(a-b)\cos(a+b) = \sin(2a) - \sin(2b)$, we have

$$\sum_{k=1}^{\infty} 2\sin(\frac{1}{k} - \frac{1}{k+1})\cos(\frac{1}{k} - \frac{1}{k+1}) = \sum_{k=1}^{\infty}(\sin\frac{2}{k} - \sin\frac{2}{k+1}) = \sin 2 - 0 = \sin 2.$$

6.1.3. a) $\cos(1/k^2) \to \cos 0 = 1$. Hence this series diverges by the Divergence Test.

b) By L'Hôpital's Rule, $(1 - 1/k)^k \to e^{-1}$. Hence this series diverges by the Divergence Test.

c) $s_n := \sum_{k=1}^{n}(k+1)/k^2 \geq t_n := \sum_{k=1}^{n} 1/k$. Since $t_n \to \infty$, it follows from the Squeeze Theorem that $s_n \to \infty$ as $n \to \infty$. Therefore, the original series diverges.

6.1.4. Since $a_{k+1} - 2a_k + a_{k-1} = (a_{k+1} - a_k) + (a_{k-1} - a_k)$, this series is the sum of two telescopic series. Hence

$$\sum_{k=1}^{\infty}(a_{k+1} - 2a_k + a_{k-1}) = \sum_{k=1}^{\infty}(a_{k+1} - a_k) + \sum_{k=1}^{\infty}(a_{k-1} - a_k) = L - a_1 + a_0 - L = a_0 - a_1.$$

6.1.5. By telescoping,

$$\sum_{k=1}^{\infty}(x^{2k} - x^{2(k-1)}) = (-1 + \lim_{k\to\infty} x^{2k}) = \begin{cases} -1 & |x| < 1 \\ \ 0 & |x| = 1 \\ \text{diverges} & |x| > 1. \end{cases}$$

6.1.6. a) Let $s_n := \sum_{k=1}^{n} a_k$. If $\sum_{k=1}^{\infty} a_k$ converges then $s_n \to s$ for some $s \in \mathbf{R}$. By Theorem 2.8, convergent sequences are bounded. Therefore, $\{s_n\}$ is bounded.

b) The partials sums of $\sum_{k=1}^{\infty}(-1)^k$ assume only the values $-1, 0$, hence are bounded. But the series itself diverges by the Divergence Test.

6.1.7. a) Let $x, y \in I$. By the Mean Value Theorem,

$$F(x) - F(y) = F'(c)(x-y) = \left(1 - \frac{f'(c)}{f'(a)}\right)(x-y).$$

Thus by hypothesis, $|F(x) - F(y)| \le r|x - y|$.

b) By a) and induction, $|x_{n+1} - x_n| = |F(x_n) - F(x_{n-1})| \le r|x_n - x_{n-1}| \le r^n|x_1 - x_0|$.

c) Since $x_0 \in I$ and $F(I) \subseteq I$, all x_n's belong to I. Thus by b) and Geometric series, if $m = n + k$ then

$$|x_m - x_n| \le (r^n + r^{n+1} + \cdots + r^{n+k-1})|x_1 - x_0| \le \frac{r^n}{1-r}|x_1 - x_0|.$$

Since $r^n \to 0$ as $n \to \infty$, we see that x_n is Cauchy, hence converges to some $b \in I$, since I is closed. Taking the limit of $x_{n+1} = x_n - f(x_n)/f'(a)$, we obtain $b = b - f(b)/f'(a)$. We conclude that $f(b) = 0$.

6.1.8. a) Since the a_k's are decreasing, $ka_{2k} = a_{2k} + \cdots + a_{2k} \le a_{k+1} + a_{k+2} + \cdots + a_{2k} = \sum_{j=k+1}^{2k} a_j$. Let $\epsilon > 0$ and choose N so large that $|\sum_{j=k+1}^{2k} a_j| < \epsilon$ for $k \ge N$. Then $|ka_{2k}| < \epsilon$ for $k \ge N$, i.e., $2k\,a_{2k} \to 0$ as $k \to \infty$. On the other hand, since $0 \le (2k+1)a_{2k+1} \le 2ka_{2k} + a_{2k} \to 0$ as $k \to \infty$, it follows from the Squeeze Theorem that $(2k+1)a_{2k+1} \to 0$ as $k \to \infty$.

b) Clearly, $s_{2n+2} = s_{2n} + 1/(2n+1) - 1/(2n+2) > s_{2n}$ and $s_{2n+1} = s_{2n-1} - 1/(2n) + 1/(2n+1) < s_{2n-1}$. Also, $0 \le s_{2n+1} - s_{2n} = 1/(2n+1) \to 0$ as $n \to \infty$. Hence by the Squeeze Theorem, $s_{2n+1} - s_{2n} \to 0$ as $n \to \infty$.

c) By part b), $\{s_{2n}\}$ is increasing and bounded above by $s_1 = 1$, $\{s_{2n+1}\}$ is decreasing and bounded below by $s_2 = 1/2$, and $s_{2n+1} - s_{2n} \to 0$ as $n \to \infty$. Hence both these sequences converge to the same value, i.e., the series $\sum_{k=1}^{\infty}(-1)^{k+1}/k$ converges. However, $k \cdot (-1)^{k+1}/k = (-1)^{k+1}$ does not converge to 0 as $k \to \infty$.

6.1.9. a) $|\sum_{k=1}^{n} b_k - nb| = |\sum_{k=1}^{n}(b_k - b)| \le \sum_{k=1}^{N}|b_k - b| + \sum_{k=N+1}^{n}|b_k - b| \le \sum_{k=1}^{N}|b_k - b| + M(n - N)$.

b) Set $B_n = (b_1 + \cdots + b_n)/n$. Let $\epsilon > 0$ and choose N so large that $|b_k - b| < \epsilon$ for $k \ge N$. By part a), $|B_n - b| \le (\sum_{k=1}^{N}|b_k - b| + \epsilon(n - N))/n$. Since N is fixed, $\sum_{k=1}^{N}|b_k - b|/n \to 0$ and $(n - N)/n \to 1$ as $n \to \infty$. Consequently, $\limsup_{n\to\infty}|B_n - b| \le \epsilon$. Since $\epsilon > 0$ was arbitrary, it follows that $\limsup_{n\to\infty}|B_n - b| = 0$. Therefore, $B_n - b$ converges to 0 as $n \to \infty$.

c) If $b_k = (-1)^k$ then $B_n = -1/n$ if n is odd and 0 if n is even, so $B_n \to 0$ as $n \to \infty$. However, b_k does not converge as $k \to \infty$.

6.1.10. a) $\sigma_n = \sum_{k=0}^{n-1}(1 - k/n)a_k = \sum_{k=0}^{n-1}(n-k)a_k/n = (na_0 + (n-1)a_1 + \cdots + a_{n-1})/n = (a_0 + (a_0 + a_1) + \cdots + (a_0 + a_1 + \cdots + a_{n-1}))/n = (s_1 + \cdots + s_n)/n$.

b) If $\sum_{k=0}^{\infty} a_k = L$ then $s_n \to L$ as $n \to \infty$. Hence by part a) and Exercise 6.1.9b, $\sigma_n \to L$ as $n \to \infty$, i.e., $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to L.

c) Since $s_n = \sum_{k=0}^{n-1}(-1)^k$ is 1 when n is odd and 0 when n is even, the corresponding averages are given by

$$\sigma_n = \begin{cases} (n+1)/(2n) & \text{when } n \text{ is odd} \\ 1/2 & \text{when } n \text{ is even.} \end{cases}$$

Therefore, $\sigma_n \to 1/2$ as $n \to \infty$ although $\sum_{k=0}^{\infty}(-1)^k$ diverges.

d) Suppose $\sum_{k=0}^{\infty} a_k$ diverges. Since $a_k \ge 0$, it follows that $s_n \to \infty$ as $n \to \infty$. Hence given $M > 0$, choose N so large that $s_n \ge M$ for $n \ge N$. Then $\sigma_n \ge \sum_{k=N}^{n-1} s_k/n \ge (n - N)M/n$. Since $(n - N)/n \to 1$ as $n \to \infty$, it follows that $\sigma_n > M/2$ for n large, i.e., $\sigma_n \to \infty$ as $n \to \infty$.

6.1.11. Let $\epsilon > 0$ and choose N so large that $\sum_{k=N+1}^{\infty}|a_k|/k < \epsilon/2$. Since N is fixed, $\sum_{k=1}^{N} a_k/(j+k) \to 0$ as $j \to \infty$. Hence we can choose J so large that $j > J$ implies $|\sum_{k=1}^{N} a_k/(j+k)| < \epsilon/2$. Consequently, if $j > J$ then (since $k + j > k$)

$$\sum_{k=1}^{\infty} \frac{a_k}{j+k} \le \left|\sum_{k=1}^{N} \frac{a_k}{j+k}\right| + \sum_{k=N+1}^{\infty} \frac{|a_k|}{k} < \epsilon.$$

6.1.12. Fix $n \ge 2$. By hypothesis,

$$na_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)},$$

so

$$a_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

By telescoping, we have

$$\sum_{k=2}^{\infty} a_k = \frac{1}{2}\left(\frac{1}{6} - 0\right) = \frac{1}{12}.$$

Since $a_1 = 2/3$, we conclude that

$$\sum_{k=1}^{\infty} a_k = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}.$$

6.2 Series with nonnegative terms.

6.2.0. a) False. If $a_k = 1/k^2$ and $b_k = 1/k$, then $a_k/b_k \to 0$ as $k \to \infty$ and $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} b_k$ does not.

b) True. By hypothesis, $0 \le a_k \le a^k$. Since $0 < a < 1$, the Geometric series $\sum_{k=1}^{\infty} a^k$ converges. thus by the Comparison Theorem, $\sum_{k=1}^{\infty} a_k$ converges.

c) True. By hypothesis, $a_{k+1} \le a_k^2$ for all $k \in \mathbf{N}$. Choose $N \in \mathbf{N}$ such that $|a_k| \le 1/2$ for $k \ge N$. Then $a_{N+1} \le a_N^2 \le 1/4$, $a_{N+2} \le a_{N+1}^2 \le 1/16$, and in general, $a_{N+k} \le 1/4^k$ for $k = 1, 2, \dots$. Since the Geometric series $\sum_{k=1}^{\infty}(1/4)^k$ converges, it follows from the Comparison Theorem that $\sum_{k=N+1}^{\infty} a_k$ converges.

d) False. Let $f(k) = 1/2^k$ and $\int_k^{k+1} f(x)\,dx \ge 1/k$. (Such a function can be constructed by making f piecewise linear on each $[k, k+1]$, its graph forming a triangle whose peak occurs at the midpoint of $[k, k+1]$ with height $2/k$.) Then $\sum_{k=1}^{\infty} f(k) = 1$ converges but $\int_1^{\infty} f(x)\,dx = \infty$.

6.2.1. a) It converges by the Limit Comparison Test, since

$$\frac{(2k+5)/(3k^3+2k-1)}{1/k^2} \to \frac{2}{3} \neq 0$$

as $k \to \infty$.

b) It converges by the Comparison Test and the Geometric Series Test, since $0 \le (k-1)/(k2^k) \le 1/2^k$.

c) Since $p > 1$, choose $\alpha > 0$ such that $p - \alpha > 1$. But $|\log x| \le Cx^{\alpha}$, so $\log k/k^p \le C/k^{p-\alpha}$. Hence the series converges by the Comparison Test and the p-Series test.

d) Since $\log k < \sqrt{k}$ for k large, $k^3 \log^2 k/e^k < k^4/e^k$ for k large. But by six applications of l'Hôpital's Rule,

$$\lim_{k\to\infty} \frac{k^4/e^k}{1/k^2} = \lim_{k\to\infty} \frac{k^6}{e^k} = \lim_{k\to\infty} \frac{6!}{e^k} = 0.$$

But $\sum_{k=1}^{\infty} e^{-k}$ is a geometric series which converges, so by Theorem 6.16ii, $\sum_{k=1}^{\infty} k^4/e^k$ converges. Thus the original series converges by the Comparison Test.

e) It converges by the Limit Comparison Test, since

$$\frac{(\sqrt{k}+\pi)/(2+k^{8/5})}{1/k^{11/10}} \to 1 \neq 0$$

as $k \to \infty$.

f) $k \ge 3$ implies $\log k \ge \log 3 > \log e = 1$, so $\log k \ge p := \log 3$. Thus $k^{\log k} \ge k^p$ for $k \ge 3$, and it follows from the Comparison Test that $\sum_{k=1}^{\infty} k^{-\log k}$ converges.

6.2.2. a) It diverges by the Limit Comparison Test since

$$\frac{(3k^3+k-4)/(5k^4-k^2+1)}{1/k} \to \frac{3}{5} \neq 0$$

as $k \to \infty$.

b) Since $(\sqrt[k]{k}/k) \ge (1/k)$, this series diverges by the Comparison Test.

c) $(k+1)/k \ge 1$ so the terms of this series are all ≥ 1. Thus the original series diverges by the Divergence Test.

d) Let $f(x) = (x(\log x)^p)^{-1}$ for $x > 0$. Since

$$f'(x) = -(x\log^p x)^{-2}(p\log^{p-1} x + \log^p x) \le 0$$

for $x > 1$, $f(x)$ is decreasing for $x > 1$. Since

$$\int_e \frac{dx}{x \log^p x} = \int_1^\infty \frac{du}{u^p} = \infty$$

for $p \le 1$, this series diverges by the Integral Test.

6.2.3. Let $M \ge a_k$ and note that $1/(k+1)^p \le 1/k^p$ for all $k \in \mathbf{N}$. Thus the series $a_k/(k+1)^p$ has nonnegative terms and is dominated by M/k^p. It follows from the Comparison Test and the p-Series Test that this series converges for all $p > 1$.

6.2.4. Since $\log^p(k+1) \ge \log^p k$, we have

$$\sum_{k=2}^\infty \frac{1}{k \log^p(k+1)} \le \sum_{k=2}^\infty \frac{1}{k \log^p k}.$$

But by the Integral Test, this last series converges when $p > 1$. Hence by the Comparison Test, the original series converges when $p > 1$. Similarly,

$$\sum_{k=2}^\infty \frac{1}{k \log^p(k+1)} \ge \sum_{k=2}^\infty \frac{1}{(k+1) \log^p(k+1)} \equiv \sum_{k=3}^\infty \frac{1}{k \log^p k}$$

diverges when $p \le 1$.

6.2.5. When $p \ge 0$ use the Comparison Test, since in this case, $k^p \ge 1$ for all $k \in \mathbf{N}$, so the series is dominated by $\sum_{k=1}^\infty |a_k|$. When $p < 0$, the result is false, since $a_k = 1/k^{1-p}$ generates a convergent series by the p-Series Test ($1-p$ is GREATER than 1 in this case), but $|a_k|/k^p = 1/k$ which generates the harmonic series, which diverges.

6.2.6. a) If $a_n/b_n \to 0$ then $a_n \le b_n$ for n large. If $\sum_{k=1}^\infty b_k$ converges, then it follows from the Comparison Test that $\sum_{k=1}^\infty a_k$ converges.

b) If $a_n/b_n \to \infty$ then $a_n \ge b_n$ for n large. If $\sum_{k=1}^\infty b_k$ diverges, then it follows from the Comparison Test that $\sum_{k=1}^\infty a_k$ diverges.

6.2.7 Since $b_k \to 0$, it surely is bounded. Thus $a_k b_k$ is nonnegative and dominated by Ma_k. Hence the product converges by the Comparison Test. Notice, we really only need that one of the series is bounded and the other convergent.

6.2.8. Notice that $ak + b \ne 0$ for $k \in \mathbf{N}$, since otherwise, $b/a = -k \in \mathbf{Z}$. Also notice that $(1/kq^k)/|1/(ak+b)q^k)| = |ak+b|/k \to |a| \ne 0$. Since $ak+b$ and a are both positive or both negative for large k, the terms $1/(ak+b)q^k$ are eventually all positive or all negative. It follows from the Limit Comparison Test that we need only consider $\sum_{k=1}^\infty (kq^k)^{-1}$.

If $0 < q \le 1$ then $1/q^k \ge 1$ so $\sum_{k=1}^\infty (kq^k)^{-1} \ge \sum_{k=1}^\infty 1/k = \infty$ diverges. If $q > 1$ then the geometric series $\sum_{k=1}^\infty (kq^k)^{-1} \le \sum_{k=1}^\infty 1/q^k < \infty$. Thus the original series diverges when $0 < q \le 1$ and converges when $q > 1$.

6.2.9. If $s_n := \sum_{k=1}^n a_k$ converges then so does s_{2n+1}. Thus

$$\sum_{k=1}^\infty (a_{2k} + a_{2k+1}) = \lim_{n\to\infty} (a_2 + a_3) + \cdots + (a_{2n} + a_{2n+1}) = \lim_{n\to\infty} s_{2n+1}$$

converges. Conversely, if $L := \sum_{k=1}^\infty (a_{2k} + a_{2k+1})$ converges then

$$\sum_{k=2}^{2n+1} a_k = \sum_{k=1}^n (a_{2k} + a_{2k+1}) \to L \quad \text{and} \quad \sum_{k=2}^{2n} a_k = -a_{2n+1} + \sum_{k=2}^{2n+1} a_k \to L$$

as $n \to \infty$. Therefore, $\sum_{k=1}^\infty a_k = a_1 + L$ converges.

6.2.10. If $p \le 0$, then the series diverges by the Divergence Test. If $p > 0$, then $\log(\log(\log k)) > 2/p$ for large k implies that $p \log(\log(\log k)) > 2$ for large k. It follows that

$$\frac{1}{\log(\log k))^p \log k} = \frac{1}{e^{p \log k \cdot \log(\log(\log k))}} < \frac{1}{e^{2 \log k}} = \frac{1}{k^2}.$$

Thus the original series converges by the Comparison Test.

6.3 Absolute Convergence.

6.3.0. a) True. Since

$$\limsup_{k\to\infty} |a_k^\alpha|^{1/k} = \limsup_{k\to\infty} |a_k|^{\alpha/k} = a_0$$

by Remark 6.22iii and $a_0 < 1$, it follows from the Root Test that $\sum_{k=1}^\infty a_k^\alpha$ is absolutely convergent.

b) False. If $a_k = 1/k^2$, then $\sum_{k=1}^\infty a_k$ is absolutely convergent, but $|a_k|^{1/k} \to 1$ as $k \to \infty$.

c) False. If $a_k = -1/k$ and $b_k = 1/k^2$, then $a_k \le b_k$ for all $k \in \mathbf{N}$ and $\sum_{k=1}^\infty b_k$ converges absolutely, but $\sum_{k=1}^\infty a_k$ diverges.

d) True. If $\sum_{k=1}^\infty a_k$ converges absolutely, then $|a_k| \le 1$ for large k. But $|a_k| \le 1$ implies $|a_k^2| \le |a_k|$. Hence, $|a_k^2| \le |a_k|$ for large k, and it follows from the Comparison Theorem that $\sum_{k=1}^\infty a_k^2$ converges absolutely.

6.3.1. a) Since $[1/(k+1)!]/[1/k!] = 1/(k+1) \to 0$ as $k \to \infty$, this series converges by the Ratio Test.

b) Since $\sqrt[k]{1/k^k} = 1/k \to 0$ as $k \to \infty$, this series converges by the Root Test.

c) Since $(\pi^{k+1}/(k+1)!)/(\pi^k/k!) = \pi/(k+1) \to 0$ as $k \to \infty$, this series converges by the Ratio Test.

d) Since by L'Hôpital's Rule $\sqrt[k]{(k/(k+1))^{k^2}} = (k/(k+1))^k \to e^{-1}$ as $k \to \infty$, this series converges by the Root Test.

6.3.2. a) The Ratio Test gives 1, but the series converges by the Comparison Test since $k > e^5$ implies $\log k > 5$ so

$$\frac{k^3}{(k+1)^{\log k}} < \frac{k^3}{(k+1)^5} < \frac{1}{k^2}.$$

b) It converges by the Ratio Test, since

$$\frac{(k+1)^{100}/e^{k+1}}{k^{100}/e^k} = \frac{((k+1)/k)^{100}}{e} \to \frac{1}{e}$$

as $k \to \infty$.

c) It converges by the Root Test, since

$$\sqrt[k]{a_k} \equiv \frac{k+1}{2k+3} \to \frac{1}{2} < 1.$$

d) It converges by the Ratio Test, since

$$\frac{|a_{k+1}|}{|a_k|} = \frac{2k+1}{(2k+1)(2k+2)} \to 0$$

as$k \to \infty$.

e) It converges by the Root Test, since

$$|a_k|^{1/k} = \frac{(k-1)!}{k!+1} < \frac{(k-1)!}{k!} = \frac{1}{k} \to 0$$

as $k \to \infty$.

f) It converges by the Root Test, since

$$\sqrt[k]{a_k} \equiv \frac{3+(-1)^k}{5}$$

has a limit supremum of $4/5$.

g) It diverges by the Root Test, since

$$\sqrt[k]{a_k} \equiv \frac{3-(-1)^k}{\pi}$$

has a limit supremum of $4/\pi$.

6.3.3. a) By the Integral Test (see Exercise 6.2.2d) it converges for all $p > 1$ and diverges for $0 < p \le 1$. It also diverges for $p \le 0$ by the Divergence Test. Therefore, this series converges if and only if $p > 1$.

b) It diverges for all $p > 0$ since $\log k \le Ck^{1/p}$ implies $1/\log^p k \ge 1/k$ for $k \ge 2$. If $p \le 0$, then the series diverges by the Divergence Test.

c) If $p = 0$, the series obviously doesn't make sense, so we can suppose that $p \neq 0$. We shall use the Ratio Test. Since

$$\left|\frac{a_{k+1}}{a_k}\right| \equiv \frac{1}{|p|}\left(\frac{k+1}{k}\right)^p \to \frac{1}{|p|}$$

as $k \to \infty$, $\sum_{k=1}^\infty k^p/p^k$ converges absolutely when $|p| > 1$ and diverges when $|p| < 1$. By inspection, it does not converge absolutely when $|p| = 1$. Therefore, the series converges absolutely if and only if $|p| > 1$.

d) Since

$$\frac{1/\sqrt{k}(k^p - 1)}{1/k^{p+1/2}} = \frac{k^p}{k^p - 1} \to 1$$

as $k \to \infty$, it follows from the Limit Comparison Test and the p-Series Test that this series converges if and only if $p + 1/2 > 1$, i.e., $p > 1/2$.

e) Rationalizing the numerator, the terms of this series look like $1/(\sqrt{k^{2p}+1} + k^p)$. By the Limit Comparison Test, $\sum_{k=1}^\infty 1/\sqrt{k^{2p}+1}$ converges if and only if $\sum_{k=1}^\infty 1/k^p$ converges, i.e., if and only if $p > 1$. Since $2k^p \leq \sqrt{k^{2p}+1} + k^p \leq 2\sqrt{k^{2p}+1}$ implies $1/(2\sqrt{k^{2p}+1}) \leq 1/(\sqrt{k^{2p}+1} + k^p) \leq 1/(2k^p)$, it follows from the Comparison Test that the original series converges if and only if $p > 1$.

f) We shall use the Ratio Test. Since

$$\left|\frac{a_{k+1}}{a_k}\right| \equiv 2^p\left(\frac{k+1}{k}\right)^k \to \frac{2^p}{e}$$

as $k \to \infty$, by L'Hôpital's Rule, $\sum_{k=1}^\infty 2^{kp}k!/k^k$ converges absolutely when $2^p < e$, i.e., when $p < \log_2(e)$, and diverges when $p > \log_2(e)$. When $p = \log_2(e)$, we compare the series with $\sqrt{k}$. Indeed, by Stirling's Formula,

$$\left(\frac{e^k k!}{k^k}\right)/\sqrt{k} \to \sqrt{2\pi} \neq 0$$

as $k \to \infty$. Therefore, the original series diverges when $p = \log_2(e)$.

6.3.4. Notice that $\sqrt[k]{|a_k x^k|} = \sqrt[k]{a_k}\,|x| \to a\,|x|$ as $k \to \infty$. Hence if $a \neq 0$, then it follows from the Root Test that this series converges absolutely when $a|x| < 1$, i.e., $|x| < 1/a$. If $a = 0$, then the limit is zero for all x, so by the Root Test the series converges absolutely for all $x \in \mathbf{R}$.

6.3.5. Notice that all $(-1)^k a_k$'s are nonnegative. Hence $\sum_{k=1}^\infty a_k$ converges absolutely by the Ratio Test, since

$$\left|\frac{a_{k+1}}{a_k}\right| \equiv \left(1 + (k+1)\sin\left(\frac{1}{k+1}\right)\right)^{-1} \to \frac{1}{2}$$

as $k \to \infty$ by L'Hôpital's Rule.

6.3.6. a) Since $a_{kj} \geq 0$, $0 \leq \sum_{j=1}^N a_{kj} \leq \sum_{j=1}^\infty a_{kj} = A_k$ for all $N \in \mathbf{N}$. Hence by the Comparison Test,

$$\sum_{j=1}^N \sum_{k=1}^\infty a_{kj} = \sum_{k=1}^\infty \sum_{j=1}^N a_{kj} \leq \sum_{k=1}^\infty A_k = \sum_{k=1}^\infty \sum_{j=1}^\infty a_{kj}.$$

Taking the limit of this inequality as $N \to \infty$ we obtain the desired inequality.

b) By part a), $\sum_{k=1}^\infty a_{kj} < \infty$. Hence by reversing the roles of k and j, we obtain the reverse inequality.

c) By inspection, $\sum_{j=1}^\infty a_{kj} = 0$ for all $k \in \mathbf{N}$ but $\sum_{k=1}^\infty a_{kj} = 1$ if $j = 1$ and 0 if $j > 1$. Therefore,

$$\sum_{k=1}^\infty \sum_{j=1}^\infty a_{kj} = 0 \neq 1 = \sum_{j=1}^\infty \sum_{k=1}^\infty a_{kj}.$$

6.3.7. a) Since $a_k \to 0$ as $k \to \infty$, $|a_k| < 1$ for k large. Hence $|a_k|^p \leq |a_k|$ for k large and it follows from the Comparison Test that $\sum_{k=1}^\infty |a_k|^p$ converges.

b) If $\sum_{k=1}^\infty k^p a_k$ converges for some $p > 1$, then $k^p a_k \to 0$ as $k \to \infty$, i.e., $k^p|a_k| < 1$ for large k. Thus $|a_k| \leq 1/k^p$ for large k. Since $p > 1$, it follows from the Comparison Test that $\sum_{k=1}^\infty |a_k|$ converges, a contradiction.

6.3.8. a) The middle inequality is obvious since the infimum of a set is always less than or equal to its supremum.

To prove the right-most inequality, suppose that $r = \limsup_{k\to\infty} a_{k+1}/a_k$. We may suppose that $r \neq \infty$. For any $r_0 > r$, by Remark 6.22i, there is an $N \in \mathbf{N}$ such that $k \ge N$ implies $a_{k+1}/a_k \le r_0$. Fix $j \in \mathbf{N}$. It follows that

$$a_{N+j} \le a_{N+j-1} r_0 \le a_{N+j-2} r_0^2 \le \cdots \le a_N r_0^j,$$

i.e., $a_k \le a_N r_0^{k-N}$ for all $k \ge N$. In particular, if $n > N$, then

$$\sup_{k>n} \sqrt[k]{a_k} \le (a_N r_0^{-N})^{1/k} \cdot r_0.$$

Taking the limit of this last inequality as $k \to \infty$, we see that $\limsup_{k\to\infty} \sqrt[k]{a_k} \le r_0$. Finally, letting $r_0 \downarrow r$, we conclude that $\limsup_{k\to\infty} \sqrt[k]{a_k} \le r$, as required.

To prove the left-most inequality, repeat the steps above, using part a) in place of Remark 6.22i, but with infimum in place of supremum and $r_1 < r$ in place of $r_0 > r$, proves part c).

d) If $|b_{k+1}/b_k| \to r$ as $k \to \infty$, then by Remark 6.22iii and part b), $\limsup_{k\to\infty} |b_{k+1}|/|b_k| = r = \liminf_{k\to\infty} |b_{k+1}|/|b_k|$. We conclude from part c that $\limsup_{k\to\infty} \sqrt[k]{|b_k|} = \liminf_{k\to\infty} \sqrt[k]{|b_k|} = r$. But if we translate this back into ϵ–δ language, we conclude that $\sqrt[k]{|b_k|} \to r$ as $k \to \infty$.

6.3.9. By hypothesis,

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{24}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

6.3.10. Since each a_k^+ and $-a_k^-$ is either a_k or 0, it suffices to show there are integers $0 < k_1 < r_1 < k_2 < r_2 < \dots$ such that if $b_1 = a_1^+$, $b_2 = a_2^+$, $\dots$, $b_{k_1} = a_{k_1}^+$, $b_{k_1+1} = -a_1^-$, $\dots$, $b_{r_1} = -a_{r_1-k_1}^-$, $b_{r_1+1} = a_{k_1+1}^+$, $\dots$, and $s_n = \sum_{j=1}^n b_j$, then $\liminf_{n\to\infty} s_n = x$ and $\limsup_{n\to\infty} s_n = y$. We suppose for simplicity that x and y are both finite.

Since $\sum_{k=1}^\infty a_k^+ = \infty$, choose an integer $k_1 \in \mathbf{N}$ least such that

$$s_{k_1} := b_1 + b_2 + \cdots + b_{k_1} := a_1^+ + a_2^+ + \cdots + a_{k_1}^+ > y.$$

Since k_1 is least, $s_{k_1-1} \le y$, hence $s_{k_1} \le y + b_{k_1}$. Similarly, since $\sum_{k=1}^\infty a_k^- = \infty$ we can choose an integer $r_1 > k_1$ least such that

$$s_{r_1} := b_1 + b_2 + \cdots + b_{r_1} := s_{k_1} - a_1^- - \cdots - a_{r_1-k_1}^- < x,$$

and $s_{r_1} \ge x + b_{r_1}$. Since the $-a_\ell^-$'s are nonpositive, it is clear that $s_\ell \le s_{k_1} \le y + b_{k_1}$ for $k_1 < \ell \le r_1$,. Therefore,

$$s_{k_1} > y \quad \text{and} \quad x + b_{r_1} \le s_\ell \le y + b_{k_1}$$

for all $k_1 \le \ell \le r_1$. By a similar argument, if $k_2 > r_1$ is least such that $s_{k_2} > y$, then $s_{r_1} < x$ and $x + b_{r_1} \le s_\ell \le y + b_{k_2}$ for all $r_1 \le \ell \le k_2$. In particular,

$$y < \sup_{k_1 \le \ell \le k_2} s_\ell \le y + \max\{b_{k_1}, b_{k_2}\} \le y + \sup_{\ell \ge k_1} b_\ell.$$

In the same way, if $r_2 > k_2$ is least such that $s_{r_2} < x$, then

$$x + \inf_{\ell \ge r_1} b_\ell \le \sup_{r_1 \le \ell \le r_2} s_\ell < x.$$

Continuing this process, we generate integers $k_1 < r_1 < k_2 < r_2 < \dots$ such that for each $j \in \mathbf{N}$,

$$y < \sup_{k_j \le \ell \le k_{j+1}} s_\ell \le y + \sup_{\ell \ge k_j} b_\ell \quad \text{and} \quad x + \inf_{\ell \ge r_j} b_\ell \le \inf_{r_j \le \ell \le r_{j+1}} s_\ell < x.$$

The first of these inequalities implies

$$y < \sup_{\ell \geq k_j} s_\ell \leq y + \sup_{\ell \geq k_j} b_\ell.$$

Taking the limit of this inequality as $j \to \infty$, bearing in mind that by the Divergence Test $b_n \to 0$ as $n \to \infty$, we conclude that

$$y \leq \limsup_{n\to\infty} s_n \leq y + \limsup_{n\to\infty} b_n = y.$$

This proves s_n has limit supremum y. A similar argument establishes that s_n has limit infimum x.

6.3.11. By Exercise 4.4.4 and the Squeeze Theorem, it suffices to show that $s_m := \sum_{k=0}^{m}(-1)^k x^{2k+1}/(2k+1)!$ converges as $m \to \infty$ for all $x \in \mathbf{R}$. But it does converge by the Ratio Test:

$$\left|\frac{(-1)^{k+1}x^{2k+3}/(2k+3)!}{(-1)^k x^{2k+1}/(2k+1)!}\right| = \frac{|x^2|}{(2k+2)(2k+3)} \to 0$$

for all $x \in \mathbf{R}$. A similar argument works for the cosine series.

6.4 Alternating series.

6.4.0. a) True. If $\sum_{k=1}^{\infty} b_k$ converges, then its partial sums are bounded. Hence apply Dirichlet's Test to $a_k \downarrow 0$ and $\sum_{k=1}^{\infty} b_k$.

b) False. Let $a_k = (-1)^k/k$. Then $\sum_{k=1}^{\infty}(-1)^k a_k = \sum_{k=1}^{\infty} 1/k$ which diverges.

c) False. Let $a_k = 1/k$ if k is odd and $a_k = 2/k$ if k is even. Then

$$a_{2k} - a_{2k+1} = \frac{2}{2k} - \frac{1}{2k+1} = \frac{k+1}{2k^2+k}.$$

The series associated with this last fraction diverges by the Limit Comparison Test (compare it with $1/k$). Therefore, $\sum_{k=1}^{\infty}(-1)^k a_k$ diverges.

d) False. Let $a_k = 1/k^2$ if k is odd and $a_k = 2/k^2$ if k is even. Since $2k+1 < k^2$ for $k \geq 3$ implies $(k+1)^2 < 2k^2$, it is easy to check that a_k is not monotone when $k > 3$. On the other hand, $\sum_{k=1}^{\infty}(-1)^k a_k$ converges absolutely by the Comparison Test since

$$\sum_{k=1}^{\infty} |a_k| \leq 2\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

6.4.1. a) Clearly, $1/k^p \downarrow 0$ as $k \to \infty$ for all $p > 0$. Therefore, the series converges by the Alternating Series Test.

b) By Example 6.32, $\sum_{k=1}^{\infty} \sin(kx)$ has bounded partial sums for all $x \in \mathbf{R}$. Hence the series converges by the Dirichlet Test.

c) Since $(1 - \cos(1/x))' = -\sin(1/x)/x^2 < 0$ for $x \geq 1$, $1 - \cos(1/k)$ is decreasing. Thus the original series converges by the Alternating Series Test.

d) Since $(x/3^x)' = (3^x \cdot 1 - x \log 3 \cdot 3^x)/3^{2x} = (1 - x\log 3)/3^x < 0$ for $x \geq 1$, $k/3^k$ is decreasing. Thus the original series converges by the Alternating Series Test.

e) Let $f(x) = \pi/2 - \arctan x$. Since $f'(x) = -1/(1+x^2) < 0$ for all $x \in \mathbf{R}$, $f(k) \downarrow 0$ as $k \to \infty$. Hence this series converges by the Alternating Series Test.

6.4.2. a) By the Ratio Test, this series converges for all $|x| < 1$ and diverges for all $|x| > 1$. It diverges at $x = 1$ (the harmonic series) and converges at $x = -1$ (an alternating series). Thus it converges if and only if $x \in [-1, 1)$.

b) Since $x^{3k}/2^k = (x^3/2)^k$, this series is geometric. Hence, it converges if and only if $|x^3| < 2$, i.e., if and only if $x \in (-\sqrt[3]{2}, \sqrt[3]{2})$.

c) By the Ratio Test, this series converges when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$ it converges by the Alternating Series Test. When $x = -1$ it diverges by the Limit Comparison Test (compare it with $1/k$).

d) The absolute value of the ratio of successive terms of this series is given by

$$k\sqrt{k+1}|x+2|/((k+1)\sqrt{k+2}).$$

Thus by the Ratio Test, this series converges when $|x+2| < 1$ (i.e., when $-3 < x < -1$) and diverges when $|x+2| > 1$. If $x = -1$ or $x = -3$, this series is $\sum_{k=1}^{\infty}(\pm 1)^k/(k\sqrt{k+1})$ which converges absolutely by the Limit Comparison Test, since $\sum_{k=1}^{\infty} k^{-3/2} < \infty$. Therefore, the original series converges if and only if $x \in [-3, -1]$.

6.4.3. a) Since $[(k+1)^3/(k+2)!]/[k^3/(k+1)!] = (k+1)^3/(k^3(k+2)) \to 0$ as $k \to \infty$, this series converges absolutely by the Ratio Test.

b) Since

$$\left|\frac{(-1)(-3)\dots(1-2k)(-1-2k)/(1\cdot 4\dots(3k-2)(3k+1))}{(-1)(-3)\dots(1-2k)/(1\cdot 4\dots(3k-2))}\right| = \left|\frac{-1-2k}{3k+1}\right| \to \frac{2}{3} < 1,$$

this series converges absolutely by the Ratio Test.

c) Since $((k+2)^{k+1}/(p^{k+1}(k+2)!))/((k+1)^k/(p^k k!)) = ((k+2)/(k+1))^{k+1} \cdot (1/p) \to e/p$ as $k \to \infty$ and $e/p < 1$, this series converges absolutely by the Ratio Test.

d) Let $f(x) = \sqrt{x}/(x+1)$ for $x > 0$. Since $f'(x) = (1-x)/(2\sqrt{x}(x+1)^2) < 0$ for $x > 1$, f is strictly decreasing on $(1, \infty)$. Thus $f(k) \downarrow 0$ as $k \to \infty$ and this series converges by the Alternating Series Test. On the other hand, $(\sqrt{k}/(k+1))/(1/\sqrt{k}) = k/(k+1) \to 1$ as $k \to \infty$. Hence it follows from the Limit Comparison Test that $\sum_{k=1}^\infty \sqrt{k}/(k+1)$ diverges. Hence the original series is conditionally convergent.

e) Since $(\sqrt{k+1}/k^{k+1/2})/(1/k^k) = \sqrt{k+1}/\sqrt{k} \to 1$ as $k \to \infty$, this series converges absolutely by the Limit Comparison Test.

6.4.4. If $b_k \downarrow b$, then $b_k - b \downarrow 0$ as $k \to \infty$. Moreover, if $\sum_{k=1}^\infty a_k$ converges, then it surely has bounded partial sums. Hence by Dirichlet's Test, $\sum_{k=1}^\infty a_k(b_k - b)$ converges, say to s. But $\sum_{k=1}^\infty a_k b$ converges, so we can add it to both sides of $s = \sum_{k=1}^\infty a_k(b_k - b) \equiv \sum_{k=1}^\infty a_k b_k - \sum_{k=1}^\infty a_k b$. We obtain

$$\sum_{k=1}^\infty a_k b_k = \sum_{k=1}^\infty a_k b + s.$$

6.4.5. By Abel's Formula, $\sum_{k=1}^n a_k b_k = b_n s_n + \sum_{k=1}^{n-1} s_k(b_k - b_{k-1})$. Take the limit of this identity as $n \to \infty$, bearing in mind that s_n is bounded and $b_n \to 0$ as $n \to \infty$. We obtain $\sum_{k=1}^\infty a_k b_k = \sum_{k=1}^\infty s_k(b_k - b_{k-1})$.

6.4.6. By Abel's Formula, $\sum_{k=m}^n a_k b_k = B_{n,m} a_n - \sum_{k=m}^{n-1} B_{k,m}(a_{k+1} - a_k)$ where $B_{n,m} := \sum_{k=m}^n b_k$. By hypothesis, $|B_{n,m}| \le 2M$. Hence

$$\left|\sum_{k=m}^n a_k b_k\right| \le 2M|a_n| + 2M \sum_{k=m}^{n-1} |a_{k+1} - a_k|.$$

Since $a_n \to 0$ and $\sum_{k=1}^\infty |a_{k+1} - a_k| < \infty$, it follows that $\sum_{k=1}^\infty a_k b_k$ is Cauchy, hence convergent.

6.4.7. Let $c_n := \sum_{k=n}^\infty a_k b_k$ for $n \in \mathbf{N}$. Given $\epsilon > 0$ choose N so large that $b_k > 0$ and $|c_k| < \epsilon/2$ for $k \ge N$. By Abel's Formula,

$$\sum_{k=m}^n a_k = \sum_{k=m}^n \frac{a_k b_k}{b_k} = \sum_{k=m}^n \frac{c_k - c_{k+1}}{b_k} = \frac{c_m - c_{n+1}}{b_n} + \sum_{k=m}^{n-1} (c_k - c_{k+1}) \left(\frac{1}{b_{k+1}} - \frac{1}{b_k}\right).$$

Now $1/b_n \to 0$ as $n \to \infty$ so

$$\sum_{k=m}^\infty a_k = \lim_{n\to\infty} \sum_{k=m}^{n-1} (c_k - c_{k+1}) \left(\frac{1}{b_{k+1}} - \frac{1}{b_k}\right)$$

and this limit must exist. Let $m > N$. Since the $1/b_k$'s are decreasing, we have by telescoping that

$$\left|\sum_{k=m}^\infty a_k\right| \le 2 \sup_{k\ge m} |c_k| \left|\frac{1}{b_n} - \frac{1}{b_m}\right| \le 2 \sup_{k\ge m} |c_k| \left(\frac{1}{b_m} - 0\right) \le 2 \sup_{k\ge N} |c_k| \left(\frac{1}{b_m}\right) < \frac{\epsilon}{b_m}.$$

We conclude that $|b_m \sum_{k=m}^\infty a_k| < \epsilon$ for $m \ge N$, i.e., $b_m \sum_{k=m}^\infty a_k \to 0$ as $m \to \infty$.

6.4.8. By a sum angle formula and telescoping, we see that $2\sin(x/2)\sum_{k=1}^n \cos(kx) = \sum_{k=1}^n (\sin((k-1/2)x) - \sin((k+1/2)x)) = \sin(x/2) - \sin((n+1/2)x)$. Thus

$$|\sum_{k=1}^n \cos(kx)| \le 1/|\sin(x/2)| < \infty$$

for each fixed $x \in (0, 2\pi)$. Hence by Dirichlet's Test, $\sum a_k \cos(kx)$ converges for each $x \in (0, 2\pi)$. When $x = 0$, the series converges if and only if $\sum_{k=1}^\infty a_k$ converges.

6.4.9. By a sum angle formula and telescoping, we see that $2\sin x \sum_{k=1}^n \sin(2k+1)x = \sum_{k=1}^n (\cos((2k)x) - \cos((2k+2)x) = \cos(2x) - \cos((2n+2)x)$. Thus

$$|\sum_{k=1}^n \sin(2k+1)x| \le 2/|\sin x| < \infty$$

for each fixed $x \in (0, \pi) \cup (\pi, 2\pi)$. Hence by Dirichlet's Test, $\sum a_k \sin(2k+1)x$ converges for each $x \in (0, \pi) \cup (\pi, 2\pi)$. Since the series is identically zero when $x = 0, \pi, 2\pi$, it converges everywhere on $[0, 2\pi]$. But $\sin(2k+1)x$ is periodic of period 2π. Hence this series converges everywhere on $\mathbf{R}$.

6.5 Estimation of series.

6.5.1. a) Let $f(x) = \pi/2 - \arctan x$. Since $f'(x) = -1/(1+x^2) < 0$ for all $x \in \mathbf{R}$, $f(k) \downarrow 0$ as $k \to \infty$. Hence this series converges by the Alternating Series Test. Since $f(100) = 0.00999$, $n = 100$ terms will estimate the value to an accuracy of 10^{-2}.

b) Let $f(x) = x^2 2^{-x} = x^2 e^{-x \log 2}$. Since $f'(x) = x2^{-x}(2 - x\log 2) < 0$ for all $x > 2/\log 2$, $f(x)$ is strictly decreasing for x large. Therefore, the series converges by the Alternating Series Test. Since $f(15) = 0.0068$, $n = 15$ terms will estimate the value to an accuracy of 10^{-2}.

c) Let $a_k = (2 \cdot 4 \ldots 2k)/(1 \cdot 3 \ldots (2k-1)k^2)$ and observe that

$$a_{k+1}/a_k = (2k+2)k^2/((2k+1)(k+1)^2) = 2k^2/(2k^2+3k+1) < 1.$$

Thus $a_{k+1} < a_k$. Moreover, $a_k = (2/3)(4/5)\ldots((2k-2)/(2k-1)) \cdot (2k/k^2) < 2/k \to 0$ as $k \to \infty$. Therefore, this series converges by the Alternating Series Test. Since $a_9 \approx= .0105$ and $a_{10} \approx .0055$, $n = 10$ terms will estimate the value to an accuracy of 10^{-2}.

6.5.2. a) $p > 1$ (see Exercise 6.2.4).

b) Let $f(x) = 1/(x \log^p(x+1)$. By Theorem 6.35,

$$-\int_n^\infty f(x)\,dx \le s_n - s \le f(n) - \int_n^\infty f(x)\,dx,$$

so

$$|s - s_n| \le f(n) + \int_n^\infty f(x)\,dx.$$

Since

$$\int_n^\infty f(x)\,dx = \int_n^\infty \frac{dx}{x\log^p(x+1)} \le \int_n^\infty \frac{dx}{x\log^p(x)} = \frac{1}{(p-1)\log^{p-1}(n)},$$

it follows that

$$|s - s_n| \le \frac{1}{n\log^p(n+1)} + \frac{1}{(p-1)\log^{p-1}(n)} \le \frac{n+p-1}{n(p-1)}\left(\frac{1}{\log^{p-1}(n)}\right).$$

6.5.3. a) Since $[1/(k+1)!]/[1/k!] = 1/(k+1) \to 0$ as $k \to \infty$, this series converges by the Ratio Test. The ratio is less than or equal to $1/3$ for $k > N = 1$. Hence by Remark 6.40, $|s_n - s|$ is dominated by $(1/3)^n/(2/3) = (1/2)(1/3)^{n-1}$. For $n = 7$, this last ratio is about 0.00069 still a little too big, but it's about $0.00023 < 0.0005$ for $n = 8$.

b) Since $\sqrt[k]{1/k^k} = 1/k \to 0$ as $k \to \infty$, this series converges by the Root Test. The root is less than or equal to $1/2$ for $k \ge N = 2$. Hence by Remark 6.40, $|s_n - s|$ is dominated by $(1/2)^{n+1}/(1/2) = (1/2)^n$ for $n \ge 2$. Since $1/2^n \approx 0.00098$ for $n = 10$ and ≈ 0.00049 for $n = 11$, choose $n = 11$.

c) Since $(2^{k+1}/(k+1)!)/(2^k/k!) = 2/(k+1) \to 0$ as $k \to \infty$, this series converges by the Ratio Test. The ratio is less than or equal to $1/2$ for $k > N = 2$. Hence by Remark 6.40, $|s_n - s|$ is dominated by $(2^2/2!)(1/2)^{n-1}/(1/2) = (1/2)^{n-3}$. Thus by the calculations in part b), choose $n = 14$.

d) Since by L'Hôpital's Rule $\sqrt[k]{(k/(k+1))^{k^2}} = (k/(k+1))^k \to e^{-1}$ as $k \to \infty$, this series converges by the Root Test. The root is less than or equal to $1/2$ for $k \ge N = 1$ by Example 4.30. Hence by Theorem 6.40, $|s_n - s|$ is dominated by $(1/2)^{n+1}/(1/2) = (1/2)^n$. Thus by the calculations in part b), choose $n = 11$.

6.5.4. Fix $n \geq N$. If $|a_{k+1}|/|a_k| \leq x$ for $k > N$, then $|a_{N+1}| \leq x|a_N|$, $|a_{N+2}| \leq x^2|a_N|$, ..., hence $|a_k| \leq |a_N|x^{k-N}$ for any $k > N$. Hence given $n \geq N$,

$$0 \leq s - s_n = \sum_{k=n+1}^{\infty} |a_k| \leq |a_N| \sum_{k=n+1}^{\infty} x^{k-N} = |a_N| \frac{x^{n-N+1}}{1-x}.$$

6.6 Additional tests.

6.6.1. a) The ratio of successive terms of this series is

$$\frac{2k+3}{2k+2} > 1.$$

Hence $a_{k+1} \geq a_k > 0$, so the series diverges by the Divergence Test.

b) The ratio of successive terms of this series is

$$\frac{2k+1}{2k+5} = 1 - \frac{4}{2k+5} = 1 - \frac{2}{k+5/2}.$$

Hence it converges absolutely by Raabe's Test.

c) Let $u = \log k$ and note that $u \to \infty$ as $k \to \infty$. Now for $k > e$,

$$\frac{\log(1/|a_k|)}{\log k} = \frac{\log(\log k^{\log\log k})}{\log k} = \frac{(\log\log k)^2}{\log k} \equiv \frac{\log^2 u}{u}.$$

But the limit of this last quotient is (by L'Hôpital's Rule twice)

$$\lim_{u\to\infty} \frac{2\log u \cdot (1/u)}{1} = \lim_{u\to\infty} \frac{2\log u}{u} = 0.$$

Hence the series diverges by the Logarithmic Test.

d) Applying L'Hôpital's Rule twice, we obtain

$$\begin{aligned}
p := \lim_{k\to\infty} \frac{k\log(\sqrt{k}/(\sqrt{k}-1))}{\log k} &= \lim_{k\to\infty} \frac{\log(\sqrt{k}/(\sqrt{k}-1))}{\log k/k} \\
&= \lim_{k\to\infty} \frac{\sqrt{k}-1}{\sqrt{k}} \lim_{k\to\infty} \frac{-k^2/2\sqrt{k}}{(\sqrt{k}-1)^2(1-\log k)} \\
&= \lim_{k\to\infty} \frac{k}{(\sqrt{k}-1)^2} \lim_{k\to\infty} \frac{-\sqrt{k}/2}{1-\log k} \\
&= \lim_{k\to\infty} \frac{-1/(4\sqrt{k})}{-1/k} = \infty.
\end{aligned}$$

Hence the series converges absolutely by the Logarithmic Test.

6.6.2. a) It converges absolutely for all $p > 0$ by the Ratio Test, since

$$\frac{(k+1)/e^{(k+1)p}}{k/e^{kp}} = \frac{k+1}{ke^p} \to \frac{1}{e^p} < 1$$

for all $p > 0$. If $p \leq 0$, this series diverges by the Divergence Test.

b) Since $\log((\log k)^{p\log k})/\log k = p\log\log k \to \infty$ if $p > 0$, this series converges absolutely for all $p > 0$ by the Logarithmic Test. It diverges for $p \leq 0$ by the Divergence Test.

c) It converges absolutely for all $|p| < 1/e$ by the Ratio Test, since

$$\left|\frac{(p(k+1))^k/(k+1)!}{(pk)^k/k!}\right| = \left|\frac{p(k+1)^{k+1}}{(k+1)k^k}\right| = |p|\left(\frac{k+1}{k}\right)^k \to |p|e < 1$$

for all $|p| < 1/e$. Similarly, if $|p| > 1/e$, this series diverges by the Ratio Test. If $p = 1/e$, then the terms of the series become

$$\frac{k^k}{e^k \cdot k!} > \frac{1}{e\sqrt{k}}$$

by Stirling's Formula. By the Comparison Test and the p-Series Test, the original series diverges. For $p = -1/e$, the series converges conditionally by the Alternating Series Test and what we just proved.

6.6.3. a) By L'Hôpital's Rule, $\sqrt[k]{1/(\log k)^{\log k}} \to e^0 = 1$ as $k \to \infty$ so the Root Test yields $r = 1$. However, the series converges by the Logarithmic Test since $\log((\log k)^{\log k})/\log k = \log\log k \to \infty$ as $k \to \infty$.

b) The ratio of consecutive terms of this series is $(2k+1)/(2k+4)$ which converges to 1 as $k \to \infty$. However, since $(2k+1)/(2k+4) = 1 - (3/2)/(k+2)$, the series converges by Raabe's Test.

6.6.4. Since the range of f is positive, $|f(k)| = f(k)$ for all $k \in \mathbf{N}$. Moreover, by L'Hôpital's Rule,

$$\lim_{k\to\infty} \frac{\log(1/f(k))}{\log k} = -\lim_{k\to\infty} \frac{f'(k)/f(k)}{1/k} \equiv -\alpha.$$

By the Logarithmic Test, if $-\alpha > 1$, then this series converges absolutely. Hence it surely converges.

6.6.5. If $p > 1$ is infinite, let $q = 2$. If $p > 1$ is finite, let $q = \sqrt{p}$. Note that in either case, $q > 1$. By hypothesis, $k(1 - |a_{k+1}/a_k|) > q$ for k large. (Indeed, in either case, $q < p$ so this expression is eventually bigger than q.) The inequality implies $|a_{k+1}/a_k| < 1 - q/k$ for k large. Since $q > 1$, it follows from Raabe's Test that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

CHAPTER 7

7.1 Uniform Convergence of Sequences.

7.1.1. a) Given $\epsilon > 0$ choose N so large that $N > \max\{|a|,|b|\}/\epsilon$. Then $n \ge N$ and $x \in [a,b]$ imply $|x/n| \le \max\{|a|,|b|\}/N < \epsilon$. Hence $x/n \to 0$ uniformly on $[a,b]$.

b) Given $x \in (0,1)$, $nx \to \infty$, hence $1/(nx) \to 0$ as $n \to \infty$. If $\{1/(nx)\}$ were uniformly convergent, then there is an $N \in \mathbf{N}$ such that $|1/(Nx)| \le 1$ for all $x \in (0,1)$. Applying this inequality to $x = 1/(2N)$ we obtain $2 = 1/(N \cdot (1/(2N)) \le 1$, a contradiction.

7.1.2. a) Since $(3^{36}+3)/N \to 0$ as $N \to \infty$, given $\varepsilon > 0$, we can choose $N \in \mathbf{N}$ so that $0 < (3^{36}+3)/N < \varepsilon$. Since $x \in [1,3]$ implies $|3 - x^{36}| \le 3 + 3^{36}$ and $x^3 + nx^{66} \ge 0 + n = n$, it follows that

$$\left| \frac{nx^{99}+3}{x^3+nx^{66}} - x^{33} \right| = \frac{|3-x^{36}|}{x^3+nx^{66}} \le \frac{3+3^{36}}{n} \le \frac{3+3^{36}}{N} < \varepsilon$$

for all $x \in [1,3]$ and $n \ge N$. Hence $(nx^{99}+3)/(x^3+nx^{66}) \to x^{33}$ uniformly on $[1,3]$, so by Theorem 7.10,

$$\lim_{n\to\infty} \int_1^3 \frac{nx^{99}+3}{x^3+nx^{66}}\,dx = \int_1^3 x^{33}\,dx = \frac{3^{34}-1}{34}.$$

b) Since $e > 1$ implies $e^{4/N} > 1$ and $e^{4/N} \to 1$ as $N \to \infty$, given $\varepsilon > 0$, we can choose $N \in \mathbf{N}$ so that $0 < e^{4/N} - 1 < \varepsilon$. Since $x \in [0,2]$ implies $e^{x^2/n} \le e^{4/n}$, it follows that

$$|e^{x^2/n} - 1| = e^{x^2/n} - 1 \le e^{4/n} - 1 \le e^{4/N} - 1 < \varepsilon$$

for all $x \in [0,2]$ and $n \ge N$. Hence $e^{x^2/n} \to 1$ uniformly on $[0,2]$, so by Theorem 7.10,

$$\lim_{n\to\infty} \int_0^2 e^{x^2/n}\,dx = \int_0^2 dx = 2.$$

c) Let $x \in [0,3]$. Since $\sin(x/n) > 0$ for $n \ge 3$, we have $g(x) := \sqrt{\sin(x/n)+x+1} + \sqrt{x+1} > \sqrt{1} + \sqrt{1} = 2$ for $n \ge 3$. Given $\varepsilon > 0$, choose $N \in \mathbf{N}$ so that $N \ge 3$ and $2/N < \varepsilon$. Since $0 < \sin(x/n) \le x/n$, it follows by rationalizing the numerator that

$$\begin{aligned}|\sqrt{\sin(x/n)+x+1} - \sqrt{x+1}| &= \left| \frac{\sin(x/n)+x+1-(x+1)}{\sqrt{\sin(x/n)+x+1}+\sqrt{x+1}} \right| \\ &< \frac{x/n}{2} \le \frac{3}{2n} < \varepsilon\end{aligned}$$

for all $x \in [0,3]$ and $n \ge N$. Hence $\sqrt{\sin(x/n)+x+1} \to \sqrt{x+1}$ uniformly on $[0,3]$, so by Theorem 7.10,

$$\lim_{n\to\infty} \int_0^3 \sqrt{\sin(x/n)+x+1}\,dx = \int_0^3 \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{3/2} \Big|_0^3 = \frac{14}{3}.$$

7.1.3. Choose N so large that $|f(x) - f_n(x)| < 1$ for all $x \in E$ and $n \ge N$. Set $M := \sup_{x\in E} |f_N(x)|$ and observe by the Triangle Inequality that $|f(x)| \le |f(x) - f_N(x)| + |f_N(x)| < 1 + M$ for all $x \in E$. Therefore, $|f_n(x)| \le |f(x)| + 1 \le (1+M) + 1 = 2 + M$ for all $n \ge N$ and $x \in E$, i.e., $\{f_n\}_{n\ge N}$ is uniformly bounded on E. In particular,

$$|f_n(x)| \le \widetilde{M} := \max\{2+M, \sup_{x\in[a,b]} |f_1(x)|, \dots, \sup_{x\in[a,b]} |f_{N-1}(x)|\} < \infty$$

for all $n \in \mathbf{N}$ and $x \in E$.

7.1.4. Since g is continuous on $[a,b]$, it is bounded by the Extreme Value Theorem, i.e., there is a $C > 0$ such that $|g(x)| \le C$ for all $x \in [a,b]$. Since f is bounded and $\{f_n\}$ is uniformly bounded, there is an $M > 0$ such that $\max\{|f_n(x) - f(x)| : x \in [a,b], n \in \mathbf{N}\} \le M$. Given $\epsilon > 0$ choose $\delta > 0$ so small that $a < x < a + \delta$ or

$b > x > b - \delta$ implies $|g(x)| < \epsilon/M$. By hypothesis, $f_n \to f$ uniformly on $[a+\delta, b-\delta]$. Thus choose N so large that $x \in [a+\delta, b-\delta]$ and $n \geq N$ imply $|f_n(x) - f(x)| < \epsilon/C$. If $n \geq N$ and $x \in [a,b]$ then

$$|f_n(x)g(x) - f(x)g(x)| = |f_n(x) - f(x)|\,|g(x)| < \begin{cases} (\epsilon/C) \cdot C = \epsilon & x \in [a+\delta, b-\delta] \\ M \cdot (\epsilon/M) = \epsilon & x \notin [a+\delta, b-\delta]. \end{cases}$$

Therefore, $f_n g \to fg$ uniformly on $[a,b]$.

7.1.5. a) Given $\epsilon > 0$ choose N so large that $n \geq N$ and $x \in E$ imply $|f_n(x) - f(x)| < \epsilon/\max\{2, |\alpha|+1\}$ and $|g_n(x) - g(x)| < \epsilon/\max\{2, |\alpha|+1\}$. Then $n \geq N$ and $x \in E$ imply

$$|(f+g)(x) - (f_n + g_n)(x)| \leq |f(x) - f_n(x)| + |g(x) - g_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and

$$|(\alpha f)(x) - (\alpha f_n)(x)| = |\alpha|\,|f(x) - f_n(x)| < |\alpha| \frac{\epsilon}{|\alpha|+1} < \epsilon.$$

b) See Theorem 2.12.

c) Given $\epsilon > 0$ choose $M > 0$ so large that $\sup\{|f(x)|, |g(x)| : x \in E\} \leq M$. Choose N_1 so large that $n \geq N_1$ and $x \in E$ imply $|f_n(x) - f(x)| < \epsilon/(3M)$ and $|g_n(x) - g(x)| < \epsilon/(3M)$. Since $g_n \to g$ and g is bounded by M, choose N_2 so large that $|g_n(x)| \leq 2M$ for all $n \geq N_2$ and $x \in E$. If $n \geq N := \max\{N_1, N_2\}$ and $x \in E$ then

$$|(fg)(x) - (f_n g_n)(x)| \leq |f(x) - f_n(x)|\,|g_n(x)| + |f(x)|\,|g(x) - g_n(x)| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

d) Let $f_n(x) = 1/n$ and $g_n(x) = 1/x$. Then $f_n \to 0$ uniformly on $\mathbf{R}$ and $g_n(x) \to 1/x$ uniformly on $(0,\infty)$, in particular, on $(0,1)$, as $n \to \infty$. However, by Exercise 7.1.1b, $f_n(x)g_n(x) = 1/(nx)$ does not converges uniformly on $(0,1)$.

7.1.6. Given $\epsilon > 0$ choose $\delta > 0$ so small that $x, y \in E$ and $|x-y| < \delta$ imply $|f_N(x) - f_N(y)| < \epsilon/3$. If $x, y \in E$ and $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon.$$

Hence f is uniformly continuous on E.

7.1.7. Let $\varepsilon > 0$ and choose δ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $x \in \mathbf{R}$ and choose N such that $n \geq N$ implies $|y_n| < \delta$. If $n \geq N$, then $|x + y_n - x| = |y_n| < \delta$, so $|f_n(x) - f(x)| = |f(x+y_n) - f(x)| < \varepsilon$.

7.1.8. Choose N so large that $[a,b] \subset [-N,N]$. Let $x \in [a,b]$ and $n \geq N$. Then $x \geq -N \geq -n$, $x/n \geq -1$, and it follows from Bernoulli's Inequality that $(1+x/n)^n \uparrow e^x$ for $n \geq N$.

Let $n > N$, $x > 0$, and set $f(x) = e^x - (1+x/n)^n$. Then

$$f'(x) = e^x - \left(1 + \frac{x}{n}\right)^{n-1} \geq e^x - \left(1 + \frac{x}{n}\right)^n > 0$$

since $1 + x/n > 1$. Thus f takes its maximum on $[a,b]$ at $x = b$. Therefore,

$$\left| e^x - \left(1 + \frac{x}{n}\right)^n \right| \leq e^b - \left(1 + \frac{b}{n}\right)^n \to 0$$

as $n \to \infty$. It follows that $(1+x/n)^n \to e^x$ uniformly on $[a,b]$. In particular,

$$\lim_{n\to\infty} \int_a^b \left(1 + \frac{x}{n}\right)^n e^{-x}\, dx = \int_a^b dx = b - a.$$

7.1.9. a) By the Extreme Value Theorem, f is bounded on $[a,b]$ and there are positive numbers ϵ_0 and M such that $\epsilon_0 < |g(x)| < M$ for all $x \in [a,b]$. Hence $1/M < 1/|g(x)| < 1/\epsilon_0$ for all $x \in [a,b]$ and it follows that $1/g$ is bounded on $[a,b]$ and $1/(2M) < 1/|g_n(x)| < \epsilon_0$ for large n and all $x \in [a,b]$, i.e., $1/g_n$ is defined and bounded on $[a,b]$. Hence by Exercise 7.1.5c, $f_n/g_n = f_n \cdot (1/g_n) \to f \cdot (1/g) = f/g$ uniformly on $[a,b]$ as $n \to \infty$.

b) Let $f_n(x) = 1/n$ and $g_n(x) = x$. Then $f_n \to 0$ uniformly on $\mathbf{R}$, $|g_n| > 0$ for $x \neq 0$, and $g_n(x) \to x$ uniformly on $(0, \infty)$, in particular, on $(0, 1)$, as $n \to \infty$. However, by Exercise 7.1.1b, $f_n(x)/g_n(x) = 1/(nx)$ does not converges uniformly on $(0, 1)$.

7.1.10. Given $\epsilon > 0$ choose N_0 so large that $k \geq N_0$ and $x \in E$ imply $|f_k(x) - f(x)| < \epsilon/2$. Since $\sum_{k=1}^{N_0} |f_k(x) - f(x)|$ is bounded on E, choose N such that $(1/n) \sum_{k=1}^{N_0} |f_k(x) - f(x)| < \epsilon/2$ for all $n \geq N$ and $x \in E$. If $x \in E$ and $n \geq \max\{N_0, N\}$ then

$$\left| \frac{1}{n} \sum_{k=1}^{N} f_k(x) - f(x) \right| \leq \frac{1}{n} \sum_{k=1}^{N_0} |f_k(x) - f(x)| + \frac{\epsilon}{2} \left(1 - \frac{N_0}{n} \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

7.1.11. Since f is integrable, there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, 1]$. Choose $n_0 \in \mathbf{N}$ so that $1 - b_{n_0} \leq \epsilon/(2M)$ and $N > n_0$ so large that $|f_n(x) - f(x)| < \epsilon/2$ for $n \geq N$ and $x \in [0, 1]$. Suppose $n \geq N$ and $x \in [0, 1]$. Since the b_n's are increasing, $b_n \leq 1$ for all $n \in \mathbf{N}$ and $n \geq n_0$ imply that $1 - b_n \leq 1 - b_{n_0}$. Therefore,

$$\begin{aligned} \left| \int_0^1 f(x)\, dx - \int_0^{b_n} f_n(x)\, dx \right| &\leq \int_0^{b_n} |f(x) - f_n(x)|\, dx + \int_{b_n}^1 |f(x)|\, dx \\ &\leq \frac{\epsilon}{2} b_n + M(1 - b_n) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

7.2 Uniform Convergence of Series.

7.2.1. a) Since $|\sin(x/k^2)| \leq |x|/k^2 \leq \max\{|a|, |b|\}/k^2$ for any $x \in [a, b]$, this series converges uniformly on $[a, b]$ by the Weierstrass M–Test.
b) Let $I = [a, \infty) \subset (0, \infty)$. Then $x \in I$ implies $e^{-kx} \leq e^{-ka}$. Since this last series converges (it's Geometric with $r = e^{-a} < 1$), the original series converges uniformly on $[a, b]$ by the Weierstrass M–Test.

7.2.2. Clearly, $|x^k| \leq r^k$ for $x \in [a, b]$ and $r = \max\{|a|, |b|\}$. Since $[a, b] \subset (-1, 1)$ implies $r < 1$ and the geometric series $\sum_{k=0}^{\infty} r^k$ converges, it follows from the Weierstrass M–Test that the original series converges uniformly on $[a, b]$.

7.2.3. a) Since $|x^{k+1}/(k+1)!|/|x^k/k!| = |x|/(k+1) \to 0$ as $k \to \infty$, this series converges pointwise on $\mathbf{R}$ by the Ratio Test. Moreover, since $x \in [a, b]$ implies $|x|^k/k! \leq c^k/k!$, where $c := \max\{|a|, |b|\}$, it follows from the Weierstrass M–Test that the original series converges uniformly on $[a, b]$.
b) Integrating term by term, we have

$$\int_a^b E(x)\, dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} \Big|_a^b = E(b) - E(a).$$

c) Clearly, $E(0) = 1$. Differentiating term by term, we obtain $E'(x) = \sum_{k=0}^{\infty} x^k/k! = E(x)$. Thus $y = E(x)$ solves the initial value problem $y' - y = 0$, $y(0) = 1$.

7.2.4. The series converges uniformly on $\mathbf{R}$ by the Weierstrass M–Test. Hence integrating term by term, we obtain

$$\int_0^{\pi/2} f(x)\, dx = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\pi/2} \cos(kx)\, dx = \sum_{k=1}^{\infty} \frac{1}{k^3} \sin\left(\frac{k\pi}{2} \right).$$

Since $\sin(k\pi/2) = -1$ when $k = 3, 7, \ldots$, $\sin(k\pi/2) = 1$ when $k = 1, 5, \ldots$, and $\sin(k\pi/2) = 0$ when $k = 2, 4, \ldots$, it follows that

$$\int_0^{\pi/2} f(x)\, dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

7.2.5. Since $|\sin(x/(k+1))/k| \leq |x|/(k(k+1))$, the series converges uniformly on any closed bounded interval $[a, b]$ by the Weierstrass M–Test. In fact, for any $x \in \mathbf{R}$,

$$|f(x)| \leq \sum_{k=1}^{\infty} \left| \frac{\sin(x/(k+1))}{k} \right| \leq \sum_{k=1}^{\infty} \frac{|x|}{k(k+1)} = |x| \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = |x|.$$

Finally, the derived series $\sum_{k=1}^{\infty} \cos(x/(k+1))/(k(k+1))$ converges uniformly on $\mathbf{R}$ by the Weierstrass M–Test. Hence differentiating term by term, we obtain

$$|f'(x)| = \left| \sum_{k=1}^{\infty} \frac{\cos(x/(k+1))}{k(k+1)} \right| \le \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

7.2.6. The series $f(x) := \sum_{k=1}^{\infty} \sin(x/k)/k$ converges uniformly on $[0,1]$ by the Weierstrass M–Test. Hence integrating term by term,

$$\int_0^1 f(x)\,dx = \int_0^1 \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)\,dx = \sum_{k=1}^{\infty} \frac{1}{k}(-k \cos\left(\frac{x}{k}\right) \Big|_0^1 = \sum_{k=1}^{\infty} \left(1 - \cos\left(\frac{1}{k}\right)\right).$$

But $|f(x)| \le \sum_{k=1}^{\infty} 1/k^2 \le 1 + \sum_{k=2}^{\infty} 1/(k(k-1)) = 2$. Hence $|\sum_{k=1}^{\infty}(1-\cos(1/k))| \le \int_0^1 |f(x)|\,dx \le 2$.

7.2.7. Let $F_{n,m} := \sum_{k=m}^{n} f_k$. Choose $M > 0$ so large that $|g_1(x)| \le M$ for $x \in E$. Since $\{g_n\}$ is decreasing and nonnegative, it follows that $|g_n(x)| \le |g_1(x)| \le M$ for all $x \in E$.

Given $\epsilon > 0$ choose N so large that $|F_{k,m}(x)| < \epsilon/(3M)$ for $x \in E$ and $m, k \ge N$. Let $x \in E$ and $m, k \ge N$. By Abel's Formula,

$$\begin{aligned} \left| \sum_{k=m}^{n} f_k(x) g_k(x) \right| &\le |F_{n,m}(x)|\,|g_n(x)| + \sum_{k=m}^{n-1} |F_{k,m}(x)|\,(g_k(x) - g_{k+1}(x)) \\ &< \frac{\epsilon}{3M} \cdot M + \frac{\epsilon}{3M}(g_m(x) - g_n(x)) \le \frac{\epsilon}{3} + \frac{\epsilon}{3M}(2M) = \epsilon. \end{aligned}$$

7.2.8. a) Fix $n \ge 0$ and $x \in \mathbf{R}$. Since the absolute value of the ratio of consecutive terms of the series defining B_n is $|x/2|/((k+1)(n+k+1)) \to 0$ as $k \to \infty$, $B_n(x)$ converges by the Ratio Test. Moreover, by the Weierstrass M–Test, $B_n(x)$ converges uniformly on each closed bounded interval $[a,b]$.

b) Differentiating term by term,

$$B_n'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)}{2k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k-1}$$

and

$$B_n''(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)(n+2k-1)}{4k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k-2}.$$

(Both these series converge uniformly on $[a,b]$ by the argument in part a).) Therefore,

$$\begin{aligned} &x^2 B_n''(x) + x B_n'(x) - n^2 B_n(x) \\ &\quad = \sum_{k=0}^{\infty} \left(\frac{(-1)^k (n+2k)}{k!(n+k)!}\right) \left((n+2k-1) + 1 - \frac{n^2}{n+2k}\right) \left(\frac{x}{2}\right)^{n+2k} \\ &\quad = \sum_{k=0}^{\infty} \left(\frac{(-1)^k (n+2k)}{k!(n+k)!}\right) \left(\frac{4nk + 4k^2}{n+2k}\right) \left(\frac{x}{2}\right)^{n+2k} \\ &\quad = -x^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!(n+k-1)!} \left(\frac{x}{2}\right)^{n+2k-2} = -x^2 B_n(x). \end{aligned}$$

Thus $x^2 B_n''(x) + x B_n'(x) + (x^2 - n^2) B_n(x) = 0$ for all $x \in \mathbf{R}$.

c) By the Product Rule,

$$\begin{aligned} (x^n B_n(x))' &= x^n B_n'(x) + n x^{n-1} B_n(x) \\ &= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)}{2k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k-1} + x^n \sum_{k=0}^{\infty} \frac{(-1)^k n}{2k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k-1} \\ &= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k-1} = x^n B_{n-1}(x). \end{aligned}$$

7.2.9. By Example 6.32, $\widetilde{D}_n(x) := \sum_{k=1}^n \sin(kx) \le 1/|\sin(x/2)|$ for $x \in (0, 2\pi)$. Since $[a,b] \subset (0, 2\pi)$, it follows that $|\widetilde{D}_n(x)| \le 1/|\sin(c/2)|$ for $c = \max\{|a|, |b|\}$. In particular, the original series converges uniformly on $[a,b]$ by the Dirichlet Test.

7.2.10. Fix $x \in [a,b]$. Since

$$(*) \qquad f_n(x) = (f_n^n(x))^{1/n} \le \left(\sum_{k=1}^n f_k^n(x)\right)^{1/n} \le n^{1/n} f_n(x)$$

and $n^{1/n} \to 1$ as $n \to \infty$, it is clear by the Squeeze Theorem that $(\sum_{k=1}^n f_k^n(x))^{1/n}$ converges pointwise to $f(x)$ as $n \to \infty$. Is it uniform?

Let $\epsilon > 0$. Since f is a uniform limit of continuous functions, f is continuous on $[a,b]$. Hence, by the Extreme Value Theorem, there is an $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$. Since $f_n \to f$ uniformly and $n^{1/n} \to 1$, as $n \to \infty$, choose an N so large that $n \ge N$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{4} \quad \text{and} \quad |n^{1/n} - 1| < \frac{\epsilon}{2M}.$$

Notice by (*) that

$$f(x) - n^{1/n} f_n(x) \le f(x) - \left(\sum_{k=1}^n f_k^n(x)\right)^{1/n} \le f(x) - f_n(x),$$

i.e., that

$$\left| f(x) - \left(\sum_{k=1}^n f_k^n(x)\right)^{1/n} \right| \le \max\{|f(x) - f_n(x)|, |f(x) - n^{1/n} f(x)|\} =: \eta(x)$$

for all $x \in [a,b]$ and all $n \in \mathbf{N}$. But $n \ge N$ implies that $|f(x) - f_n(x)| < \epsilon/4 < \epsilon$ and

$$|f(x) - n^{1/n} f_n(x)| \le |f(x)|\,|1 - n^{1/n}| + n^{1/n}|f(x) - f_n(x)| < M\frac{\epsilon}{2M} + 2\frac{\epsilon}{4} = \epsilon$$

for all $x \in [a,b]$. (We have used the fact that $n^{1/n} \le 2$ for all $n \in \mathbf{N}$. If you don't want to verify this, note that since $n^{1/n} \to 1$, $n^{1/n}$ is surely ≤ 2 for large n.) Thus $\eta(x) < \epsilon$ for all $x \in [a,b]$. It follows that $(\sum_{k=1}^n f_k^n(x))^{1/n} \to f(x)$ uniformly on $[a,b]$ as $n \to \infty$. We conclude that the integrals of this sequence converges to the integral of $f(x)$ as required.

7.3 Power Series.

7.3.1. a) $R = 1$ since

$$\frac{|a_k|}{|a_{k+1}|} = \frac{k}{(2k+1)^2} \cdot \frac{(2k+2)^2}{k+1} \to 1.$$

b) Since

$$|a_j| = \begin{cases} 0 & j \text{ odd} \\ 3^j & j = 2k \text{ where } k \text{ is even} \\ 1^j & j = 2k \text{ where } k \text{ is odd} \end{cases}$$

it is clear that $\limsup_{j\to\infty} |a_j|^{1/j} = 3$, so $R = 1/3$.

c) Since

$$|a_j| = \begin{cases} 0 & j \ne k^2 \\ 3^{k^2} & j = k^2 \end{cases}$$

it is clear that $\limsup_{j\to\infty} |a_j|^{1/j} = 3$, so $R = 1/3$.

d) Since

$$|a_j| = \begin{cases} 0 & j \ne k^3 \\ k^{k^2} & j = k^3 \end{cases}$$

it is clear that $\limsup_{j\to\infty} |a_j|^{1/j} = \lim_{k\to\infty} k^{1/k} = 1$, so $R = 1$.

7.3.2. a) The radius of convergence is $R = \lim_{k\to\infty} \sqrt[k]{2^k} = 2$. If $x = \pm 2$ then $x^k/2^k = \pm 1$ and the series diverges by the Divergence Test. Therefore, the interval of convergence is $(-2, 2)$.

b) Since $\sqrt[k]{|(-1)^k + 3|^k} = 2$ when k is odd and 4 when k is even, the radius of convergence is $R = 1/4$. If $x = 3/4$ or $x = 5/4$ then the even terms of the series are 1 and the series diverges by the Divergence Test. Therefore, the interval of convergence is $(3/4, 5/4)$.

c) By Theorem 7.22 and L'Hôpital's Rule, the radius of convergence is $R = \lim_{k\to\infty} \log((k+1)/k)/\log((k+2)/(k+1)) = 1$. At $x = 1$, the series telescopes to $\lim_{n\to\infty} \log n = \infty$, hence diverges. At $x = -1$ the series alternates. Since $(\log((x+1)/x))' = -(x(x+1))^{-1} < 0$ for $x > 0$, the terms of this series decrease monotonically to 0. Hence by the Alternating Series Test, the series converges for $x = -1$. Therefore, the interval of convergence is $[-1, 1)$.

d) By Theorem 7.22, the radius of convergence is $R = \lim_{k\to\infty}(k+2)/(2k+1) = 1/2$, i.e., the series converges when $x^2 < 1/2$. At $x = \pm 1/\sqrt{2}$ the terms satisfy

$$\frac{a_{k+1}}{a_k} = \frac{2k+1}{2k+4} = 1 - \frac{3/2}{k+2}.$$

Hence the series converges at $x = \pm 1/\sqrt{2}$ by Raabe's Test. Therefore, the interval of convergence is $[-1/\sqrt{2}, 1/\sqrt{2}]$.

7.3.3. a) Since $|a_j| = |a_k|$ when $j = 2k$ and $|a_j| = 0$ when j is odd, we have $\limsup_{j\to\infty} |a_j|^{1/j} = = \limsup_{k\to\infty} \sqrt{|a_k|^{1/k}} = 1/\sqrt{R}$ so the radius of convergence is $\sqrt{R}$.

b) $\limsup_{k\to\infty} |a_k^2|^{1/k} = \limsup_{k\to\infty}(|a_k|^{1/k})^2 = 1/R^2$ so the radius of convergence is R^2.

7.3.4. Let R_a represent the radius of convergence of $\sum_{k=0}^\infty a_k x^k$ and R_b represent the radius of convergence of $\sum_{k=0}^\infty b_k x^k$. Since $|a_k|^{1/k} \le |b_k|^{1/k}$ for large k, it is clear by definition that $R_a \ge R_b$. Hence if $\sum_{k=0}^\infty b_k x^k$ converges on an open interval I then $I \subseteq (-R_b, R_b) \subseteq (-R_a, R_a)$. Thus $\sum_{k=0}^\infty a_k x^k$ converges on I.

Let $I = [-1, 1)$, $a_k = (-1)^k/k$, $b_k = 1/k$ for $k \in \mathbf{N}$. Then $|a_k| = |b_k|$ and $\sum_{k=1}^\infty b_k x^k$ converges on I. However, $\sum_{k=1}^\infty a_k x^k$ does not converge for $x = -1$, hence does not converge on I.

7.3.5. Since $|a_k|^{1/k} \le M^{1/k} \to 1$ as $k \to \infty$, the radius of convergence of f satisfies $R \ge 1$.

7.3.6. a) Let $f(x) = \sum_{k=0}^\infty a_k x^k$. By hypothesis and observation, $f(x)$ converges for $x = 0$ and $x = 1$. Hence by Theorem 7.21, $R \ge 1$ and $f(x)$ converges for all $x \in [0, 1]$. In particular, it follows from Abel's Theorem that $L := f(1) = \lim_{r\to 1-} f(r)$. Hence $\sum_{k=0}^\infty a_k$ is Abel summable to L.

b) The Geometric Series

$$\sum_{k=0}^\infty (-1)^k r^k = \sum_{k=0}^\infty (-r)^k = \frac{1}{1+r}$$

converges for any $0 < r < 1$. Thus $\sum_{k=0}^\infty (-1)^k$ is Abel summable to $1/2$.

7.3.7. a) For $|x| < 1$, $f(x) = (3/x)\sum_{k=1}^\infty (x^3)^k = (3/x)(x^3/(1-x^3)) = 3x^2/(1-x^3)$. It does not converge at $x = \pm 1$. Therefore, $f(x) = 3x^2/(1-x^3)$ for $x \in (-1, 1)$.

b) For $|x| < 1$, $xf(x) = \sum_{k=2}^\infty kx^{k-1}$ so

$$g(x) := \int_0^x tf(t)\,dt = \sum_{k=2}^\infty x^k = \frac{x^2}{1-x}.$$

Hence $xf(x) = g'(x) = (x^2/(1-x))' = (2x - x^2)/(1-x)^2$. The series does not converge at $x = \pm 1$. Therefore, $f(x) = (2-x)/(1-x)^2$ for $x \in (-1, 1)$.

c) For $|1-x| < 1$, $(1-x)f(x) = \sum_{k=1}^\infty (2k/(k+1))(1-x)^{k+1}$ so $((1-x)f(x))' = -2\sum_{k=1}^\infty k(1-x)^k$. Let $g(x) = \sum_{k=1}^\infty k(1-x)^k$. Then

$$\int_1^x \frac{g(t)}{1-t}\,dt = \sum_{k=1}^\infty k\int_1^x (1-t)^{k-1}\,dt = -\sum_{k=1}^\infty (1-x)^k = \frac{x-1}{x}.$$

(Note: $g(x)/(1-x)$ is defined at $x = 1$ and equals 1.) Thus $g(x)/(1-x) = (1-1/x)' = 1/x^2$, i.e., $g(x) = (1-x)/x^2$. Hence,

$$(1-x)f(x) = -2\int_1^x g(x)\,dx = 2\int_1^x \left(\frac{1}{t} - \frac{1}{t^2}\right)dt = 2(\log x + \frac{1}{x} - 1).$$

The series does not converge at $x = 0, 2$. Therefore, $f(x) = 2(\log x + 1/x - 1)/(1 - x)$ for $x \in (0, 2)$, $x \neq 1$, and $f(1) = 0$.

d) For $|x| < 1$, $x^3 f(x) = \sum_{k=0}^{\infty} x^{3(k+1)}/(k+1)$ so $(x^3 f(x))' = 3\sum_{k=0}^{\infty} x^{3k+2} = 3x^2/(1 - x^3)$. Hence

$$x^3 f(x) = 3\int_0^x \frac{t^2}{1 - t^3}\, dt = -\int_1^{1-x^3} \frac{du}{u} = -\log|1 - x^3|.$$

The series converges at $x = -1$. Therefore, $f(x) = \log(1/(1 - x^3))/x^3$ for $x \in [-1, 1)$, $x \neq 0$, and $f(0) = 1$.

7.3.8. Suppose $\limsup |a_k/a_{k+1}| < R$. By Exercise 2.5.8 $\liminf |a_{k+1}/a_k| > 1/r$ for some $r < R$. Hence it follows from definition that $|a_{k+1}/a_k| > 1/r = r^k/r^{k+1}$ for k large, say $k \geq N$. In particular, $k \geq N$ implies $|a_k r^k| \geq |a_{k-1} r^{k-1}| \geq \cdots \geq |a_N r^N| > 0$. Therefore, $\sum_{k=0}^{\infty} a_k r^k$ diverges by the Divergence Test, which contradicts the fact that $r < R$ and R is the radius of convergence. On the other hand, if $\liminf |a_k/a_{k+1}| > R$ then $\limsup |a_{k+1}/a_k| < 1/r$ for some $r > R$, i.e., $|a_{k+1}/a_k| < 1/r = r^k/r^{k+1}$ for k large. Hence $a_k r^k$ is eventually decreasing, in particular, bounded above, say by M. If $R < r_0 < r$ then it follows that $a_k r_0^k \leq M(r_0/r)^k$ for k large. Since the geometric series $\sum_{k=0}^{\infty}(r_0/r)^k < \infty$, it follows from the Comparison Test that $\sum_{k=0}^{\infty} a_k r_0^k$ converges, which contradicts the fact that $r_0 > R$ and R is the radius of convergence.

7.3.9. The coefficients of this power series are given by $a_k = ((-1)^k + 4)^{-k}$. Since $\sqrt[k]{a_k} = 1/3$ if k is odd and $1/5$ if k is even, the radius of convergence of this series is $R = 3$. Thus the series converges uniformly on any closed subinterval of $(-3, 3)$. Differentiating term by term, we obtain

$$f'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{((-1)^k + 4)^k} \leq \sum_{k=1}^{\infty} \frac{kx^{k-1}}{3^k} =: g(x)$$

for $0 \leq x < 3$. Now $\int_0^x g(t)\, dt = \sum_{k=1}^{\infty} x^k/3^k = x/(3 - x)$ by Theorem 6.7, so $g(x) = (x/(3 - x))' = 3/(3 - x)^2$. Hence $|f'(x)| \leq 3/(3 - x)^2$ for $0 \leq x < 3$.

7.3.10. Since $a_k \downarrow 0$ as $k \to \infty$, the radius of convergence of the power series

$$f(x) := \sum_{k=0}^{\infty} (-1)^k a_k x^k$$

is greater than or equal to 1 (see the proof of Exercise 7.3.5), i.e., $f(x)$ converges for all $x \in [0, 1)$. By the Alternating Series Test, $\sum_{k=0}^{\infty}(-1)^k a_k$ converges, so $f(1)$ converges. Hence by Abel's Theorem, f is uniformly continuous on $[0, 1]$. In particular, given $\epsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ and $x, y \in [0, 1]$ imply

$$\left|\sum_{k=0}^{\infty} (-1)^k a_k (x^k - y^k)\right| = |f(x) - f(y)| < \epsilon.$$

7.3.11. a) Since

$$\sum_{k=1}^{n} \log k = \log(n!) \quad \text{and} \quad \int_1^n \log x = n \log n - n + 1,$$

we have by Theorem 6.35 that

$$0 \leq \log(n!) - n \log n + n - 1 \leq \log n.$$

Exponentiating this inequality, we have

$$1 \leq \frac{n!}{n^n} e^{n-1} \leq n.$$

Therefore,

$$n^n \leq n!\, e^{n-1} \leq n^{n+1}$$

for all $n \in \mathbf{N}$.

b) The radius of convergence of this power series is $1/e$. The series diverges at $x = 1/e$ because its terms satisfy $n^n/(n! e^n) \geq 1/(ne)$ by part a). It converges at $x = -1/e$ by the Alternating Series Test but evidently does not converge absolutely. Thus the series converges absolutely on $(-1/e, 1/e)$.

7.4 Analytic Functions.

7.4.1. a) By Example 7.44, $\cos x = \sum_{k=0}^{\infty}(-1)^k x^{2k}/(2k)!$ for $x \in \mathbf{R}$. Substituting $2x$ for x, we have $\cos(2x) = \sum_{k=0}^{\infty}(-4)^k x^{2k}/(2k)!$ for $x \in \mathbf{R}$. Thus

$$x^2 + \cos(2x) = 1 - x^2 + \sum_{k=2}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!}$$

for $x \in \mathbf{R}$.

b) By Example 7.45, $3^x = e^{x \log 3} = \sum_{k=0}^{\infty} x^k \log^k 3/k!$ for all $x \in \mathbf{R}$. Thus by Theorem 7.33

$$x^2 3^x = \sum_{k=2}^{\infty} \frac{x^k \log^{k-2} 3}{(k-2)!}$$

for $x \in \mathbf{R}$.

c) Since $\cos^2 x - \sin^2 x = \cos(2x)$, it follows from Example 7.44 that

$$\cos^2 x - \sin^2 x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!}$$

for $x \in \mathbf{R}$.

d) By Example 7.45, $e^x - 1 = \sum_{k=1}^{\infty} x^k/k! = x \sum_{k=0}^{\infty} x^k/(k+1)!$. Hence $(e^x - 1)/x = \sum_{k=0}^{\infty} x^k/(k+1)!$ for $x \in \mathbf{R}$.

7.4.2. a) For $|x| < 1$ we have by the Geometric series and Theorem 7.33 that

$$\frac{x}{1+x^5} = x \sum_{k=0}^{\infty} (-x)^5 = \sum_{k=0}^{\infty} (-1)^k x^{5k+1}.$$

b) By Theorem 7.33 and Example 7.45,

$$\frac{e^x}{1+x} = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} (-1)^k x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!}\right) x^k$$

for $|x| < 1$.

c) Since $|x| < 1$ implies $t = |x^2 - 1| = 1 - x^2 \in (0,1)$ and $(-1)^{k+1}(-1)^k = -1$, we have by Example 7.49

$$\log(|x^2 - 1|^{-1}) = -\log(1 - x^2) = -\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-x^2)^k}{k} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k}$$

for $|x| < 1$.

d) Integrating the binomial series term by term,

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \sum_{k=0}^{\infty} \int_0^x \binom{-1/2}{k} (-1)^k t^{2k}\, dt = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k \frac{x^{2k+1}}{2k+1}$$

for all $|x| < 1$.

7.4.3. a) Since $f(x) = e^x$ implies $f^{(k)}(x) = e^x$ for all $k \geq 1$, the Taylor expansion of e^x at $x = 1$ is $e^x = \sum_{k=0}^{\infty} e(x-1)^k/k!$ valid on $\mathbf{R}$ by Theorem 7.43.

b) By Example 7.49, $\log_2 x^5 = 5 \log x/\log 2 = 5 \sum_{k=1}^{\infty} (-1)^{k+1}(x-1)^k/(k \log 2)$ for $0 < x < 2$. The series converges at $x = 2$ by the alternating series test but diverges at $x = 0$. Hence this expansion is valid on $(0, 2]$.

c) If $f(x) = x^3 - x + 5$ then $f'(x) = 3x^2 - 1$, $f''(x) = 6x$, $f^{(3)}(x) = 6$, and $f^{(k)}(x) = 0$ for all $k \geq 4$. Thus $f(1) = 5$, $f'(1) = 2$, $f''(1) = 6$, $f^{(3)}(1) = 6$, and $f^{(k)}(1) = 0$ for all $k \geq 3$. Hence f is analytic on every bounded interval by Theorem 7.43 and $x^3 - x + 5 = 5 + 2(x-1) + 3(x-1)^2 + (x-1)^3$. In particular, this expansion is valid on $\mathbf{R}$.

d) Since $f(1) = 1$, $f'(1) = 1/2$, $f''(1) = -1/2^2$ and $f^{(n)}(1) = (-1)^{n-1} 1 \cdot 3 \cdots (2n-3)/2^n$ for $n \ge 2$, we have

$$\sqrt{x} = 1 + \frac{x-1}{2} + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} 1 \cdot 3 \cdots (2k-3)}{2 \cdot 4 \cdots (2k)} (x-1)^k.$$

The radius of convergence is $R = 1$, i.e., the endpoints of the interval of convergence are 0 and 2. Since the ratio of successive coefficients is $(2k-1)(2k+2) = 1 - (3/2)/(k+1)$, it follows from Raabe's Test that the series converges absolutely at both endpoints. Thus the expansion is valid for $x \in [0, 2]$.

7.4.4. Since $P^{(k)}(x) = 0$ for $k > n$ and $x \in \mathbf{R}$, the Taylor series truncates. Thus set $\beta_k = P^{(k)}(x_0)/k!$.

7.4.5. We begin with a general observation. If f is even and differentiable, then $f'(0) = 0$. Indeed,

$$f'(0) = \lim_{h \to 0-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0+} \frac{f(-h) - f(0)}{-h} = -\lim_{h \to 0+} \frac{f(h) - f(0)}{h} = -f'(0).$$

a) Suppose that f is odd. By iterating Exercise 4.1.9, we see that all derivatives of f of even order are even functions. Hence, by our opening observation, $f^{(2k)}(0) = 0$ for all $k \in \mathbf{N}$. Thus the Taylor series contains only odd terms.

b) Suppose that f is even. By iterating Exercise 4.1.9, we see that all derivatives of f of odd order are even functions. Hence, by our opening observation, $f^{(2k-1)}(0) = 0$ for all $k \in \mathbf{N}$. Thus the Taylor series contains only even terms.

7.4.6. Using the substitution $u = a - x$, $du = -dx$, we have

$$\int_0^a x^n f^{(n+1)}(a-x)\, dx = \int_a^0 (a-u)^n f^{(n+1)}(u)\, du.$$

Thus by Theorem 7.50 (with $x_0 = 0$), $R_n^{f,0}(a) \to 0$ as $n \to \infty$, for all $a \in \mathbf{R}$. Hence by definition, $f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) x^k/k!$ for $x \in \mathbf{R}$.

7.4.7. a) Fix $x \in [-1, 1]$. By Theorem 4.24, $e^{x^2} = \sum_{k=0}^{n-1} x^{2k}/k! + e^c x^n/n!$ for some c between 0 and x. Since $|e^c x^n/n!| \le 3/n!$, it follows that $|e^{x^2} - \sum_{k=0}^{n-1} x^{2k}/k!| \le 3/n!$ for all $x \in [-1, 1]$. Therefore,

$$\left| \int_0^1 e^{x^2} - \sum_{k=0}^{n-1} \int_0^1 \frac{x^{2k}}{k!} \right| \le \frac{3}{n!}.$$

Since $\int_0^1 x^{2k}/k!\, dx = 1/((2k+1)k!)$, this completes the proof of part a).

b) Notice that $3/n! < 10^{-5}$ holds when $n \ge 9$. Since $\sum_{k=0}^{9} 1/((2k+1)k!) = 1.4626713$, it follows from part a) that $1.4626613 < \int_0^1 e^{x^2}\, dx < 1.4626813$. By symmetry, then, $2.9253226 = 2 \cdot (1.4626613) < \int_{-1}^1 e^{x^2}\, dx < 2.9253626$.

7.4.8. If f is analytic on (a, b) then given $x_0 \in (a, b)$, $f(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0)(x - x_0)^k/k!$ has a positive radius of convergence. Hence by Theorem 7.30, $f'(x) = \sum_{k=1}^{\infty} f^{(k)}(x_0)(x - x_0)^{k-1}/(k-1)!$ also has a positive radius of convergence. Thus $f'(x)$ is analytic on (a, b). The converse follows similarly integrating term by term.

7.4.9. Modifying the proof of Theorem 7.43, we see that f is analytic. Thus the Taylor polynomials f_n of f converge to f uniformly on $[a, b] \subset I$. Let $C > 0$ satisfy $|f(x)| \le C$ for $x \in I$. Given $\varepsilon > 0$, choose N such that $n \ge N$ implies $|f_n(x) - f(x)| < \varepsilon/C$ for $x \in [a, b]$. Then $n \ge N$ implies $|(f \cdot f_n)(x) - f^2(x)| = |f(x)|\,|f_n(x) - f(x)| < \varepsilon$ for $x \in [a, b]$, i.e., $f \cdot f_n \to f^2$ uniformly on $[a, b]$ as $n \to \infty$. But by hypothesis,

$$\int_a^b f(x) \cdot f_n(x)\, dx = \sum_{k=0}^{n} a_k \int_a^b f(x) x^k\, dx = 0$$

for all $n \in \mathbf{N}$. Therefore, $\int_a^b f^2(x)\, dx = 0$. Since $f^2 \ge 0$, we conclude that f^2, hence f, is identically zero on $[a, b]$.

7.4.10. Since f is continuous, the hypothesis implies $|f(x)| = 0$ for all $x \in (a, b)$, i.e., $f = 0$ on (a, b). But 0 is analytic on $(-\infty, \infty)$, so by Theorem 7.56, $f = 0$ on $\mathbf{R}$.

7.4.11. Choose $k \in \mathbf{N}$ such that $k \le \beta < k+1$ and let $x \in (0, 1)$. Since $\binom{\beta}{k} > 1$ and $x^k > x^\beta$, it follows from the Binomial Series expansion that

$$(1+x)^\beta > 1 + \binom{\beta}{k} x^k > 1 + x^\beta.$$

Let $a, b \in \mathbf{R}$. If $a = 0$, $b = 0$, or $|a| = |b|$, then

$$(*) \qquad (|a| + |b|)^\beta \ge |a|^\beta + |b|^\beta$$

obviously holds. If $a \ne 0 \ne b$, then we may suppose $|a| < |b|$. Applying the above inequality to $x = |a|/|b|$ verifies (*) in this case as well. Thus (*) holds for all $a, b \in \mathbf{R}$. By induction,

$$\sum_{k=1}^{n} |a_k|^\beta \le \left(\sum_{k=1}^{n} |a_k| \right)^\beta$$

for $n \in \mathbf{N}$. Taking the limit of this inequality as $n \to \infty$ establishes the given inequality.

7.5 Applications.

7.5.1. Let $f(x) = x^3 + 3x^2 + 4x + 1$. Since $f'(x) = 3x^2 + 6x + 4$, f is increasing and has only one real root. By Newton's method,

$$x_n = x_{n-1} - \frac{x_{n-1}^3 + 3x_{n-1}^2 + 4x_{n-1} + 1}{3x_{n-1}^2 + 6x_{n-1} + 4} = \frac{2x_{n-1}^3 + 3x_{n-1}^2 + 1}{3x_{n-1}^2 + 6x_{n-1} + 4}.$$

Using an initial guess of $x_0 = 0$, we obtain $x_1 = -.25$, $x_2 = -.313953\ldots$, $x_3 = -.31766\ldots$, $x_4 = -.3176721\ldots$, $x_5 = -.3176721\ldots$.

7.5.2. b) Let $a = 3$, $b = 4$, and $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$. Therefore, we can set $M = 0.75$ and $\epsilon_0 = 0.65364$. Then $\epsilon_0/M = 0.8636912$, $r_0 = .15$ and $r = 0.1736732$. If $x_0 = 3$ is the initial guess, then $|x_0 - \pi| < .15$ and by the proof of Theorem 7.58, $|x_n - \pi| \le (0.1736732)^n(.15)$, i.e., $|x_4 - \pi| < 0.000136465$. Using a calculator, we see that $x_1 = 3.142546543$, $x_2 = 3.141592653$, $x_3 = 3.141592654$, and $x_4 = 3.141592654$. My calculator will not show more than nine places, so I cannot tell how many more digits we picked up going from x_3 to x_4. Nevertheless, it is clear that $|x_4 - \pi| < 0.0000000005$ which is much smaller than 0.000136465.

7.5.3. The proof is by induction on n. It is true for $n = 0$ by Weierstrass' Theorem. Suppose there is a function $g \in \mathcal{C}^{n-1}$ such that $g^{(n)}$ exists nowhere on $\mathbf{R}$. Set $f(x) = \int_0^x g(t)\, dt$. Then $f^{(n)}(x) = g^{(n-1)}(x)$ is continuous but $f^{(n+1)} = g^{(n)}$ exists nowhere on $\mathbf{R}$. Hence by induction, this result holds for all $n \in \mathbf{N}$.

7.5.4. The line tangent to $y = f(x)$ at $(x_{n-1}, f(x_{n-1}))$ has equation $y = f(x_{n-1}) + f'(x_{n-1})(x - x_{n-1})$. To find its x-intercept, set $y = 0$. Solving for x, we obtain $x = x_{n-1} - f(x_{n-1})/f'(x_{n-1})$ as promised.

7.5.5. Suppose there exist $q, p \in \mathbf{N}$ such that

$$\frac{q}{p} = \cos(1) = \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k)!} = \sum_{k=0}^{p} \frac{(-1)^k}{(2k)!} + \sum_{k=p+1}^{\infty} \frac{(-1)^k}{(2k)!}.$$

Then

$$(*) \qquad (-1)^{p+1}(2p)!q/p - (-1)^{p+1}(2p)! \sum_{k=0}^{p} (-1)^k/(2k)! = \sum_{k=p+1}^{\infty} (-1)^{k+p+1}(2p)!/(2k)!.$$

Now $(2p)!/(2k)! \in \mathbf{N}$ for each $0 \le k \le p$, so the left side of (*) is an integer. On the other hand, the right side lies between $1/((2p+1)(2p+2))$ and $1/((2p+1)(2p+2)) - 1/((2p+1)(2p+2)(2p+3)(2p+4))$, i.e., is a number between 0 and 1. This contradiction proves that $\cos(1)$ is irrational.

7.5.6. Let $|f''(x)| \le M$ and choose $r_0 < \epsilon_0/M$. Let $\delta > 0$ be so small that $\delta/\epsilon_0 < r^2$ and suppose $|f(x_0)| \le \delta$. We claim that $|x_n - x_{n-1}| < r_0^{n+1}$ for $n \in \mathbf{N}$. Note by (19) that

$$|x_1 - x_0| = \left| \frac{f(x_0)}{f'(x_0)} \right| \le \frac{\delta}{\epsilon_0} < r^2.$$

Hence the claim holds for $n = 1$. If it holds for some $n \in \mathbf{N}$ then by Taylor's Formula,

$$\begin{aligned}|f(x_n)| &= |f(x_n) - f(x_{n-1}) + f(x_{n-1})| \\ &= |f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1})| \le M|x_n - x_{n-1}|^2.\end{aligned}$$

Therefore, $|x_{n+1} - x_n| = |f(x_n)/f(x_n)| \le M|x_n - x_{n-1}|^2/\epsilon_0 < r_0^{2n+1} \le r_0^{n+2}$. Thus the claim holds for all $n \in \mathbf{N}$.

Now $|x_{n+1} - x_m| \le |x_{n+1} - x_n| + \cdots + |x_{m+1} - x_m| \le \sum_{k=m+1}^{\infty} r^{k+1} = r^{m+2}/(1-r)$. Therefore, $\{x_n\}$ is Cauchy, so converges to some $c \in \mathbf{R}$. Taking the limit of (19) as $n \to \infty$, we obtain $c = c - f(c)/f'(c)$, i.e., $f(c) = 0$.

7.5.7. a) Since $f(\beta_n) - f(\alpha_n) = f(\beta_n) - f(x) + f(x) - f(\alpha_n)$, and

$$1 = \frac{\beta_n - x}{\beta_n - \alpha_n} + \frac{x - \alpha_n}{\beta_n - \alpha_n},$$

We have

$$\begin{aligned}\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - \gamma = \left(\frac{f(\beta_n) - f(x)}{\beta_n - x} - \gamma\right)\left(\frac{\beta_n - x}{\beta_n - \alpha_n}\right) + \\ + \left(\frac{f(x) - f(\alpha_n)}{x - \alpha_n} - \gamma\right)\left(\frac{x - \alpha_n}{\beta_n - \alpha_n}\right).\end{aligned}$$

b) Let $\gamma = f'(x)$. Then both terms on the right side of part a) which end in $-\gamma$ converge to zero as $n \to \infty$. Since $x \in [\alpha_n, \beta_n)$, it is also clear that

$$\frac{\beta_n - x}{\beta_n - \alpha_n} \le 1 \text{ and } \frac{x - \alpha_n}{\beta_n - \alpha_n} \le 1.$$

Hence, by the Squeeze Theorem, the right side of part a) converges to $0 + 0 = 0$. We conclude that

$$\lim_{n \to \infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \gamma = f'(x).$$

SOLUTIONS TO EXERCISES

Chapter 8

8.1 Algebraic Structure.

8.1.1 a) $\|\mathbf{x}-\mathbf{y}\| \le \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{y}\| < 2+3 = 5$.
b) By vector algebra and the Cauchy-Schwarz inequality, $|\mathbf{x}\cdot\mathbf{y}-\mathbf{x}\cdot\mathbf{z}| = |\mathbf{x}\cdot(\mathbf{y}-\mathbf{z})| \le \|\mathbf{x}\|\,\|\mathbf{y}-\mathbf{z}\| < 2\cdot(3+4) = 14$.
c) By vector algebra and Cauchy-Schwarz, $|\mathbf{x}\cdot(\mathbf{y}-\mathbf{z}) - \mathbf{y}\cdot(\mathbf{x}-\mathbf{z})| = |(\mathbf{y}-\mathbf{x})\cdot\mathbf{z}| \le \|\mathbf{x}-\mathbf{y}\|\,\|\mathbf{z}\| < 2\cdot 3 = 6$.
d) By vector algebra and Cauchy-Schwarz, $|\,\|\mathbf{x}-\mathbf{y}\|^2 - \mathbf{x}\cdot\mathbf{x}| = |-2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}| = |(-2\mathbf{x}+\mathbf{y})\cdot\mathbf{y}| \le 2\cdot 1 = 2$.
e) By Theorem 8.9 and Remark 8.10, $\|\mathbf{x}\times\mathbf{z}-\mathbf{y}\times\mathbf{z}\| = \|(\mathbf{x}-\mathbf{y})\times\mathbf{z}\| \le 2\cdot 3 = 6$.
f) By Cauchy-Schwarz and Remark 8.10, $|\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})| \le \|\mathbf{x}\|\,\|\mathbf{y}\|\,\|\mathbf{z}\| < 1\cdot 2\cdot 3 = 6$.

8.1.2. a) By Cauchy-Schwarz, $\|3\mathbf{v}\| \le |\mathbf{a}\cdot\mathbf{b}|\,\|\mathbf{c}\| + |\mathbf{a}\cdot\mathbf{c}|\,\|\mathbf{b}\| + |\mathbf{c}\cdot\mathbf{b}|\,\|\mathbf{a}\| \le 3\|\mathbf{a}\|\,\|\mathbf{b}\|\,\|\mathbf{c}\| \le 3$, so $\|\mathbf{v}\| \le 3/3 = 1$.
b) By Cauchy-Schwarz, $|\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{d}| = |\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{d}| \le \|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{c}\| + \|\mathbf{b}\|\,\|\mathbf{a}-\mathbf{d}\| \le \|\mathbf{b}-\mathbf{c}\| + \|\mathbf{a}-\mathbf{d}\|$.
c) By Theorem 8.9 and Cauchy-Schwarz, $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|^2 = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|^2 \le \|\mathbf{a}\times\mathbf{b}\|^2\,\|\mathbf{c}\|^2 = (\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2)\,\|\mathbf{c}\|^2 \le (1-|\mathbf{a}\cdot\mathbf{b}|^2)\,\|\mathbf{c}\|^2 \le 1-|\mathbf{a}\cdot\mathbf{b}|^2$. Thus $\sqrt{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|^2+|\mathbf{a}\cdot\mathbf{b}|^2} \le 1$.

8.1.3. It is clear that equality holds if either $\mathbf{x}=0$ or $\mathbf{y}=0$. Are there any others?
Suppose that neither $\mathbf{x}$ nor $\mathbf{y}$ is zero. By the proof of Theorem 8.5, the only place an inequality slipped in is on the left side of (3); all other steps in the proof were identities. Thus we get equality in the Cauchy-Schwarz inequality if and only if $\|\mathbf{x}-t\mathbf{y}\| = 0$, i.e., if and only if $\mathbf{x} = t\mathbf{y}$. But this is exactly what it means for $\mathbf{x}$ to be parallel to $\mathbf{y}$.

8.1.4. a) Let θ be the angle between $\phi(t_1)-\phi(t_0)$ and $\phi(t_2)-\phi(t_0)$. Since $\phi(t_1)-\phi(t_0) = (t_1-t_0)\mathbf{b}$ and $\phi(t_2)-\phi(t_0) = (t_2-t_0)\mathbf{b}$, it follows that

$$\cos\theta = \frac{(t_1-t_0)(t_2-t_0)\|\mathbf{b}\|^2}{|t_1-t_0|\,|t_2-t_0|\,\|\mathbf{b}\|^2} = \pm 1.$$

Thus θ is 0 or π.
b) By definition, $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a}=t\mathbf{b}$, i.e., $|\mathbf{a}\cdot\mathbf{b}| = |t|\,\|\mathbf{a}\|^2 = \|\mathbf{a}\|\,\|\mathbf{b}\|$. In view of (2), this happens if and only if $|\cos\theta| = 1$, i.e., if and only if θ is 0 or π,
On the other hand, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b}=0$. By (2), this occurs if and only if the angle between them is $\pi/2$.

8.1.5. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ denote the vertices of Δ, and C be the line segment between $\mathbf{a}$ and $\mathbf{b}$. Since L is the line segment between $(\mathbf{a}+\mathbf{c})/2$ and $(\mathbf{b}+\mathbf{c})/2$, L has direction

$$\mathbf{v} := \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{\mathbf{b}+\mathbf{c}}{2} = \frac{\mathbf{a}-\mathbf{b}}{2}$$

and length $\|\mathbf{v}\|$. Since $\mathbf{v}$ is parallel to $\mathbf{b}-\mathbf{a}$, L is parallel to C. Finally, the length of L equals $\|\mathbf{a}-\mathbf{b}\|/2$, which is exactly half the length of C.

8.1.6. a) $(4,5,6)-(1,2,3) = (3,3,3)$, $(0,4,2)-(1,2,3) = (-1,2,-1)$, and $(3,3,3)\cdot(-1,2,-1) = 0$, so the sides of this triangle emanating from $(1,2,3)$ are orthogonal.
b) Let (a,b,c) be a nonzero vector in the plane $z=x$ orthogonal to $(1,-1,0)$. Then $a=c$ and $0 = (a,b,c)\cdot(1,-1,0) = a-b$, i.e., such a vector must have the form (a,a,a), $a\ne 0$.
c) Let (a,b,c) be a nonzero vector orthogonal to $(3,2,-5)$, i.e., $0 = (a,b,c)\cdot(3,2,-5) = 3a+2b-5c$. If $a+b+c=4$ then $b = (20-8a)/7$, $c = (8+a)/7$. Thus such a vector has the form $(a,(20-8a)/7,(8+a)/7)$, $a\ne 0$.

8.1.7. By symmetry, we may use any side of Q. The longest diagonal of this cube is $\mathbf{x} := (b,b,\dots,b)-(a,a,\dots,a) = (b-a,\dots,b-a)$. If θ is the angle between this longest side and the "first" side of Q, $\mathbf{y} := (a-b,0,\dots,0)$, then by definition,

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{(b-a)^2}{\sqrt{n}(b-a)^2}.$$

Thus $\theta = \arccos(1/\sqrt{n})$. For $n = 3$, this is about 54.74 degrees.

8.1.8. a) The Associative Property is proved in the text. By the Commutative Property of real numbers, $\mathbf{x}+\mathbf{y} = (x_1,\dots,x_n)+(y_1,\dots,y_n) = (x_1+y_1,\dots,x_n+y_n) = (y_1+x_1,\dots,y_n+x_n) = \mathbf{y}+\mathbf{x}$ and $\mathbf{x}\cdot\mathbf{y} = x_1\cdot y_1+\cdots+x_n\cdot y_n = y_1\cdot x_1+\cdots+y_n\cdot x_n = \mathbf{y}\cdot\mathbf{x}$.

By the Distributive and Commutative Properties of real numbers, $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = (x_1,\dots,x_n)\cdot(y_1+z_1,\dots,y_n+z_n) = x_1(y_1+z_1)+\cdots+x_n(y_n+z_n) = x_1y_1+\cdots+x_ny_n+x_1z_1+\cdots+x_nz_n = \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}$. The rest of these identities follow in a similar way from corresponding properties of real numbers.

b) By definition, $\mathbf{x}\times\mathbf{x} = (x_2x_3-x_2x_3, x_1x_3-x_1x_3, x_1x_2-x_1x_2) = 0$, and $\mathbf{x}\times\mathbf{y} = (x_2y_3-x_3y_2, x_3y_1-x_1y_3, x_1y_2-x_2y_1) = -(-x_2y_3+x_3y_2, -x_3y_1+x_1y_3, -x_1y_2+x_2y_1) = -\mathbf{y}\times\mathbf{x}$. This proves i)

$(\alpha\mathbf{x})\times\mathbf{y} = (\alpha x_2y_3-\alpha x_3y_2, \alpha x_3y_1-\alpha x_1y_3, \alpha x_1y_2-\alpha x_2y_1) = \alpha(\mathbf{x}\times\mathbf{y}) = (x_2(\alpha y_3)-x_3(\alpha y_2), x_3(\alpha y_1)-x_1(\alpha y_3), x_1(\alpha y_2)-x_2(\alpha y_1)) = \mathbf{x}\times(\alpha\mathbf{y})$, so ii) holds.

By parts iv) and v), $\|\mathbf{x}\times\mathbf{y}\|^2 = (\mathbf{x}\times\mathbf{y})\cdot(\mathbf{x}\times\mathbf{y}) = \mathbf{x}\cdot(\mathbf{y}\times(\mathbf{x}\times\mathbf{y})) = \mathbf{x}\cdot((\mathbf{y}\cdot\mathbf{y})\mathbf{x}-(\mathbf{y}\cdot\mathbf{x})\mathbf{y}) = (\mathbf{y}\cdot\mathbf{y})(\mathbf{x}\cdot\mathbf{x})-(\mathbf{x}\cdot\mathbf{y})^2$. Thus vi) holds

c) $\|\mathbf{x}\times\mathbf{y}\| = \sin\theta\|\mathbf{x}\|\,\|\mathbf{y}\| \le 1\cdot\|\mathbf{x}\|\,\|\mathbf{y}\|$.

8.1.9. By the Cauchy–Schwarz Inequality,

$$x_n := \sum_{k=0}^n |a_kb_k| \equiv (|a_1|,\dots,|a_n|)\cdot(|b_1|,\dots,|b_n|) \le \left(\sum_{k=0}^n |a_k|^2\right)^{1/2}\left(\sum_{k=0}^n |b_k|^2\right)^{1/2}$$

for all $n\in\mathbf{N}$. Notice that $|a_k|^2 = a_k^2$ and the sequence x_n is monotone increasing. Hence it follows from the Monotone Convergence Theorem and hypothesis that x_n converges to a finite real number as $n\to\infty$, i.e., $\sum_{k=1}^\infty a_kb_k$ is absolutely convergent.

8.1.10. Homogeneity and positive definiteness are obvious. To show the triangle inequality holds, let $\mathbf{a} = (a_1,\dots,n)$ and $\mathbf{b} = (b_1,\dots,b_n)$. Then

$$\|\mathbf{a}+\mathbf{b}\|_1 = \sum_{k=1}^n |a_k+b_k| \le \sum_{k=1}^n |a_k|+|b_k| = \|\mathbf{a}\|_1+\|\mathbf{b}\|_1,$$

and

$$\|\mathbf{a}+\mathbf{b}\|_\infty = \sup_{1\le k\le n}|a_k+b_k| \le \sup_{1\le k\le n}|a_k| + \sup_{1\le k\le n}|b_k| = \|\mathbf{a}\|_\infty+\|\mathbf{b}\|_\infty.$$

8.2 Planes and Linear Transformations.

8.2.1. a) By definition, $\mathbf{a}-\mathbf{b}$ and $\mathbf{a}-\mathbf{c}$ lie in the plane. Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ do not lie on the same straight line, Remark 8.10 implies that $\mathbf{d} := (\mathbf{a}-\mathbf{b})\times(\mathbf{a}-\mathbf{c})$ is nonzero. Thus by Theorem 8.9vii, $\mathbf{d}$ is a normal to the plane. Hence, by the point-normal form, $\mathbf{d}\cdot(\mathbf{x}-\mathbf{a}) = 0$ is an equation of the plane through $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

b) The line is parallel to $\mathbf{a}$, so $\mathbf{a}$ "lies in the plane." Since $\mathbf{b}-\mathbf{c}$ is another vector that lies in the plane, it follows from part a) that an equation of the plane is given by $\mathbf{d}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{d}$, where $\mathbf{d} = \mathbf{a}\times(\mathbf{b}-\mathbf{c})$.

8.2.2. a) Since $(1,0,0,0)$ lies on the plane, the constant term in the equation of this plane must be nonzero. By dividing by it, we may suppose that the equation of the plane looks like $ax+by+cz+dw = 1$. Since this plane contains $(2,1,0,0)$, $(0,1,1,0)$, and $(0,4,0,1)$, it follows that $a = 1$, $2a+b = 1$, $b+c = 1$, and $4b+d = 1$, i.e., $a = 1$, $b = -1$, $c = 2$, and $d = 5$. Thus an equation is $x-y+2z+5w = 1$.

b) As in part a), we may suppose that $ax+by+cz+dw = 1$. Since the plane contains $\phi(0) = (0,0,0,1)$ and $\phi(1) = (1,1,1,1)$, we have $d = 1$ and $a+b+c+d = 1$, i.e., $a+b+c = 0$. Since it also contains $\psi(0) = (1,0,1,0)$ and $\psi(1) = (1,1,2,1)$, it follows that

$$\begin{aligned} a+b+c &= 0 \\ a+c &= 1 \\ a+b+2c+d &= 1. \end{aligned}$$

Solving these simultaneous equations, we have $b = -1$, $c = 0$, and $a = d = 1$, i.e., $x-y+w = 1$.

c) If the plane is parallel to $x_1+\cdots+x_n = \pi$, then a normal is given by $\mathbf{n} = (1,1,\dots,1)$. Since $\sum_{k=1}^n k = n(n+1)/2$, it follows that an equation of this plane is $x_1+\cdots+x_n = n(n+1)/2$.

8.2.3. All we have to do is find two lines which lie in parallel planes. We will choose two planes with normal $(0,0,1)$, e.g., $z = 0$ and $z = 1$. Let $\phi(t) = (0,0,0) + t(1,1,0)$ and $\psi(t) = (0,0,1) + t(3,4,0)$. These lines are not parallel because their "direction vectors" $(1,1,0)$ and $(3,4,1)$ are not parallel. If they intersect, say $\phi(t) = \psi(u)$, then $t = 3u$, $t = 4u$, and $0 = 1$, a contradiction. It follows that the lines do not intersect.

8.2.4. a) The columns of B are $T(e_1) = (0,1,1,1)$, $T(e_2) = (0,1,0,1)$, $T(e_3) = (0,0,-1,0)$, and $T(e_4) = (0,0,0,1)$.

b) $B = [1 \quad -1 \quad 1]$.

c) The columns of B are $T(e_1) = (1,-1)$, $T(e_2) = \cdots = T(e_{n-1}) = (0,0)$, and $T(e_n) = (-1,1)$.

8.2.5. a) $T(1,0) = T(1,1) - T(0,1) = (-1,\pi,-1)$, and $T(0,1) = (4,0,1)$. Thus

$$A = \begin{bmatrix} -1 & 4 \\ \pi & 0 \\ -1 & 1 \end{bmatrix}.$$

b) $(1,0,0) = a(1,1,0) + b(0,-1,1) + c(1,1,-1)$ implies $b = c = 1$ and $a = 0$. Thus $T(1,0,0) = T(0,-1,1) + T(1,1,-1) = (1,0) + (1,2) = (2,2)$. Similarly, $T(0,1,0) = (e-2, \pi-2)$ and $T(0,0,1) = (e-1, \pi-2)$. Thus

$$A = \begin{bmatrix} 2 & e-2 & e-1 \\ 2 & \pi-2 & \pi-2 \end{bmatrix}.$$

c) Let $T(1,0,0,0) = (a,b)$. Note that $T(0,0,1,0) = 0.5(T(0,1,1,0) - T(0,1,-1,0)) = 0.5((3,5) - (5,3)) = (-1,1)$ and $T(0,1,0,0) = 0.5(T(0,1,1,0) + T(0,1,-1,0)) = 0.5((3,5) + (5,3)) = (4,4)$. Therefore,

$$A = \begin{bmatrix} a & 4 & -1 & -\pi \\ b & 4 & 1 & -3 \end{bmatrix}.$$

b) Let $T(1,0,0,0) = (a,b,c)$. Then $T(0,1,0,0) = T(1,1,0,0) - T(1,0,0,0) = (5,4,1) - (a,b,c) = (5-a, 4-b, 1-c)$, so

$$A = \begin{bmatrix} a & 5-a & 1 & -\pi \\ b & 4-b & 2 & -3 \\ c & 1-c & 0 & 1 \end{bmatrix}.$$

8.2.6. By Theorem 8.9vii, a normal to the plane Π is given by $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. Hence an equation of the plane is given by

$$((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) \cdot (x,y,z) = ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) \cdot \mathbf{a}.$$

By Theorem 8.2, this can be rewritten as $((x,y,z) - \mathbf{a}) \cdot ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) = 0$ which is equivalent to

$$\det \begin{pmatrix} x - a_1 & y - a_2 & z - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{pmatrix} = 0$$

by Theorem 8.9.

8.2.7. a) Let A represent the area of $\mathcal{P}$ and θ represent the angle between $\mathbf{a}$ and $\mathbf{b}$. The base of $\mathcal{P}$ is $\|\mathbf{b}\|$ and its altitude is $\|\mathbf{a}\| \sin\theta$. Hence $A = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta = \|\mathbf{a} \times \mathbf{b}\|$ by Remark 8.10.

b) Let V represent the volume of $\mathcal{P}$ and θ represent the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$. By part a), $V = \|\mathbf{a} \times \mathbf{b}\| \cdot h$, where $h = \|\mathbf{c}\| \cdot |\cos\theta|$. But by (2), $\cos\theta = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} / (\|\mathbf{a} \times \mathbf{b}\|\,\|\mathbf{c}\|)$. Therefore,

$$V = \|\mathbf{a} \times \mathbf{b}\| \cdot h = \|\mathbf{a} \times \mathbf{b}\|\,\|\mathbf{c}\| \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{\|\mathbf{a} \times \mathbf{b}\|\,\|\mathbf{c}\|} = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.$$

8.2.8. If (x_0, y_0, z_0) lies on the plane Π then the distance is zero, and by definition, $ax_0 + by_0 + cz_0 - d = 0$. Thus the formula is correct for this case.

If (x_0, y_0, z_0) does not lie on the plane Π then the distance h from Π to (x_0, y_0, z_0) is defined to be $\|\mathbf{v}\|$ where $\mathbf{v} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ is orthogonal to Π and (x_1, y_1, z_1) lies in Π. Let (x_2, y_2, z_2) be a point on Π different from (x_1, y_1, z_1) and θ represent the angle between $\mathbf{w} := (x_0 - x_2, y_0 - y_2, z_0 - z_2)$ and the normal (a, b, c) of Π. Then we can compute $\cos\theta$ two ways:

$$\frac{|\mathbf{w} \cdot (a, b, c)|}{\|\mathbf{w}\| \, \|(a, b, c)\|} = |\cos\theta| = \frac{h}{\|\mathbf{w}\|}.$$

Since $\mathbf{w} \cdot (a, b, c) = ax_0 + by_0 + cz_0 - d$, it follows that $h = |ax_0 + by_0 + cz_0 - d|/\sqrt{a^2 + b^2 + c^2}$.

8.2.9. By definition, $B(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$. Thus

$$\|B(x, y)\|^2 = x^2(\sin^2\theta + \cos^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = x^2 + y^2 = \|(x, y)\|^2.$$

If φ is the angle between (x, y) and $B(x, y)$ then by (2) and what we just proved,

$$\cos\varphi = \frac{(x, y) \cdot B(x, y)}{\|(x, y)\| \, \|B(x, y)\|} = \frac{(x^2 + y^2)\cos\theta}{\|(x, y)\| \, \|B(x, y)\|} = \cos\theta.$$

8.2.10. Since T is linear, if the components of f are differentiable, then

$$\frac{f(x + h) - f(x) - T(h)}{h} = \frac{f(x + h) - f(x)}{h} - T(1) = (f_1'(x), \ldots, f_m'(x)) - (b_1, \ldots, b_m).$$

It follows that $T = [\, f_1'(x) \quad \ldots \quad f_m'(x)\,]$. Therefore,

a) $T = \begin{bmatrix} 2x \\ \cos x \end{bmatrix}$

b) $T = \begin{bmatrix} e^x \\ 1/(3\sqrt[3]{x^2}) \\ -2x \end{bmatrix}$

c) $T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2x + 1 \\ 2x - 1 \end{bmatrix}$

8.2.11. a) If $\|\mathbf{x}\| = 1$, Theorem 8.17 implies $\|T(\mathbf{x})\| \le \|T\|$. Taking the supremum of this inequality over all $\|\mathbf{x}\| = 1$, we obtain $M_1 \le \|T\|$.

b) If $\mathbf{x} \ne \mathbf{0}$, then the norm of $\mathbf{x}/\|\mathbf{x}\|$ is 1, so

$$\frac{\|T(\mathbf{x})\|}{\|\mathbf{x}\|} = \left\| T\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \right\| \le M_1.$$

c) Taking the supremum of this last inequality over all $\mathbf{x} \ne \mathbf{0}$, we obtain $\|T\| \le M_1$. Combining this with part a), we have $M_1 = \|T\|$.

On the other hand, by part b) and Theorem 8.17, $\|T(\mathbf{x})\| \le M_1\|\mathbf{x}\|$. Thus the set used to define M_2 is nonempty, bounded above by M_1 and bounded below by 0. Hence, by the Completeness Postulate, M_2 exists and satisfies $M_2 \le M_1$.

Finally, use the Approximation Property to choose $C_k > 0$ such that $C_k \downarrow M_2$ as $k \to \infty$ and take the limit of $\|T(\mathbf{x})\| \le C_k\|\mathbf{x}\|$ as $k \to \infty$. We obtain $\|T(\mathbf{x})\| \le M_2\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{R}^n$. Taking the supremum of this last inequality over all $\|\mathbf{x}\| = 1$, we conclude that $M_1 \le M_2$.

8.3 Topology of $\mathbf{R}^n$.

8.3.1. a) This is the plane without the x-axis. It is open but not connected since it is the union of the open upper half plane and the open lower half plane.

b) This is the set of points on or inside the ellipse $x^2 + 4y^2 = 1$. It is closed because its complement $\{(x, y) : x^2 + 4y^2 > 1\}$ is open. It is evidently connected.

c) This is the set of points on or above the parabola which lie below the line $y = 1$. It is neither open nor closed. However, it is connected.

d) This is the set of points inside the two branches of the hyperbola $x^2 - y^2 = 1$ which lie above the line $y = -1$ and below the line $y = 1$. It is open but not connected.

e) This is the set of points on the circle $(x-1)^2 + y^2 = 1$ or on the x axis between $x = 2$ and $x = 3$. It is closed and connected.

8.3.2. Let $y \in V = \{x \in \mathbf{R}^n : s < \|x - a\| < r\}$ and let $\epsilon < \min\{\|y-a\| - s, r - \|y - a\|\}$. If $w \in B_\epsilon(y)$ then

$$\|w - a\| \le \|w - y\| + \|y - a\| < r - \|y - a\| + \|y - a\| = r$$

and

$$\|w - a\| \ge \|y - a\| - \|w - y\| > \|y - a\| + s - \|y - a\| = s.$$

Hence $w \in V$ and V is open by definition.

A similar argument shows that $\{x \in \mathbf{R}^n : \|x - a\| > r\}$ and $\{x \in \mathbf{R}^n : \|x - a\| < s\}$ are both open, hence

$$E := \{x \in \mathbf{R}^n : s \le \rho(x, a) \le r\} = \{x \in \mathbf{R}^n : \|x - a\| > r\}^c \cap \{x \in \mathbf{R}^n : \|x - a\| < s\}^c$$

is closed.

8.3.3. a) It is connected (see Remark 9.34 for proof).

b) The set is "dumbbell" shaped. It almost looks connected, except that $(-1, 0)$ and $(1, 0)$ do not belong to the set. Hence a separation can be made, e.g., by using the open sets $V = \{(x, y) : x < -1\}$ and $U = \{(x, y) : x > -1\}$, and applying Remark 8.29.

8.3.4. Since E_1 is closed and E_2 is open, and $U = E_1 \cap E_2$, it is clear by definition that U is relatively open in E_1 and U is relatively closed in E_2.

8.3.5. By completing the square, $\{x^2 - 4x + y^2 + 2 < 0\} = B_{\sqrt{2}}(2, 0)$. Thus $U = \overline{B_1(0,0)} \cap B_{\sqrt{2}}(2, 0)$. It follows that U is relatively open in $\overline{B_1(0,0)}$ and relatively closed in $B_{\sqrt{2}}(2, 0)$.

8.3.6. a) If C is relatively closed in E, then there is a closed set A such that $C = E \cap A$. Since E and A are closed, it follows that C is closed. Conversely, if C is closed, then $C = E \cap C$ implies that C is relatively closed in E.

b) If C is relatively closed in E then $C = E \cap A$ for some closed set A. But $E \setminus C = E \cap A^c$. Since A^c is open, it follows that $E \setminus C$ is relatively open in E. Conversely, if $E \setminus C = E \cap V$ for some open V, then $C = E \cap V^c$, so C is relatively closed in E.

8.3.7. b) Suppose E is not connected. Then there exists a pair of open sets U, V which separates E. Let $x \in \cap_{\alpha \in A} E_\alpha$. Since $E \subseteq U \cup V$ we may suppose $x \in U$. Choose $\alpha_0 \in A$ such that $V \cap E_{\alpha_0} \ne \emptyset$. Since $x \in E_{\alpha_0}$, we also have $U \cap E_{\alpha_0} \ne \emptyset$. Therefore, the pair U, V separates E_{α_0}, a contradiction.

a) Use part b) with $A = \{1, 2\}$.

c) If E is connected in $\mathbf{R}$ then E is an interval, hence E^o is either empty or an interval, hence connected by definition or Theorem 8.30.

d) The set $E = B_1(0,0) \cup B_1(3,0) \cup \{(x, 0) : 1 \le x \le 2\}$ is connected in $\mathbf{R}^2$, but $E^o = B_1(0,0) \cup B_1(3,0)$ is not.

8.3.8. a) Given $x \in V$ choose $\epsilon := \epsilon_x > 0$ such that $B_\epsilon(x) \subseteq V$. Then $V \subseteq \cup_{x \in V} B_\epsilon(x)$. On the other hand, $\cup_{x \in V} B_\epsilon(x) \subseteq V$ since each $B_\epsilon \subseteq V$. Therefore $V = \cup_{x \in V} B_\epsilon(x)$ as required.

It is even easier for closed sets. Since every singleton is closed (see Remark 8.22), $E = \cup_{x \in E} \{x\}$ is a decomposition of E into closed sets.

8.3.9. Suppose E is closed and $a \notin E$, but $\inf_{x \in E} \|x - a\| = 0$. Then by the Approximation Property, there exist $x_j \in E$ such that $\|x_j - a\| \to 0$, i.e., such that $x_j \to a$. But E is closed, so the limit of the x_j's, namely a, must belong to E, a contradiction.

8.3.10. In $\mathbf{R}^2$, an ℓ^1 ball at the origin is $\{(x, y) : |x| + |y| < 1\}$. Since $y = 1 \pm x$ are lines with y- intercept 1, it is easy to see that this ball is a diamond with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

In $\mathbf{R}^2$, an ℓ^∞ ball at the origin is $\{(x, y) : \max\{|x|, |y|\} < 1\}$, i.e., $|x| \le 1$ and $|y| \le 1$. Thus this ball is a square with vertices $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$.

8.4 Interior, Closure, and Boundary.

8.4.1. a) The closure is $E \cup \{0\}$, the interior is $\emptyset$, the boundary is $E \cup \{0\}$.

b) The closure is $[0,1]$, the interior is E, the boundary is $\{1/n : n \in \mathbf{N}\} \cup \{0\}$.
c) The closure is $\mathbf{R}$, the interior is $\mathbf{R}$, the boundary is $\emptyset$.
d) The closure is $\mathbf{R}$, the interior is $\emptyset$, the boundary is $\mathbf{R}$.

8.4.2. a) This is the set of points on or inside the ellipse $x^2 + 4y^2 = 1$. It is closed because its complement $\{(x,y) : x^2 + 4y^2 > 1\}$ is open. $E^o = \{(x,y) : x^2 + 4y^2 < 1\}$ and $\partial E = \{(x,y) : x^2 + 4y^2 = 1\}$.
b) This is the set of points on the circle $(x-1)^2 + y^2 = 1$ or on the x axis between $x = 2$ and $x = 3$. It is closed. $E^o = \emptyset$ and $\partial E = E$.
c) This is the set of points on or above the parabola which lie below the line $y = 1$. It is neither open nor closed. $E^o = \{(x,y) : y > x^2, 0 < y < 1\}$, $\overline{E} = \{(x,y) : y \geq x^2, 0 \leq y \leq 1\}$, and $\partial E = \{(x,y) : y = x^2, 0 \leq y \leq 1\} \cup \{(x,1) : -1 \leq x \leq 1\}$.
d) This is the set of points between the two branches of the hyperbola $x^2 - y^2 = 1$ which lie above the line $y = -1$ and below the line $y = 1$. It is open. $\overline{E} = \{x^2 - y^2 \leq 1, -1 \leq y \leq 1\}$ and $\partial E = \{x^2 - y^2 = 1, -1 \leq y \leq 1\} \cup \{(x,1) : -\sqrt{2} \leq x \leq \sqrt{2}\} \cup \{(x,-1) : -\sqrt{2} \leq x \leq \sqrt{2}\}$.

8.4.3. If $A \subseteq B$ then A^o is an open set contained in B. Hence by Theorem 8.32, $A^o \subseteq B^o$. Similarly, $\overline{B}$ is a closed set containing A, hence $\overline{A} \subseteq \overline{B}$.

8.4.4. First, we prove that relatively open sets are closed under arbitrary unions, and relatively closed sets are closed under arbitrary intersections.
Let B_α be relatively open in E, i.e., $B_\alpha = E \cap V_\alpha$ for open sets V_α in $\mathbf{R}^n$. Since

$$\bigcup_{\alpha \in A} B_\alpha = E \cap \bigcup_{\alpha \in A} V_\alpha,$$

and the union of V_α's is open by Theorem 8.24, it is clear that the union of the B_α's is relatively open in E. Similarly for intersections of relatively closed sets.
Now, repeating the proof of Theorem 8.32, we see that the largest relatively open set which is a subset of A is the union of all sets $U \subset A$ such that U is relatively open in E, and the smallest relatively closed set which contains A is the intersection of all sets $B \supset A$ such that B is relatively closed in E.

8.4.5. Suppose $x \notin E^o$ but $B_r(x) \subset E$. Then by Theorem 8.32, $B_r(x) \subseteq E^o$ so $x \in E^o$, a contradiction.
Conversely, if $B_r(x) \cap E^c \neq \emptyset$ for all $r > 0$, then $x \notin E^o$ because E^o is open.

8.4.6. a) If E is connected in $\mathbf{R}$ then E is an interval, hence E^o is either empty or an interval, hence connected by definition or Theorem 8.30.
b) The set $E = B_1(0,0) \cup B_1(3,0) \cup \{(x,0) : 1 \leq x \leq 2\}$ is connected in $\mathbf{R}^2$, but $E^o = B_1(0,0) \cup B_1(3,0)$ is not.

8.4.7. Suppose A is not connected. Then there is a pair of open sets U, V which separates A. We claim that $E \cap U \neq \emptyset$. If $E \cap U = \emptyset$ then since $A \cap U \neq \emptyset$, there exists a point $x \in U \cap (A \setminus E)$. But $E \subseteq A \subseteq \overline{E}$ implies $A \setminus E \subseteq \overline{E} \setminus E = \partial E$. Thus $x \in \partial E \cap U$. Since U is open it follows that $E \cap U \neq \emptyset$, a contradiction. This verifies the claim. Similarly, $E \cap V \neq \emptyset$. Thus the pair U, V separates E, which contradicts the fact that E is connected.

8.4.8. a) By Remark 8.23, $\emptyset$ and $\mathbf{R}^n$ are clopen.
b) Suppose A is clopen and $\emptyset \subset A \subset E$. Then $U = A$ and $V = E \setminus A$ are nonempty relatively open subsets of E, $U \cap V = \emptyset$, and $E = U \cup V$. Therefore, E is not connected.
Conversely, if E is not connected then there exist nonempty relatively open subsets U and V of E such that $U \cap V = \emptyset$ and $E = U \cup V$. Thus $A := U = E \setminus V$ is clopen and $\emptyset \subset A \subset E$. In particular, E contains more than two clopen sets.
c) Let E be a nonempty, proper subset of $\mathbf{R}^n$. By Theorem 8.32, E has no boundary if and only if $\overline{E} \setminus E^o = \partial E = \emptyset$, i.e., if and only if $\overline{E} = E^o$. Thus E has no boundary if and only if E is clopen. This happens, by part b), if and only if $\mathbf{R}^n$ is not connected.

8.4.9. a) If $A = (0,1)$ and $B = [1,2]$ then $(A \cup B)^o = (0,2)$ but $A^o \cup B^o = (0,1) \cup (1,2) \neq (0,2)$.
b) If $A = \mathbf{Q}$ and $B = A^c$ then $\overline{A \cap B} = \emptyset$ but $\overline{A} \cap \overline{B} = \mathbf{R} \cap \mathbf{R} = \mathbf{R}$.
c) If A and B are as in part a), then $\partial(A \cup B) = \{0,2\} \neq \{0,1,2\} = \partial A \cup \partial B$ and $\partial(A \cap B) = \emptyset \neq \{0,1,2\} = \partial A \cup \partial B$.

8.4.10. a) Let $x \in \partial(A \cap B) \cap (A^c \cup (\partial B)^c)$.
Case 1. $x \in A^c$. Since $B_r(x)$ intersects A, it follows that $x \in \partial A$.

Case 2. $x \in (\partial B)^c$. Since $B_r(x)$ intersects B, it follows that $B_r(x) \subseteq B$ for small $r > 0$. Since $B_r(x)$ also intersects $A^c \cup B^c$, it must be the case that $B_r(x)$ intersects A^c. In particular, $x \in \partial A$.

b) Suppose $x \in \partial(A \cap B)$; i.e., suppose $B_r(x)$ intersects $A \cap B$ and $(A \cap B)^c$ for all $r > 0$. If $x \notin (A \cap \partial B) \cup (B \cap \partial A)$, then $x \in (A^c \cup (\partial B)^c) \cap (B^c \cup (\partial A)^c)$. But by part a), $A^c \cup (\partial B)^c \subseteq \partial A$ and $B^c \cup (\partial A)^c \subseteq \partial B$. Hence the intersection is a subset of $\partial A \cap \partial B$.

c) Suppose $x \in \partial(A \cap B)$. If $x \in (A \cap \partial B) \cup (B \cap \partial A)$, then there is nothing to prove. If $x \notin (A \cap \partial B) \cup (B \cap \partial A)$, then by part b), $x \in \partial A \cap \partial B$. Hence $x \in (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

d) If $A = (0,1)$ and $B = [1,2]$, then $\partial(A \cap B) = \emptyset \neq \{1\} = \partial A \cap \partial B \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

8.4.11. Part a) follows directly from Remark 8.27ii. To prove part b), let $\mathbf{x} \in U \cap \partial A$ and suppose for a moment that $\mathbf{x} \in E^o$. Since $U \cap E^o$ is open (see Remark 8.27ii), there is an $r > 0$ such that $B_r(\mathbf{x}) \subset U \cap E^o \subset A$. Thus $\mathbf{x} \notin \partial U$, a contradiction. It follows that $\mathbf{x} \in E \setminus E^o \subseteq \partial E$.

Conversely, suppose $\mathbf{x} \in U \cap \partial E$. Since $U \subseteq E$ implies $U^o \subseteq E^o$ (see Exercise 8.4.3), it follows that $\mathbf{x} \notin U^o$. Thus, $\mathbf{x} \in U \setminus U^o \subseteq \partial U$. ∎

Chapter 9

9.1 Limits of Sequences.

9.1.1. a) Let $\varepsilon > 0$ and choose (by Archimedes) an $N \in \mathbf{N}$ such that $k > N$ implies $1/k < \varepsilon/2$. Notice that $k^2 \le k^4$ for all $k \in \mathbf{N}$. If $k > N$, then

$$\|(1/k, 1 - 1/k^2) - (0,1)\|^2 = 1/k^2 + 1/k^4 \le 1/k^2 + 1/k^2 < \varepsilon^2.$$

Thus $k > N$ implies $\|(1/k, 1 - 1/k^2) - (0,1)\| < \varepsilon$.

b) Let $\varepsilon > 0$ and choose (by Archimedes) an $N \in \mathbf{N}$ such that $k > N$ implies $1/k < \varepsilon^2/2$. Recall that $|\sin(k^3)| \le 1$ for all $k \in \mathbf{N}$. If $k > N$, then

$$\|(k/(k+1), \sin(k^3)/k) - (1,0)\|^2 = 1/(k+1)^2 + \sin^2(k^3)/k < 1/k^2 + 1/k < 2/k < \varepsilon^2.$$

Thus $k > N$ implies $\|(k/(k+1), \sin(k^3)/k) - (1,0)\| < \varepsilon$.

c) Let $\varepsilon > 0$ and choose (by Archimedes) an $N_1 \in \mathbf{N}$ such that $k > N_1$ implies $1/k < \varepsilon/2$. Recall that $1/2^k < 1/k$ and $\log(k+1) - \log k = \log((k+1)/k) \to 0$ as $k \to \infty$, so choose N_2 so that $\log((k+1)/k < \varepsilon/2$. If $k > N := \max\{N_1, N_2\}$, then

$$\|\log(k+1) - \log k, 1/2^k) - (0,0)\|^2 = \log^2((k+1)/k) + 1/2^{2k} < \varepsilon^2/4 + 1/k^2 < \varepsilon^2.$$

Thus $k > N$ implies $\|(\log(k+1) - \log k, 2^{-k}) - (0,0)\| < \varepsilon$.

9.1.2. a) By Theorem 9.2, $(1/k, (2k^2 - k + 1)/(k^2 + 2k - 1)) \to (0,2)$ as $k \to \infty$.

b) Since $\sin \pi k = 0$ for all $k \in \mathbf{N}$ and $\cos(0) = 1$, $(1, \sin \pi k, \cos(1/k)) \to (1,0,1)$ as $k \to \infty$.

c) Since

$$k - \sqrt{k^2 + k} = \frac{(k - \sqrt{k^2 + k})(k + \sqrt{k^2 + k})}{k + \sqrt{k^2 + k}} = \frac{-k}{k + \sqrt{k^2 + k}} \to -\frac{1}{2}$$

as $k \to \infty$ and by l'Hôpital's Rule, $k^{1/k} \to e^0 = 1$ as $k \to \infty$, we see by Theorem 9.2 that $(k - \sqrt{k^2 - k}, k^{1/k}, 1/k) \to (-1/2, 1, 0)$ as $k \to \infty$.

9.1.3. a) By the Cauchy-Schwarz inequality and the Squeeze Theorem, $\|\mathbf{x}_k \cdot \mathbf{y}_k\| \le \|\mathbf{x}_k\| \, \|\mathbf{y}_k\| \le M\|\mathbf{x}_k\| \to 0$ as $k \to \infty$.

b) By Remark 8.10 and the Squeeze Theorem, $\|\mathbf{x}_k \times \mathbf{y}_k\| \le \|\mathbf{x}_k\| \, \|\mathbf{y}_k\| \le M\|\mathbf{x}_k\| \to 0$ as $k \to \infty$.

9.1.4. Let $\varepsilon > 0$ and choose N so that $k \ge N$ implies $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon/2$ and $\|\mathbf{x}_k - \mathbf{y}_k\| < \varepsilon/2$. Then $k \ge N$ implies

$$\|\mathbf{y}_k - \mathbf{a}\| \le \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{x}_k - \mathbf{y}_k\| < \varepsilon.$$

9.1.5. a) Repeat the proofs of Remark 2.4 and Theorem 2.6, replacing the absolute value by the norm sign.

b) Repeat the proofs of Theorem 2.8 and Remark 2.28, replacing the absolute value by the norm sign.

c) Repeat the proof of Theorem 2.12, replacing the absolute value by the norm sign.

d) Repeat the proof of Theorem 2.29.

9.1.6. a) Let $\mathbf{x}_k \in E$ converge to some point $\mathbf{a}$. Let U be relatively open in E, i.e., $U = E \cap V$ for some open V in $\mathbf{R}^n$. If $\mathbf{a} \in U$, then by Theorem 9.7, $\mathbf{x}_k \in V$ for large k. Since $\mathbf{x}_k \in E$ too, it follows that $\mathbf{x}_k \in U$ for large k.

Conversely, if $\mathbf{x}_k$ belongs to any relatively open set for large k then, since $U = E \cap B_\varepsilon(\mathbf{a})$ is relatively open in E and contains $\mathbf{a}$, $\mathbf{x}_k \in U \subseteq B_\varepsilon(\mathbf{a})$ for large k, i.e., $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$.

b) Let C be relatively closed in E, i.e., there is a closed set B such that $C = E \cap B$. By Theorem 9.8, $\mathbf{x}_k \to \mathbf{a}$ and $\mathbf{x}_k \in C \subseteq B$ implies $\mathbf{a} \in B$. Since $\mathbf{a} \in E$, it follows that $\mathbf{a} \in C$.

Conversely, suppose $C \subseteq E$ contains all its limit points which stay in E. By Exercise 8 in 8.3, $E \setminus C$ is relatively open in E, i.e., $E \setminus C = E \cap V$ for some open V. Hence, $C = E \cap V^c$. Since V^c is closed in $\mathbf{R}^n$, it follows that C is relatively closed.

9.1.7. a) Let B be a closed ball of radius R and center $\mathbf{a}$. If $\mathbf{x}_k \in B$, then

$$(*) \qquad \|\mathbf{x}_k - \mathbf{a}\| \le M$$

for all $k \in \mathbf{N}$. Hence by the triangle inequality, $\|\mathbf{x}_k\|$ is bounded (by $M + \|\mathbf{a}\|$). By Theorem 9.5, there is a subsequence $\mathbf{x}_{k_j}$ which converges, say to $\mathbf{b}$. Taking the limit of (*), as $k_j \to \infty$, we see that $\|\mathbf{b} - \mathbf{a}\| \le M$, i.e., $\mathbf{b} \in B$. We conclude that B is sequentially compact.

b) The sequence $(k, 1, \dots, 1)$ belongs to $\mathbf{R}^n$ but has no convergent subsequence.

9.1.8. a) If $E \cap B_r(\mathbf{a})$ contains infinitely many points, then surely $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty. Conversely, suppose $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is nonempty for every $r > 0$. Fix $r > 0$ and let $\mathbf{x}_1 \in E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$. Suppose distinct points $\mathbf{x}_j \in E \cap B_r(\mathbf{a})$ have been chosen for each $1 \le j < k$. Let $r_0 = \min\{r, \|\mathbf{a} - \mathbf{x}_j\| : 1 \le j < k\}$. Since $r_0 > 0$, there is a point $\mathbf{x}_k \in E \cap B_{r_0}(\mathbf{a}) \setminus \{\mathbf{a}\}$. By the choice of r_0, $\mathbf{x}_k$ belongs to $E \cap B_r(\mathbf{a})$ and is distinct from the $\mathbf{x}_j$'s, $1 \le j < k$. By induction, there exist infinitely many points in $E \cap B_r(\mathbf{a})$.

b) If E is infinite, then it contains a sequence $\mathbf{x}_k$ of distinct points. Since E is bounded, the Bolzano–Weierstrass Theorem implies that some subsequence $\mathbf{x}_{k_j}$ converges to a point $\mathbf{a}$. It follows from Definition 9.1i), given any $r > 0$, there is an N such that $k \ge N$ implies $\|\mathbf{x}_k - \mathbf{a}\| < r$. Since the points of the original sequence were distinct, it follows that $\mathbf{a}$ is a cluster point of E.

9.2 Heine-Borel Theorem.

9.2.1. If E is compact, then by Heine-Borel, E is closed (and bounded). If E is closed, then it is already bounded because it is a subset of K. Thus by the Heine-Borel Theorem, E is compact.

9.2.2. Since E is bounded, there is an M such that $\|\mathbf{x}\| < M$ for all $\mathbf{x} \in E$. Hence, $\|\mathbf{x}\| \le M$ for all $\mathbf{x} \in K := \overline{E}$. In particular, K is closed and bounded, hence compact by the Heine-Borel Theorem.

Since $g(\mathbf{x}) > f(\mathbf{x})$, it is clear that

$$\mathbf{x} \in B_{f(\mathbf{x})}(\mathbf{x}) \subset \overline{B_{f(\mathbf{x})}(\mathbf{x})} \subset B_{g(\mathbf{x})}(\mathbf{x})$$

for all $\mathbf{x} \in E$. Thus by Theorem 8.37ii,

$$\overline{E} \subset \bigcup_{\mathbf{x} \in E} \overline{B_{f(\mathbf{x})}(\mathbf{x})} \subset \bigcup_{\mathbf{x} \in E} B_{g(\mathbf{x})}(\mathbf{x}).$$

Since $\overline{E}$ is closed and bounded, we conclude by the Heine-Borel Theorem that there exist $\mathbf{x}_1, \dots, \mathbf{x}_N \in E$ such that

$$E \subset \overline{E} \subset \bigcup_{j=1}^{N} B_{g(\mathbf{x}_j)}(\mathbf{x}_j).$$

9.2.3. For each $x \in E$, choose $r = r_x > 0$ and $f_x \ge 0$ such that f is $\mathcal{C}^\infty$ on $\mathbf{R}$, $f(t) = 1$ for $t \in I_r(x) := (x - r, x + r)$, and $f(t) = 0$ for $t \notin J_r(x) := (x - 2r, x + 2r)$. Since $\{I_r(x)\}_{x \in E}$ covers the compact set E, there exist finitely many x_j's in E such that

$$E \subset \bigcup_{j=1}^{N} I_{r_j}(x_j)$$

for $r_j = r(x_j)$. Set $f = \sum_{k=1}^N f_{x_j}$ and $V = \bigcup_{j=1}^N J_{r_j}(x_j)$. Since f is a finite sum of $\mathcal{C}^\infty$ functions, it is $\mathcal{C}^\infty$ on $\mathbf{R}$. V is open since it is a union of open intervals. If $x \in E$, then $x \in I_{r_j}(x_j)$ for some j, so $f_{x_j}(x) = 1$. Thus

$f(x) \geq 0+\cdots+f_{x_j}(x)+\cdots+0=1$ for all $x \in E$. Moreover, since f_{x_k} is continuous on $E^* := \bigcup_{k=1}^N [x_k - r_k, x_k + r_k]$, the Extreme Value Theorem implies that there are constants M_k that $|f_{x_k}| \leq M_k$ on E^* for all k. Thus $f(x) \leq\leq M_1+\cdots+M_N =: M$ for all $x \in E^* \supset E$. Finally, if $x \notin V$, then $x \notin J_{r_j}(x_j)$ for all j. Thus $f(x) = 0+0+\cdots+0 = 0$.

9.2.4. For each $\mathbf{x} \in K$, $B_{r(\mathbf{x})} \cap K = \{\mathbf{x}\}$. Since K is compact and is covered by $\{B_{r(\mathbf{x})}(\mathbf{x})\}_{\mathbf{x}\in K}$, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_N$ such that

$$K \subset \bigcup_{j=1}^N B_{r(\mathbf{x}_j)}(\mathbf{x}_j).$$

Thus

$$K = \bigcup_{j=1}^N B_{r(\mathbf{x}_j)}(\mathbf{x}_j) \cap K = \bigcup_{j=1}^N \{\mathbf{x}_j\} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}.$$

9.2.5. Let $x \in E$. Since $f_x(x) > 0$ and f_x is continuous, choose by the Sign Preserving Property an $r(x) > 0$ such that $f_x(y) > 0$ for $y \in B_{r(x)}(x)$. By the Heine–Borel Theorem, there are points $x_j \in E$ such that $E \subset \bigcup_{j=1}^N B_{r(x_j)}(x_j)$. Set $f(y) = \sum_{j=1}^N f_{x_j}(y)$. Since f is a finite sum of $\mathcal{C}^\infty$ functions, it is $\mathcal{C}^\infty$. Let $y \in E$. Then $y \in B_{r(x_j)}(x_j)$ for some j. Since each f_{x_j} is nonnegative on $B_{r(x_j)}(x_j)$, it follows from the choice of the $r(x_j)$'s that $f(y) \geq f_{x_j}(y) > 0$. On the other hand, if $y \notin E$, then $f'_{x_j}(y) = 0$ for all j. In particular, $f'(y)$, a sum of zeros, is itself zero. Finally, since each f_{x_j} is increasing and nonconstant, there is a $t \in E$ such that $f'_{x_j}(t) > 0$ Thus $f'(t) = \sum_{k=1}^N f'_{x_k}(t) \geq f'_{x_j}(t) > 0$. In particular, f is nonconstant.

9.2.6. For each $\mathbf{x} \in K$, f is constant on $B_{\delta_{\mathbf{x}}(\mathbf{x})} \cap K$. Since K is compact and is covered by $\{B_{\delta_{\mathbf{x}}}(\mathbf{x})\}_{\mathbf{x}\in K}$, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_N$ such that

$$K \subset \bigcup_{j=1}^N B_{\delta_{\mathbf{x}_j}}(\mathbf{x}_j).$$

Let $\mathbf{x} \in K$. Then $\mathbf{x} \in B_{\delta_{\mathbf{x}_j}}(\mathbf{x}_j)$ for some $1 \leq j \leq N$, so $f(\mathbf{x}) = f(\mathbf{x}_j)$. It follows that $f(\mathbf{x}) \in \{f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N)\}$ for all $\mathbf{x} \in K$. In particular, $f(K) \subset \{f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N)\}$, so $f(K)$ is finite, say $K = \{\mathbf{y}_1, \ldots, \mathbf{y}_M\}$. But a nonempty, finite set is nonempty connected if and only if it is a single point. Indeed, if $M > 1$, then set $r = \min\{\|\mathbf{y}_j - \mathbf{y}_k\| : j, k \in [1, M]\}$ and notice that the $B_r(\mathbf{y}_j)$'s are open, nonempty, and disjoint, hence separate $f(K)$. Thus $M = 1$ and $f(\mathbf{x}) = f(\mathbf{x}_1)$ for all $\mathbf{x} \in K$. Since $\mathbf{a} \in K$, we conclude that $f(\mathbf{x}) = f(\mathbf{a})$ for all $\mathbf{x} \in K$.

9.2.7. a) Since both sets are nonempty and $\|\mathbf{x} - \mathbf{y}\|$ is bounded below by 0, the dist (A, B) exists and is finite. By the Approximation Property for Infima, choose $\mathbf{x}_k \in A$ and $\mathbf{y}_k \in B$ such that $\|\mathbf{x}_k - \mathbf{y}_k\| \to$ dist (A, B). Since A and B are closed and bounded, use the Bolzano-Weierstrass Theorem to choose subsequences such that $\mathbf{x}_{k_j} \to \mathbf{x}_0 \in A$ and $\mathbf{y}_{k_j} \to \mathbf{y}_0 \in B$. Since $A \cap B = \emptyset$, $\mathbf{x}_0 \neq \mathbf{y}_0$. Hence dist $(A, B) = \|\mathbf{x}_0 - \mathbf{y}_0\| > 0$.

b) Let $A = \{(x, y) : y = 0\}$ and $B = \{(x, y) : y = 1/x\}$. Then A and B are closed, $A \cap B = \emptyset$, but dist $(A, B) = 0$ because $1/x \to 0$ as $x \to \infty$.

9.2.8. Suppose that $a < b$. Set

$$f(t) := \begin{cases} e^{-1/(t-a)^2} e^{-1/(t-b)^2} & t \neq 0 \\ 0 & t = 0, \end{cases}$$

and observe by Exercise 4.4.7 that f is nonnegative and $\mathcal{C}^\infty$ on $\mathbf{R}$, f is positive on (a, b), and $f = 0$ on $(a, b)^c$. Since for each $x \in K := \overline{E} \subset V$ there exists an open interval I_x such that $x \in I_x \subset V$, it follows that for each $x \in K$ there is a $\mathcal{C}^\infty$ function $f_x \geq 0$ such that $f_x > 0$ for $x \in I_x$ and $f_x = 0$ for $x \notin I_x$.

Clearly, K is compact and $\{I_x\}_{x\in K}$ is an open covering of K. Thus by the Heine-Borel Theorem, there exist $x_1, \ldots, x_N \in K$ such that $K \subset \bigcup_{j=1}^N I_{x_j}$. Set $f = f_{x_1} + \cdots + f_{x_N}$. Then f is $\mathcal{C}^\infty$ on $\mathbf{R}$. If $x \in K$, then $x \in I_{x_j}$ for some j, so $f(x) \geq 0 + \cdots + f_{x_j}(x) + \cdots + 0 > 0$. Finally, if $x \notin V$, then $x \notin I_{x_j}$ for any j, so $f(x) = 0 + \cdots + 0 = 0$.

9.3 Limits of Functions.

9.3.1. a) The domain of f is all $(x, y) \in \mathbf{R}^2$ such that $x \neq 1$ and $y \neq 1$. By Theorem 9.16,

$$\lim_{(x,y)\to(1,-1)} \left(\frac{x-1}{y-1}, x+2\right) = (0, 3).$$

b) The domain of f is all $(x,y)\in\mathbf{R}^2$ such that $x\neq 0$, $y\neq 0$, and $x/y\neq(2k+1)\pi/2$ for $k\in\mathbf{Z}$ (for example, $\tan(\pi/2)$ is undefined). By Theorem 9.16 and L'Hôpital's Rule,

$$\lim_{(x,y)\to(0,1)}\left(\frac{y\sin x}{x},\tan\frac{x}{y},x^2+y^2-xy\right)=(1,0,1).$$

c) The domain of f is all $(x,y)\in\mathbf{R}^2$ such that $(x,y)\neq(0,0)$. Since $x^4+y^4\le(x^2+y^2)x^2+(x^2+y^2)y^2=(x^2+y^2)^2$, $(x^4+y^4)/(x^2+y^2)\le x^2+y^2\to 0$ as $(x,y)\to(0,0)$. Since $2|xy|\le(x^2+y^2)$, $\sqrt{|xy|}/\sqrt[3]{x^2+y^2}\le(x^2+y^2)^{1/2-1/3}\to 0$ as $(x,y)\to(0,0)$. Therefore,

$$\lim_{(x,y)\to(0,1)}\left(\frac{x^4+y^4}{x^2+y^2},\frac{\sqrt{|xy|}}{\sqrt[3]{x^2+y^2}}\right)=(0,0).$$

d) The domain of f is all $(x,y)\in\mathbf{R}^2$ such that $(x,y)\neq(1,1)$. The second component factors:

$$\left|\frac{x^2y-2xy+y-(x-1)^2}{x^2+y^2-2x-2y+2}\right|=\left|\frac{(y-1)(x-1)^2}{(x-1)^2+(y-1)^2}\right|\le|y-1|.$$

Hence it converges to 0 as $(x,y)\to(1,1)$ by the Squeeze Theorem. Therefore,

$$\lim_{(x,y)\to(1,1)}\left(\frac{x^2-1}{y^2+1},\frac{x^2y-2xy+y-(x-1)^2}{x^2+y^2-2x-2y+2}\right)=(0,0).$$

9.3.2. a) The iterated limits are 0. If $x=y$, then $f(x,y)=\sin^2 x/(2x^2)\to 1/2$ as $x\to 0$. Thus this function has no limit as $(x,y)\to(0,0)$.

b) Since $\lim_{y\to 0}\lim_{x\to 0}f(x,y)=1/2$ and $\lim_{x\to 0}\lim_{y\to 0}f(x,y)=1$, this function has no limit as $(x,y)\to(0,0)$.

c) Since $|x|$ and $|y|$ are $\le\sqrt{x^2+y^2}$, $|f(x,y)|\le 2(x^2+y^2)^{1/2-\alpha}$. This last term converges to 0 as $(x,y)\to(0,0)$ since $\alpha<1/2$. Therefore, the limit exists and is 0.

9.3.3. a) Since

$$\left|\frac{x^3-y^3}{x^2+y^2}\right|\le|x|\frac{x^2}{x^2+y^2}+|y|\frac{y^2}{x^2+y^2}\le|x|+|y|\to 0$$

as $(x,y)\to(0,0)$, the limit exists and is 0.

b) Since $y^4/(x^2+y^4)\le 1$ and $\alpha>0$, $|f(x,y)|\le|x^\alpha|\to 0$ as $(x,y)\to(0,0)$. Therefore, the limit exists and is 0.

9.3.4. Let $\mathbf{a}\in\mathbf{R}^n$. Suppose for a moment that the projection function $f_k(\mathbf{x})=x_k$ has a limit as $\mathbf{x}\to\mathbf{a}$ and satisfies $f_k(\mathbf{x})\to f_k(\mathbf{a})$ as $\mathbf{x}\to\mathbf{a}$ for each $k\in\{1,\dots,n\}$. Then by Theorem 9.15 (the limit of the product is the product of the limits), $f_1^{j_1}(\mathbf{x})\dots f_n^{j_n}(\mathbf{x})=x_1^{j_1}\dots x_n^{j_n}\to f_1^{j_1}(\mathbf{a})\dots f_n^{j_n}(\mathbf{a})=a_1^{j_1}\dots a_n^{j_n}$ as $\mathbf{x}\to\mathbf{a}$ for any nonnegative integers $j_1,\dots,j_n$. Hence by Theorem 9.15 (the limit of the sum is the sum of the limits), $P(\mathbf{x})\to P(\mathbf{a})$ as $\mathbf{x}\to\mathbf{a}$. In particular, it suffices to prove that $f_j(\mathbf{x})\to f_j(\mathbf{a})$.

Let $\epsilon>0$ and set $\delta=\epsilon$. If $\|\mathbf{x}-\mathbf{a}\|\le\delta$ then by Theorem 8.5, $|f_j(\mathbf{x})-f_j(\mathbf{a})|=|x_j-a_j|<\delta=\epsilon$. Therefore, $f_j(\mathbf{x})\to f_j(\mathbf{a})$ as $\mathbf{x}\to\mathbf{a}$.

9.3.5. Let $\varepsilon=1$ and choose $\delta>0$ such that $0<\|\mathbf{x}-\mathbf{a}\|<\delta$ implies $\|f(\mathbf{x}-\mathbf{L}\|<1$. Then $\|f(\mathbf{x})\|<\|\mathbf{L}\|+1$ for all $\mathbf{x}\in B_\delta(\mathbf{a})\setminus\{\mathbf{a}\}$. Thus set $V=B_\delta(\mathbf{a})$ and $M=\max\{\|f(\mathbf{a})\|,\|\mathbf{L}\|+1\}$.

9.3.6. a) We begin by proving that if $g(x,y):=f(x)$ and $f(x)\to f(a)$ as $x\to a$, then $g(x,y)\to f(a)$ as $(x,y)\to(a,b)$ no matter what b is. Indeed, let $\varepsilon>0$ and choose $\delta>0$ such that $|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$. Then $0<\|(x,y)-(a,b)\|<\delta$ implies $|x-a|<\delta$, so $g(x,y)-f(a)|=|f(x)-L|<\varepsilon$.

Iterating what we just proved, using the fact that the limit of the product is the product of the limits, we see that $g(\mathbf{x})\to f_1(a_1)\cdots f_n(a_n)$ as $\mathbf{x}\to\mathbf{a}$.

b) Define f on $\mathbf{R}$ by $f(x)=x$ for $x\neq 0$ and $f(0)=1$. Then $f(x)\to 0$ as $x\to 0$, but $g(x,y)=f(x)(y+1)$ does not have a limit as $(x,y)\to(0,0)$. Indeed, the vertical path $x=0$ yields $g(0,y)=f(0)(y+1)\to 1$ as $y\to 0$ but the horizontal path $y=0$ yields $g(x,y)=f(x)(0+1)\to 0$ as $x\to 0$.

9.3.7. By the Mean Value Theorem, $g(x)=g(x)-g(1)=g'(c)(x-1)$ for some c between x and 1. Thus $|g(x)|>|x-1|$. It follows that

$$|f(x,y)|<\frac{|x-1|^2\,|y+1|}{|x-1|\,|y|}=|x-1|\left|\frac{y+1}{y}\right|.$$

If $(x, y) \to (1, b)$ for some $b > 0$, we can assume that $0 < y_0 \le y \le y_1$ for some $y_0 < b < y_1$, so $|(y+1)/y| \le (1+y_1)/y_0 =: M$. Thus $|f(x,y)| \le M|x-1|$, and it follows from the Squeeze Theorem that $f(x,y) \to 0 =: L$ as $(x,y) \to (1,b)$. A similar argument works for $b < 0$.

9.3.8. a) Repeat the proof of Remark 3.4, replacing the absolute value by the norm sign.
b) Repeat the proof of Theorem 3.6, replacing the absolute value by the norm sign.
c) Repeat the proof of Theorem 3.8, replacing the absolute value by the norm sign.
d) Repeat the proof of Theorem 3.9, replacing the absolute value by the norm sign.

9.4 Continuous Functions.

9.4.1. a) $f(0,\pi) = (0,1]$ is not open and we don't expect it to be; $f[0,\pi] = [0,1]$ is compact and connected as Theorems 9.29 and 9.30 say it should; $f(-1,1) = (-\sin 1, \sin 1)$ is open, big deal; $f[-1,1] = [-\sin 1, \sin 1]$ is compact and connected as Theorems 9.29 and 9.30 say it should.

$g(0,\pi) = \{1\}$ is connected as Theorem 9.30 says it should; $g[0,\pi] = \{0,1\}$ is compact but not connected–note that Theorem 9.29 does not apply since g is not continuous; $g(-1,1) = \{-1,0,1\}$ is not open and we don't expect it to be; $g[-1,1] = \{-1,0,1\}$ is compact but not connected–note that Theorem 9.29 does not apply since g is not continuous.

b) $f^{-1}(0,\pi) = \dots(0,\pi)\cup(2\pi,3\pi)\cup\dots$ is open as Theorem 9.26 says it should; $f^{-1}[0,\pi] = \dots[0,\pi]\cup[2\pi,3\pi]\cup\dots$ is closed as Exercise 9.4.4 says it should; $f^{-1}(-1,1) = \mathbf{R} \setminus \{x : x = (2k+1)\pi/2, k \in \mathbf{Z}\}$ is open as Theorem 9.26 says it should; $f^{-1}[-1,1] = \mathbf{R}$ is closed as Exercise 9.4.4 says it should.

$g^{-1}(0,\pi) = (0,\infty)$ is open, no big deal; $g^{-1}[0,\pi] = [0,\infty)$ is closed–note that Exercise 9.4.4 does not apply since g is not continuous; $g^{-1}(-1,1) = \{0\}$ is not open and we don't expect it to be; $g^{-1}[-1,1] = \mathbf{R}$ is closed–note that Exercise 9.4.4 does not apply since g is not continuous.

9.4.2. a) $f(0,1) = (0,1)$ is open, no big deal; $f[0,1) = [0,1)$ is neither open nor closed; $f[0,1] = [0,1]$ is compact and connected as Theorems 9.29 and 9.30 say it should.

$g(0,1) = (1,\infty)$ is connected as Theorem 9.30 says it should; $g[0,1) = \{0\} \cup (1,\infty)$ is neither compact nor connected–note that Theorems 9.29 and 9.30 do not apply since g is not continuous; $g[0,1] = \{0\} \cup [1,\infty)$ is neither compact nor connected–note that Theorems 9.29 and 9.30 do not apply since g is not continuous.

b) $f^{-1}(-1,1) = [0,1)$ is relatively open in $[0,\infty)$, the domain of f as Theorem 9.26 says it should; $f^{-1}[-1,1] = [0,1]$ is relatively closed in $[0,\infty)$ as Exercise 9.4.5a says it should.

$g^{-1}(-1,1) = (-\infty,-1)\cup(1,\infty)\cup\{0\}$ is not open and g is not continuous; $g^{-1}[-1,1] = (-\infty,-1]\cup[1,\infty)\cup\{0\}$ is closed, no big deal–note that Exercise 9.4.4 does not apply since g is not continuous.

9.4.3. Recall that $f^{-1}(V)$ is relatively open in A if and only if $f^{-1}(E) = O \cap A$ for some open O in $\mathbf{R}^n$. But the intersection of two open sets is an open set. Thus if A is open, then $f^{-1}(V)$ is relatively open in A if and only if it is open in $\mathbf{R}^n$.

9.4.4. Suppose f is continuous on B, and that E is a closed subset of $\mathbf{R}^m$. If $\mathbf{x}_k \in f^{-1}(E) \cap B$ and $\mathbf{x}_k \to \mathbf{a}$, then $\mathbf{x}_k \in B$ and $f(\mathbf{x}_k) \in E$. Since f is continuous, and both B and E are closed, it follows that $\mathbf{a} \in B$ and $f(\mathbf{a}) \in E$. Thus $\mathbf{a} \in f^{-1}(E) \cap B$. By Theorem 9.8, then, $f^{-1}(E) \cap B$ is closed.

Conversely, suppose $f^{-1}(E) \cap B$ is closed for every closed E in $\mathbf{R}^m$ but f is NOT continuous at some $\mathbf{a} \in B$. Then there is a sequence $\mathbf{x}_k \in B$ such that $f(\mathbf{x}_k)$ does NOT converge to $f(\mathbf{a})$. Hence, there is an $\varepsilon_0 > 0$ and k_j such that $\|f(\mathbf{x}_{k_j}) - f(\mathbf{a})\| \ge \varepsilon_0$. Now $f(\mathbf{x}_{k_j} \in B^c_{\varepsilon_0}(\mathbf{a})$ so $\mathbf{x}_{k_j} \in f^{-1}(B^c_{\varepsilon_0}(\mathbf{a})) \cap B$. Since this set is closed, the limit $\mathbf{a}$ also belongs to it. In particular, $f(\mathbf{a}) \notin B_{\varepsilon_0}(\mathbf{a})$, a contradiction.

9.4.5. a) By Exercise 8.3.8b, a set E is relatively open in some set B if and only if its complement $B \setminus E$ is relatively closed in B.

Suppose f is continuous on E and A is closed in $\mathbf{R}^m$. Then A^c is open in $\mathbf{R}^m$, so by Theorem 9.26, $A_0 := f^{-1}(A^c) \cap E$ is relatively open in E. This means that there is an open set V in $\mathbf{R}^n$ such that $A_0 = V \cap E$. Since

$$E \cap f^{-1}(A) = E \setminus A_0 = E \cap V^c$$

and V^c is closed, it follows that $f^{-1}(A) \cap E$ is relatively closed in E. A similar argument proves that if $f^{-1}(A) \cap E$ is relatively closed in E for all closed sets A in $\mathbf{R}^m$, then f is continuous on E.

b) Let V be relatively open in $f(E)$, i.e., $V = U \cap f(E)$ for some U open in Y. By Theorem 9.26, $f^{-1}(U) \cap E$ is relatively open in E. But by Theorem 1.37,

$$f^{-1}(V) \cap E = f^{-1}(V \cap f(E)) = f^{-1}(U \cap f(E)) = f^{-1}(U) \cap E.$$

Hence $f^{-1}(V)$ is relatively open in E.

A similar proof using part b) in place of Theorem 9.26 proves that $f^{-1}(A) \cap E$ is relatively closed in E for every A relatively closed in $f(E)$.

9.4.6. By Theorem 9.39, $f(x, y)$ is continuous at every point (x, y) which satisfies $x \neq y$. Let (x_0, y_0) be a point where $x_0 = y_0$. Since $e^{-1/t} \to 0$ as $t \to 0+$, given $\epsilon > 0$ choose $\delta > 0$ such that $0 < t < \delta$ implies $e^{-1/t} < \epsilon$. If $\|(x, y) - (x_0, y_0)\| < \delta/2$ then $|x - y| = |x - x_0 + y_0 - y| \leq |x - x_0| + |y - y_0| \leq 2\|(x, y) - (x_0, y_0)\| < \delta$. Hence $e^{-1/|x-y|} < \epsilon$, i.e., $f(x, y) \to 0 = f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$. Thus f is continuous on $\mathbf{R}^2$.

9.4.7. a) Since f is continuous, so is $\|f\|$. Therefore, $\|f\|_H$ is finite and attained by the Extreme Value Theorem.

b) By definition, $f_n \to f$ in $\mathcal{C}[a, b]$ if and only if given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \geq N$ implies

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \varepsilon.$$

Since this last statement is equivalent to $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$, we have that $f_n \to f$ in $\mathcal{C}[a, b]$ if and only if $f_n \to f$ uniformly on $[a, b]$.

c) By part b), if f_k converges uniformly, then $\|f_k - f_j\|$ is small when k and j are large. Conversely, if $\|f_k - f_j\| < \varepsilon$ for $k, j \geq N$, then $f_k(\mathbf{x})$ is Cauchy in Y for each $\mathbf{x} \in H$. By Theorem 9.6, it follows that $f_k \to$ some function f pointwise on H. Letting $j \to \infty$ in the inequality $\|f_k - f_j\| < \varepsilon$, we obtain $\|f_k - f\| \leq \varepsilon$ for $k \geq N$. By part b), then, $f_k \to f$ uniformly on H.

9.4.8. By the proof of Lemma 3.38, if f is uniformly continuous, then f takes a Cauchy sequence in E to a Cauchy sequence in $\mathbf{R}^m$. So, let $x \in E$ and choose $x_k \in D$ such that $x_k \to x$. Then $f(x_k)$ is Cauchy, hence convergent in $\mathbf{R}^m$. Define $g(x) := \lim_{k \to \infty} f(x_k)$. By the argument of Theorem 3.40, this definition is independent of the sequence x_k chosen to approximate x. Thus g is well defined on all of E. Moreover, g is continuous by the Sequential Characterization of Limits.

9..4.9. Suppose without loss of generality that $f(a) < f(b)$. By Theorem 9.30, $f(E)$ is connected in $\mathbf{R}$, which by Theorem 8.30 means $f(E)$ is an interval. Since $f(a), f(b) \in f(E)$, it follows that $[f(a), f(b)] \subset f(E)$. In particular, $y \in f(E)$.

9.4.10. a) Suppose E is polygonally connected but some pair of open sets U, V separates E. Let $\mathbf{x}_1 \in E \cap U$, $\mathbf{x}_2 \in E \cap V$. Since E is polygonally connected, there is a continuous function $f : [0, 1] \to E$ with $f(0) = \mathbf{x}_1$ and $f(1) = \mathbf{x}_2$. By Theorems 8.30 and 9.30, $f([0, 1])$ is connected. But since $f([0, 1]) \subseteq E$, U, V separates $f([0, 1])$, a contradiction.

b) Let $\mathbf{x} \in U$. Since E is open, choose $r > 0$ such that $B_r(\mathbf{x}) \subset E$. Let $\mathbf{y} \in B_r(\mathbf{x})$ and let P be a polygonal path from $\mathbf{x}_0$ to $\mathbf{x}$ which lies in E. Then the path $P \cup L(\mathbf{x}; \mathbf{y})$ goes from $\mathbf{x}_0$ to $\mathbf{y}$ and lies in E, i.e., $\mathbf{y} \in U$. Therefore, $B_r(\mathbf{x}) \subseteq U$ and U is open.

c) Suppose E is open and connected but not polygonally connected. By part b), given any $\mathbf{x} \in E$ the set $U_{\mathbf{x}}$ which can be polygonally connected to $\mathbf{x}$ through E is open. Since E is not polygonally connected, there exist points $\mathbf{x}_0 \neq \mathbf{y}_0$ in E such that $U_{\mathbf{x}_0} \cap U_{\mathbf{y}_0} = \emptyset$. Let $U := U_{\mathbf{x}_0}$ and $V := \cup\{U_{\mathbf{y}} : U_{\mathbf{y}} \cap U_{\mathbf{x}_0} = \emptyset\}$. Then U, V are nonempty open sets, $U \cap V = \emptyset$ and $U \cup V = E$. In particular, the pair U, V separates E, a contradiction.

9.5 Compact Sets.

9.5.1. a) Since $1/k \to 0$ as $k \to \infty$, this set is closed and bounded, hence compact.

b) This set is closed and bounded, hence compact.

c) This set is bounded but not closed. If we set $H := \{(0, y) : -1 \leq y \leq 1\}$ then $E \cup H$ is closed and bounded, hence compact.

d) The set is closed but not bounded (since $(n, 1/n) \in E$ for all $n \in \mathbf{N}$). Therefore, E is neither compact nor is it contained in any compact set.

9.5.2. Let A, B be compact sets. If $\{V_\alpha\}$ is an open covering of $A \cup B$, then it is a covering of A and B. Since these are compact sets, we can choose $V_1, \dots, V_N$ to cover A and $V_{N+1}, \dots, V_M$ to cover B. Clearly, $V_1, \dots, V_M$ covers $A \cup B$. Thus $A \cup B$ is compact.

By Remark 9.37, A and B are closed sets, so by Theorem 8.24, $A \cap B$ is a closed subset of the compact set A. It follows from Remark 9.38 that $A \cap B$ is compact.

9.5.3. Since E is compact, it is bounded by the Heine-Borel Theorem. Since it is nonempty, it follows from the Completeness Axiom that E has a finite supremum. By the Approximation Property, choose $x_k \in E$ such that

$x_k \to \sup E$. Since E is closed, we have by Theorem 9.8 that $\sup E = \lim_{k\to\infty} x_k \in E$. A similar argument works for $\inf E$.

9.5.4. Suppose A is uncountable. Since each V_α is nonempty, choose a point $x_\alpha \in V_\alpha$ for each $\alpha \in A$. Since $V_\alpha \cap V_\beta = \emptyset$ for $\alpha \neq \beta$, the set $E = \{x_\alpha : \alpha \in A\}$ is uncountable. But $E \subset \cup_{\alpha\in A} V_\alpha$, hence by Lindelöf's Theorem, $E \subset \cup_{\alpha\in A_0} V_\alpha$ for some countable subset A_0 of A. Since each V_α contains at most one x_β, it follows that E is countable, a contradiction.

The result is false if "open" is omitted. By Remark 1.41, the unit interval $(0,1)$ is uncountable, hence $\{x\}_{x\in(0,1)}$ is an uncountable collection of pairwise disjoint nonempty sets which covers the unit interval $(0,1)$.

9.5.5. By Exercise 8.3.8, there exist $\epsilon := \epsilon_x > 0$ such that $V = \cup_{x\in V} B_\epsilon(x)$. But by Lindelöf (or using rational centers and rational radii as in the proof of the Borel Covering Lemma), we can find open balls $B_j := B_{\epsilon_j}(x_j)$ such that $V \subseteq \cup_{j=1}^\infty B_j$. On the other hand, $\cup_{j=1}^\infty B_j \subseteq V$ since each $B_j \subseteq V$. Therefore $V = \cup_{j=1}^\infty B_j$ as required.

9.5.6. a) Suppose E is compact, and let $x_k \in E$. By the Heine-Borel Theorem, x_k is bounded, hence (by Bolzano–Weierstrass) has a convergent subsequence. Since all compact sets are closed, the limit of this subsequence must belong to E. Thus E is sequentially compact.

b) By definition and Theorem 9.8, E is closed. It must also be bounded. Indeed, if not, e.g., if $\|x_k\| \to \infty$ for some $x_k \in E$, then choose (by sequential compactness) a convergent subsequence of x_k, say x_{k_j}. Since it converges, it must be bounded. But $\|x_{k_j}\| \to \infty$ so it cannot be bounded.

c) If E is compact, then by part a), E is sequentially compact. If E is sequentially compact, then by part b), E is closed and bounded. Finally, if E is closed and bounded then (by the Heine-Borel Theorem) E is compact.

9.5.7. a) Suppose H is compact. Let $\mathcal{E} := \{U_\alpha\}_{\alpha\in A}$ be a relatively open covering of H. Choose V_α, open in $\mathbf{R}^n$, such that $U_\alpha = H \cap V_\alpha$. Then $\{V_\alpha\}_{\alpha\in A}$ is an open covering of H. Since H is compact, there exists a finite subset A_0 of A such that $\{V_\alpha\}_{\alpha\in A_0}$ covers H. In particular, $\{U_\alpha\}_{\alpha\in A_0}$ is a finite subcovering of $\mathcal{E}$ which covers H.

Conversely, suppose every relatively open covering of H has a finite subcover. If $\{V_\alpha\}_{\alpha\in A}$ is an open covering of H then $\{H \cap V_\alpha\}_{\alpha\in A}$ is a relatively open covering of H. Therefore, there exists a finite subset A_0 of A such that $\{H \cap V_\alpha\}_{\alpha\in A_0}$ covers H. In particular, $\{V_\alpha\}_{\alpha\in A_0}$ covers H and H is compact.

b) If $\{V_\alpha\}$ is an open covering of $f(H)$, then $\{f^{-1}(V_\alpha)\}$ is a relatively open covering of H. By part a), there exist $\alpha_1, \dots, \alpha_N$ such that

$$H \subset \bigcup_{j=1}^N f^{-1}(V_{\alpha_j}).$$

Thus $f(H) \subset \bigcup_{j=1}^N V_{\alpha_j}$, i.e., $f(H)$ is compact.

9.6 Applications.

9.6.1. Since $f_k \geq 0$, the partial sums of $\sum_{k=1}^\infty f_k$ are increasing on $[a,b]$. Hence by Dini's Theorem the series converges uniformly on $[a,b]$ and can be integrated term by term.

9.6.2. By the Extreme Value Theorem, f_1 is bounded on E and by Dini's Theorem, $\sum_{k=1}^\infty g_k = g$ uniformly on E. Hence by Exercise 7.2.7, $\sum_{k=1}^\infty f_k g_k$ converges uniformly on E.

9.6.3. Given $\epsilon > 0$, choose M so large that $|f(x)| < \epsilon/2$ for $|x| > M$. By Dini's Theorem, $f_k \to f$ uniformly on $[-M,M]$. Hence there is an N so large that $k \geq N$ and $x \in [-M,M]$ imply $|f_k(x) - f(x)| < \epsilon$. Let $k \geq N$ and $x \in \mathbf{R}$. If $x \in [-M,M]$ then $|f_k(x) - f(x)| < \epsilon$. If $x \notin [-M,M]$ then $|f_k(x) - f(x)| \leq |f_k(x)| + |f(x)| \leq 2|f(x)| < \epsilon$. Therefore, $f_k \to f$ uniformly on $\mathbf{R}$.

9.6.4. Let $h > 0$ and $t \in \mathbf{R}$.

a) $\Omega_f(t-h, t+h) = 1$ so $\omega_f(t) = 1$ for all t.

b) $\Omega_f(t-h, t+h) = 0$ for $t \neq 0$ when h is small, and $= 1$ when $t = 0$. Thus $\omega_f(t) = 0$ if $t \neq 0$ and $\omega_f(0) = 1$.

c) $\Omega_f(-h, h) = 2$ for all $h \neq 0$ so $\omega_f(0) = 2$. Since f is continuous at any $t \neq 0$, $\Omega_f(t-h, t+h)$ gets smaller as $h \to 0$, so $\omega_f(t) = 0$ for $t \neq 0$.

9.6.5. Set $\delta = x/k$. If $x \geq -1$ then $\delta \geq -1$. Hence by Bernoulli's Inequality, $(1+x/k)^{k/(k+1)} \leq 1 + x/(k+1)$ for all $x \geq -1$. Thus $(1-x/k)^k$ is an increasing sequence of continuous functions. By L'Hôpital's Rule, this sequence converges to e^{-x} as $k \to \infty$ for all $x \in \mathbf{R}$. Hence it follows from Dini's Theorem that $(1-x/k)^k \to e^{-x}$ as $k \to \infty$ uniformly on any compact subset of $\mathbf{R}$. Here is a different argument which does not use Bernoulli's Inequality.

Let $\phi(t) = t\log(1 - x/t)$. Notice that $\phi'(t) = t(1 - x/t)^{-1}(x/t^2) + \log(1 - x/t) = u/(1 - u) + \log(1 - u)$ for $u = x/t$. Let $\psi(u) = u/(1 - u) + \log(1 - u)$, $u \in (-\infty, 1)$. Since $\psi'(u) = u/(1 - u)^2$, ψ has an absolute minimum of 0 at 0. Thus $\psi(u) \geq 0$ for all $u \in (-\infty, 1)$, i.e., $\phi'(t) \geq 0$ for all $t > x$. Thus $\phi(t)$ is increasing for $t > x$, so $e^{\phi(k)} = (1 - x/k)^k \uparrow e^{-x}$ as $k \to \infty$ for all $x \in \mathbf{R}$.

9.6.6. Since g is continuous, any point of discontinuity of f is a point of discontinuity of $g \circ f$ and vice versa. By Lebesgue's Theorem, f is almost everywhere continuous. Thus $g \circ f$ is almost everywhere continuous, i.e., $g \circ f$ is integrable by Lebesgue's Theorem.

9.6.7. a) Fix $x \in [0, \pi/2]$ and let $f(t) = 2t/(4t - 3x)$, $t \geq 2$. Since $f'(t) = -6x/(4t - 3x)^2 < 0$ for all t and $\sqrt{u}$ is increasing in u, the sequence $\sin x \sqrt{2k/(4k - 3x)}$ is decreasing for each $x \in [0, \pi/2]$. It converges to the continuous function $\sin x \sqrt{1/2}$ as $k \to \infty$. Therefore, by Dini's Theorem and Theorem 7.10,

$$\lim_{k\to\infty} \int_0^{\pi/2} \sin x \sqrt{\frac{2k}{4k - 3x}}\, dx = \sqrt{\frac{1}{2}} \int_0^{\pi/2} \sin x\, dx = \sqrt{\frac{1}{2}}.$$

b) Since $f'(0) > 0$ and f' is continuous, $f'(x) > 0$ for x in some $[0, a]$, $a > 0$. In particular, f is increasing on $[0, a]$. Fix $x \in [0, 1]$ and choose k_0 so large that $1/k_0 < a$. Then $k/(k^2 + x) \leq k/(k^2 + 0) = 1/k < a$ for $k \geq k_0$. Moreover, if $g(t) = t/(t^2 + x)$, $t \geq 1$, then $g'(t) = (x - t^2)/(t^2 + x)^2 < 0$ for all $t > \sqrt{x}$. It follows that the sequence $x^2 f(k/(k^2 + x))$ is decreasing for each $x \in [0, 1]$ for $k \geq k_0$. It converges to the continuous function $x^2 f(0)$ as $k \to \infty$. Therefore, by Dini's Theorem and Theorem 7.10,

$$\lim_{k\to\infty} \int_0^1 x^2 f\left(\frac{k}{k^2 + x}\right) dx = \int_0^1 x^2 f(0)\, dx = \frac{f(0)}{3}.$$

c) Fix $x \in [0, 1]$. Set $f(t) = (\log t + x)/(t + x)$ and $g(t) = 1 + x/t - \log t - x$, for $t > 1$. Notice that $f'(t) = g(t)/(t + x)^2$ and $g'(t) = -x/t^2 - 1/t$. Since $g'(t) < 0$ for all $x > 0$ and $g(t) \to -\infty$ as $t \to \infty$, it follows that $g(t)$, hence $f'(t)$, is negative for large t. Therefore, the sequence $(\log k + x)/(k + x)$ is eventually decreasing for each $x \in [0, 1]$. It converges to the continuous function 0 as $k \to \infty$, so by Dini's Theorem, $(\log k + x)/(k + x) \to 0$ uniformly on $[0, 1]$ as $k \to \infty$. Thus $\cos((\log k + x)/(k + x)) \to \cos 0 = 1$ uniformly on $[0, 1]$ as $k \to \infty$. We conclude by Theorem 7.10 that

$$\lim_{k\to\infty} \int_0^1 x^3 \cos\left(\frac{\log k + x}{k + x}\right) dx = \int_0^1 x^3\, dx = \frac{1}{4}.$$

d) Set $\delta = x/k$. If $x \geq -1$ then $\delta \geq -1$. Hence by Bernoulli's Inequality, $(1 + x/k)^{k/(k+1)} \leq 1 + x/(k + 1)$ for all $x \geq -1$. It follows from L'Hôpital's Rule that $(1 + x/k)^k \uparrow e^x$ for $x \in [-1, 1]$. Hence by Dini's Theorem and Theorem 7.10,

$$\lim_{k\to\infty} \int_{-1}^1 \left(1 + \frac{x}{k}\right)^k e^x\, dx = \int_{-1}^1 e^{2x}\, dx = \frac{e^4 - 1}{2e^2}.$$

9.6.8. a) Since $[0, 1] \cap \mathbf{Q}$ is countable, it can be covered by such a collection of intervals by Remark 9.42.

b) Since $[0, 1]$ is compact, $[0, 1] \subseteq \cup_{k=1}^N I_k$ for some $N \in \mathbf{N}$. If $\sum_{k=1}^N |I_k| < 1$ then some point of $[0, 1]$ is uncovered. Thus $\sum_{k=1}^N |I_k| \geq 1$.

9.6.9. a) By construction, E_k, hence E, can be covered by a finite collection of intervals of total length $2^k/3^k$. Since $(2/3)^k \to 0$ as $k \to \infty$, it follows that E is of measure zero.

b) By construction, to pass from E_{k-1} to E_k we eliminate each point which has a 1 as the kth digit in all of its ternary expansions. Thus $x \in E$ if and only if x has a ternary expansion whose digits are never 1.

c) Given $x \in E$, let $x = \sum_{k=1}^\infty b_k/3^k$ where $b_k \neq 1$. Consider the function $f(x) = \sum_{k=1}^\infty (b_k/2)/2^k$. As x ranges over E, the b_k's exhibit all possible combinations of 0's and 2's, hence the binary coefficients of $f(x)$ exhibit all possible combinations of 0's and 1's. In particular, f takes E onto $[0, 1]$. It is clear by construction that f is 1–1. Thus E is uncountable.

d) It is clear by construction that f is increasing on $[0, 1]$. Suppose f has a point of discontinuity $x_0 \in [0, 1]$. By Theorem 4.18, $0 \leq f(x_0-) < f(x_0+) \leq$, i.e., $(f(x_0-), f(x_0+)) \subseteq [0, 1]$ but $(f(x_0-), f(x_0+)) \cap f([0, 1]) = \emptyset$. This contradicts the fact that f takes $[0, 1]$ onto $[0, 1]$. Therefore, f is continuous at each point $x \in [0, 1]$.

10.1 Introduction.

10.1.1. Since $\rho(a,b)$ is a nonnegative real number, it follows from Theorem 1.9 that $\rho(a,b) = 0$. Since ρ is positive definite, we conclude that $a = b$.

10.1.2. a) Suppose x_n is bounded. By Definition 10.13, there is a $b \in X$ and an $M > 0$ such that $x_n \in B_M(b)$, i.e., $\rho(x_n, b) < M$ for all $n \in \mathbf{N}$. It follows from the triangle inequality that $M + \rho(a,b)$ is an upper bound for the nonempty set $\{\rho(x_n, a) : n \in \mathbf{N}\}$. Hence by the Completeness Axiom, this set has a finite supremum.

Conversely, if this set has a finite supremum s for all $a \in X$, then let $b = a \in X$. Then $s \geq \rho(x_n, b)$ for all $n \in \mathbf{N}$, i.e., $x_n \in B_s(b)$ for all $n \in \mathbf{N}$. Hence x_n is bounded by Definition 10.13

b) Apply part a) with $a = 0$ and $\rho(a,b) = \|a - b\|$.

10.1.3. a) By Remark 8.7, for all $j \in \{1, 2, \dots, n\}$ we have $|x_k^{(j)}| \leq \|\mathbf{x}_k\| \leq \sqrt{n}\|\mathbf{x}\|_\infty$. Thus $x_k^{(j)}$ is bounded in k for all j if and only if $\|\mathbf{x}_k\|$ is bounded in k. By Exercise 10.1.2 (with $a = \mathbf{0}$), this is equivalent to $\{\mathbf{x}_k\}$ is bounded in $\mathbf{R}^n$.

b) See the proof of Theorem 9.2.

10.1.4. a) If $x_n = a$ for all n, then $\rho(x_n, a) = 0$ is less than any positive ε for all $n \in \mathbf{N}$.

b) If $x_n \to a$ in the discrete space, then for n large, $\sigma(x_n, a) < 1$. But by definition, $\sigma(x_n, a) < 1$ implies $\sigma(x_n, a) = 0$, i.e., $x_n = a$.

10.1.5. a) Let a be the common limit point. Given $\varepsilon > 0$, choose $N \in \mathbf{N}$ such that $n \geq N$ implies $\rho(x_n, a), \rho(y_n, a) < \varepsilon/2$. By the Triangle Inequality, $n \geq N$ implies

$$\rho(x_n, y_n) \leq \rho(x_n, a) + \rho(y_n, a) < \varepsilon.$$

By definition, $\rho(x_n, y_n) \to 0$ as $n \to \infty$.

b) Let $x_n = n$ and $y_n = n + 1/n$. Then $|x_n - y_n| = 1/n \to 0$ as $n \to \infty$, but neither x_n nor y_n converges.

10.1.6. By Theorem 10.14, if $x_n \to a$ then $x_{n_k} \to a$. Conversely, if x_n is Cauchy and $x_{n_k} \to a$, then given $\varepsilon > 0$ there is an N such that $n, k \geq N$ implies $\rho(x_n, x_{n_k}) < \varepsilon/2$. Hence, $\rho(x_n, a) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, a) \leq \varepsilon$ for k large. By definition, then, $x_n \to a$ as $n \to \infty$.

10.1.7. If x_n is Cauchy, then there is an N such that $n \geq N$ implies $\sigma(x_N, x_n) < 1$. Since this last inequality is satisfied only when $x_n = x_N$, it follows that $x_n = x_N := a$ for large n. In particular, $x_n \to a$ as $n \to \infty$.

10.1.8. a) If f_n is Cauchy in $\mathcal{C}[a,b]$, then given $\varepsilon > 0$ there is an N such that $m, n \geq N$ implies

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \varepsilon$$

for all $x \in [a,b]$. Thus f_n is uniformly Cauchy. It follows from Lemma 7.11 that $f_n \to f$ uniformly on $[a,b]$, i.e., that $\|f_n - f\| \to 0$ as $n \to \infty$.

b) Clearly, $\|f\|_1 \geq 0$ and $\|f\|_1 = 0$ if and only if $f = 0$ (see Exercise 5.1.4b). Thus $\|f - g\|_1$ is positive definite. Also, by the homogeneous property of integration, $\|\alpha f\|_1 = |\alpha|\, \|f\|_1$, so $\|f - g\|_1$ is homogeneous. Finally, by the Comparison Theorem for Integrals, $|f - g| \leq |f - h| + |h - g|$ implies that $\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1$, so $\|f - g\|_1$ satisfies the triangle inequality.

c) Let $a = 0$ and $b = 1$. By elementary integration, it is easy to see that if $f_n(x) = x^n$, then $\|f_n\|_1 = 1/(n+1) \to 0$ as $n \to \infty$. However, the limit of f_n is not continuous on $[0,1]$ (see Remark 7.3).

10.1.9. a) Repeat the argument of Remark 10.9 with $\varepsilon/2$ in place of ε. This works since $\overline{B_{\varepsilon/2}}(x) \subset B_\varepsilon(x)$.

b) Let $r = \rho(a,b)/2$. If x belongs to the intersection of these balls, then

$$\rho(a,b) \leq \rho(x,a) + \rho(x,b) < 2r = \rho(a,b),$$

a contradiction.

c) By Remark 10.9, choose r_0, s_0 such that $B_{r_0}(x) \subseteq B_r(a)$ and $B_{s_0}(x) \subseteq B_s(b)$. Let $c := \min\{r_0, s_0\}$ and $d := \max\{2r, 2s\}$. If $y \in B_c(x)$ then, since $B_c(x) \subseteq B_{r_0}(x) \subseteq B_r(a)$, it is clear that $y \in B_r(a)$. Similarly, $y \in B_s(b)$. Thus $y \in B_r(a) \cap B_s(b)$.

On the other hand, let $y \in B_r(a) \cup B_s(b)$. If $y \in B_r(a)$ then, since $x \in B_r(a) \cap B_s(b)$, we have

$$\rho(x,y) \leq \rho(x,a) + \rho(a,y) < r + r = 2r < d.$$

If $y \in B_s(b)$ then, since $x \in B_r(a) \cap B_s(b)$, we have

$$\rho(x, y) \le \rho(x, b) + \rho(b, y) < s + s = 2s < d.$$

It follows that $B_r(a) \cup B_s(b) \subseteq B_d(x)$.

10.1.10. a) Let E be sequentially compact. By Theorem 10.16, E must be closed. Suppose E is not bounded, i.e., choose $x_n \in E$ and $a \in X$ such that $\rho(x_n, a) > n$ for all $n \in \mathbf{N}$. Since E is sequentially compact, choose $x_{n_k} \to b$ as $k \to \infty$. Then

$$n_k < \rho(x_{n_k}, a) \le \rho(x_{n_k}, b) + \rho(a, b) < 1 + \rho(a, b)$$

for k large. Letting $k \to \infty$, we conclude that $1 + \rho(a, b) = \infty$, a contradiction.

b) $\mathbf{R}$ is closed by Theorem 10.16. On the other hand $\{n\}$ is a sequence in $\mathbf{R}$ which has no convergent subsequence.

c) By the Bolzano–Weierstrass Theorem and Theorem 10.16, every closed bounded subset is sequentially compact.

10.1.12. Modify the proofs of Remark 2.4, Theorems 2.6, 2.8, and Remark 2.28 by replacing the absolute value signs by the metric ρ.

10.2 Limits of Functions.

10.2.1. a) Let $\epsilon > 0$ and $x \in \mathbf{R}$. By the Density Theorem for Irrationals, there are infinitely many points in $(x - \epsilon, x + \epsilon) \cap (\mathbf{R} \setminus \mathbf{Q})$. Thus each point $x \in \mathbf{R}$ is a cluster point of $\mathbf{R} \setminus \mathbf{Q}$.

b) Let $\epsilon > 0$ and $x \in [a, b]$. If $c = \max\{x - \epsilon, a\}$ and $d = \min\{x + \epsilon, b\}$ then $c < d$ and $(x - \epsilon, x + \epsilon) \cap [a, b) \supseteq (c, d)$. Since nondegenerate intervals always contain infinitely many points, it follows that every point in $[a, b]$ is a cluster point of $[a, b)$. On the other hand, if $x \notin [a, b]$ then $(x - \epsilon, x + \epsilon) \cap [a, b] = \emptyset$ for $\epsilon < \min\{|x - a|, |x - b|\}$. Therefore, x is not a cluster point of $[a, b)$.

c) Since $E \subset \mathbf{Z}$, given any $x \in \mathbf{R}$, $(x - 1/2, x + 1/2) \cap E$ contains at most one point. Therefore, E has no cluster points.

d) E has no cluster points if E is finite. Suppose E is infinite. Given $\epsilon > 0$ choose $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in (x - \epsilon, x + \epsilon)$. Then $(x - \epsilon, x + \epsilon) \cap E$ contains infinitely many points, so x is a cluster point of E. Let $y \in \mathbf{R}$ with $y \ne x$, and set $\epsilon = |x - y|/2$. Choose $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in (x - \epsilon, x + \epsilon)$. Then $(y - \epsilon, y + \epsilon) \cap E \subset \{x_1, \dots, x_N\}$, i.e., contains only finitely many points. Thus y is not a cluster point of E.

e) Since $E \subset \mathbf{N}$, it has no cluster points. (See the argument which appears in c) above.)

10.2.2. a) If a is not a cluster point, then some $B_r(a)$ contains only finitely many points of $E \setminus \{a\}$, say $x_1, \dots, x_N$. If we let $s := \min\{r, \rho(x_1, a), \dots, \rho(x_N, a)\}$, then $B_s(a) \cap E \subseteq \{a\}$. But $a \in E$, so $B_s(a) \cap E = \{a\}$. Conversely, if $B_s(a) \cap E = \{a\}$, then this set does not contain infinitely many points, so a is not a cluster point by definition.

b) Let $a \in \mathbf{R}$. If $r < 1$, then in the discrete space, $B_r(a) = \{a\}$. Thus by part a), a cannot be a cluster point of $\mathbf{R}$.

10.2.3. If a is a cluster point for E, then let $x_n \in (B_{1/n}(a) \cap E) \setminus \{a\}$. (Such points exist since this intersection contains infinitely many points, hence at least one different from a.) Since $\rho(x_n, a) < 1/n$, it follows from the Squeeze Theorem that $x_n \to a$ as $n \to \infty$.

Conversely, suppose $x_n \in E \setminus \{a\}$ and $x_n \to a$ as $n \to \infty$. Given $r > 0$, $\rho(x_n, a)$ is eventually smaller than r, e.g., $B_r(a) \cap E$ contains x_{n_1} for some n_1. Suppose distinct points $x_1, \dots, x_{n-1}$ have been chosen in $B_r(a) \cap E$. Let $s < \min\{\rho(x_1, a), \dots, \rho(x_{n-1}, a)\}$. Then none of the x_j's chosen so far belong to $B_s(a)$. But since $x_n \to a$ as $n \to \infty$, there is an $x_n \in B_s(a) \cap E$. But $s < r$, so $B_s(a) \subset B_r(a)$. Thus $x_n \in B_r(a) \cap E$. By induction, then, there exist infinitely many points x_{n_k} in $B_r(a) \cap E$. In particular, a is a cluster point of E.

10.2.4. a) Surely a set which has infinitely many points is nonempty. Conversely, if $E \cap B_s(a) \setminus \{a\}$ is always nonempty for all $s > 0$ and $r > 0$ is given, choose $x_1 \in E \cap B_r(a)$. If distinct points $x_1, \dots, x_k$ have been chosen so that $x_k \in E \cap B_r(a)$ and $s := \min\{\rho(x_1, a), \dots, \rho(x_k, a)\}$, then by hypothesis there is an $x_{k+1} \in E \cap B_s(a)$. By construction, x_{k+1} does not equal any x_j for $1 \le j \le k$. Hence $x_1, \dots, x_{k+1}$ are distinct points in $E \cap B_r(a)$. By induction, there are infinitely many points in $E \cap B_r(a)$.

b) If E is a bounded infinite set, then it contains distinct points $x_1, x_2, \dots$. Since $\{x_n\} \subseteq E$, it is bounded. It follows from the Bolzano–Weierstrass Theorem that x_n contains a convergent subsequence, i.e., there is an $a \in \mathbf{R}$ such that given $r > 0$ there is an $N \in \mathbf{N}$ such that $k \ge N$ implies $|x_{n_k} - a| < r$. Since there are infinitely many x_{n_k}'s and they all belong to E, a is by definition a cluster point of E.

10.2.5. Modify the proofs of Remark 3.4, Theorems 3.6, 3.8, 3.9, and 3.10, replacing the absolute value signs with the metric ρ.

10.2.6. Modify the proofs of Theorems 3.21 and 3.22, replacing the absolute value signs with the metric ρ.

10.2.7. Modify the proofs of Theorem 3.24, replacing the absolute value signs with the metric ρ.

10.2.8. a) If $x_n \in E$, then x_n is bounded. Hence by the Bolzano–Weierstrass Property, there is an $a \in X$ such that $x_{n_k} \to a$ as $k \to \infty$. But E is closed, so by Theorem 10.16, $a \in E$.

b) Repeat the proof of Theorem 3.26.

c) $M := \sup_{x\in E} f(x)$ is finite by part b). Hence by the Approximation Property for Suprema, choose $x_k \in E$ such that $f(x_k) \to M$. By part a), there is an $a \in E$ such that $x_{n_k} \to a$. Since f is continuous, it follows that $f(x_{n_k}) \to f(a) = M$. A similar argument works for the infimum as well.

10.3 Interior, closure, and boundary.

10.3.1. a) The closure is $E \cup \{0\}$, the interior is $\emptyset$, the boundary is $E \cup \{0\}$.

b) The closure is $[0, 1]$, the interior is E, the boundary is $\{1/n : n \in \mathbf{N}\} \cup \{0\}$.

c) The closure is $\mathbf{R}$, the interior is $\mathbf{R}$, the boundary is $\emptyset$.

d) By Theorem 1.18, the closure of $\mathbf{Q}$ is $\mathbf{R}$ and the interior of $\mathbf{Q}$ is the empty set, so the boundary of $\mathbf{Q}$ is $\mathbf{R}$.

10.3.2. a) This is the set of points on or inside the ellipse $x^2 + 4y^2 = 1$. It is closed because its complement $\{(x, y) : x^2 + 4y^2 > 1\}$ is open. $E^o = \{(x, y) : x^2 + 4y^2 < 1\}$ and $\partial E = \{(x, y) : x^2 + 4y^2 = 1\}$.

b) This is the set of points on the circle $(x-1)^2 + y^2 = 1$ or on the x axis between $x = 2$ and $x = 3$. It is closed. $E^o = \emptyset$ and $\partial E = E$.

c) This is the set of points on or above the parabola which lie below the line $y = 1$. It is neither open nor closed. $E^o = \{(x, y) : y > x^2, 0 < y < 1\}$, $\overline{E} = \{(x, y) : y \geq x^2, 0 \leq y \leq 1\}$, and $\partial E = \{(x, y) : y = x^2, 0 \leq y \leq 1\} \cup \{(x, 1) : -1 \leq x \leq 1\}$.

d) This is the set of points between the two branches of the hyperbola $x^2 - y^2 = 1$ which lie above the line $y = -1$ and below the line $y = 1$. It is open. $\overline{E} = \{x^2 - y^2 \leq 1, -1 \leq y \leq 1\}$ and $\partial E = \{x^2 - y^2 = 1, -1 \leq y \leq 1\} \cup \{(x, 1) : -\sqrt{2} \leq x \leq \sqrt{2}\} \cup \{(x, -1) : -\sqrt{2} \leq x \leq \sqrt{2}\}$.

10.3.3. Let $y \in V = \{x \in X : s < \rho(x, a) < r\}$ and let $\epsilon < \min\{\rho(y, a) - s, r - \rho(y, a)\}$. If $w \in B_\epsilon(y)$ then

$$\rho(w, a) \leq \rho(w, y) + \rho(y, a) < r - \rho(y, a) + \rho(y, a) = r$$

and

$$\rho(w, a) \geq \rho(y, a) - \rho(w, y) > \rho(y, a) + s - \rho(y, a) = s.$$

Hence $w \in V$ and V is open by definition.

A similar argument shows that $\{x \in X : \rho(x, a) > r\}$ and $\{x \in X : \rho(x, a) < s\}$ are both open, hence

$$E := \{x \in X : s \leq \rho(x, a) \leq r\} = \{x \in X : \rho(x, a) > r\}^c \cap \{x \in X : \rho(x, a) < s\}^c$$

is closed.

10.3.4. If $A \subseteq B$ then A^o is an open set contained in B. Hence by Theorem 10.34, $A^o \subseteq B^o$. Similarly, $\overline{B}$ is a closed set containing A, hence $\overline{A} \subseteq \overline{B}$.

10.3.5. Suppose E is closed and $a \notin E$. Then there is an $\epsilon > 0$ such that $B_\epsilon(a) \cap E = \emptyset$. Thus $\rho(x, a) \geq \epsilon$ for all $x \in E$. Taking the infimum of this inequality over all $x \in E$, we conclude that $\inf_{x\in E} \rho(x, a) \geq \epsilon > 0$.

10.3.6. Suppose $x \notin E^o$ but $B_r(x) \subset E$. Then by Theorem 10.34, $B_r(x) \subseteq E^o$ so $x \in E^o$, a contradiction. Conversely, if $B_r(x) \cap E^c \neq \emptyset$ for all $r > 0$, then $x \notin E^o$ because E^o is open.

10.3.7. a) If $A = (0, 1)$ and $B = [1, 2]$ then $(A \cup B)^o = (0, 2)$ but $A^o \cup B^o = (0, 1) \cup (1, 2) \neq (0, 2)$.

b) If $A = \mathbf{Q}$ and $B = A^c$ then $\overline{A \cap B} = \emptyset$ but $\overline{A} \cap \overline{B} = \mathbf{R} \cap \mathbf{R} = \mathbf{R}$.

c) If A and B are as in part a), then $\partial(A \cup B) = \{0, 2\} \neq \{0, 1, 2\} = \partial A \cup \partial B$ and $\partial(A \cap B) = \emptyset \neq \{1\} = \partial A \cap \partial B$.

10.3.8. a) If V is open in Y, then given $x \in V$ there is ball $B_Y(x) = B_X(x) \cap Y$, open in Y, which contains x and is a subset of V. It follows that

$$V = \bigcup_{x\in V} B_X(x) \cap Y =: U \cap Y.$$

But U is open in X by Theorem 10.31.

Conversely, if $x \in V = U \cap Y$, then there is ball B_X, open in X, such that $x \in B_X \subset U$. Hence $x \in B_X \cap Y \subset V$, i.e., V is open in Y.

b) If E is closed in Y then $Y \setminus E$ is open in Y, so by part a), $Y \setminus E = U \cap Y$ for some U open in X. Hence $E = A \cap Y$, where $A = Y \setminus U$ is closed in Y.

10.3.9. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and $I = (a, b)$. Let $x \in f^{-1}(I)$. By definition, $f(x) \in I$. Since I is open, there is an $\epsilon > 0$ such that $(f(x)-\epsilon, f(x)+\epsilon) \subset I$. Since f is continuous at x, choose $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then $y \in (x - \delta, x + \delta)$ implies $|f(x) - f(y)| < \epsilon$, i.e., $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset I$. Thus $(x - \delta, x + \delta) \subseteq f^{-1}(I)$ and $f^{-1}(I)$ is open by definition.

Conversely, let $a \in \mathbf{R}$, $\epsilon > 0$, and set $I = (f(a) - \epsilon, f(a) + \epsilon)$. Since $a \in f^{-1}(I)$, choose by hypothesis a $\delta > 0$ such that $(a - \delta, a + \delta) \subset f^{-1}(I)$. Then $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Therefore, f is continuous at a.

10.3.10. a) Given $x \in V$ choose $\epsilon := \epsilon_x > 0$ such that $B_\epsilon(x) \subseteq V$. Then $V \subseteq \cup_{x \in V} B_\epsilon(x)$. On the other hand, $\cup_{x \in V} B_\epsilon(x) \subseteq V$ since each $B_\epsilon \subseteq V$. Therefore $V = \cup_{x \in V} B_\epsilon(x)$ as required.

It is even easier for closed sets. Since every singleton is closed (see Remark 10.10), $E = \cup_{x \in E}\{x\}$ is a decomposition of E into closed sets.

10.3.11. a) Let $x \in \partial E$. Then there exist points $x_k \in B_{1/k}(x) \cap E$ for each $k \in \mathbf{N}$. By construction $x_k \in E$ and $x_k \to x$ as $k \to \infty$. Since E is closed, it follows from Theorem 10.16 that $x \in E$.

b) Suppose $\partial E = E$. If $E^o \neq \emptyset$ then there exists a point $x \in E^o \subset E$ such that $B_r(x) \subset E$ for some $r > 0$, i.e., $B_r(x) \cap E^c = \emptyset$. Therefore, $x \notin \partial E$, a contradiction.

Conversely, suppose $E^o = \emptyset$. By part a), $\partial E \subseteq E$. On the other hand, if $x \in E$ then since $E^o = \emptyset$, $B_r(x)$ is not contained in E for any $r > 0$, i.e., $B_r(x) \cap E^c \neq \emptyset$ for all $r > 0$. Hence $x \in \partial E$.

c) Let $E = \{(x, y) : x^2 + y^2 = 1, x \neq 1\}$. Then $(1, 0)$ is a boundary point of E which does not belong to E.

10.4 Compact Sets.

10.4.1. a) Since $1/k \to 0$ as $k \to \infty$, this set is closed and bounded, hence compact.

b) This set is closed and bounded, hence compact.

c) This set is bounded but not closed. If we set $H := \{(0, y) : -1 \leq y \leq 1\}$ then $E \cup H$ is closed and bounded, hence compact.

d) The set is closed but not bounded (since $(n, 1/n) \in E$ for all $n \in \mathbf{N}$). Therefore, E is neither compact nor is it contained in any compact set.

10.4.2. Let A, B be compact sets. If $\{V_\alpha\}$ is an open covering of $A \cup B$, then it is a covering of A and B. Since these are compact sets, we can choose $V_1, \dots, V_N$ to cover A and $V_{N+1}, \dots, V_M$ to cover B. Clearly, $V_1, \dots, V_M$ covers $A \cup B$. Thus $A \cup B$ is compact.

By Remark 10.44, A and B are closed sets, so by Theorem 10.31, $A \cap B$ is a closed subset of the compact set A. It follows from Remark 10.45 that $A \cap B$ is compact.

10.4.3. Since E is compact, it is bounded by Theorem 10.46. Since it is nonempty, it follows from the Completeness Axiom that E has a finite supremum. By the Approximation Property, choose $x_k \in E$ such that $x_k \to \sup E$. Since E is closed, we have by Theorem 10.16 that $\sup E = \lim_{k \to \infty} x_k \in E$. A similar argument works for $\inf E$.

10.4.4. Suppose A is uncountable. Since each V_α is nonempty, choose a point $x_\alpha \in V_\alpha$ for each $\alpha \in A$. Since $V_\alpha \cap V_\beta = \emptyset$ for $\alpha \neq \beta$, the set $E = \{x_\alpha : \alpha \in A\}$ is uncountable. But $E \subset \cup_{\alpha \in A} V_\alpha$, hence by Lindelöf's Theorem, $E \subset \cup_{\alpha \in A_0} V_\alpha$ for some countable subset A_0 of A. Since each V_α contains at most one x_β, it follows that E is countable, a contradiction.

The result is false if "open" is omitted. By Remark 1.41, the unit interval $(0, 1)$ is uncountable, hence $\{x : x \in (0, 1)\}$ is an uncountable collection of pairwise disjoint nonempty sets which covers the unit interval $(0, 1)$.

10.4.5. By Exercise 10.3.10, there exist $\epsilon := \epsilon_x > 0$ such that $V = \cup_{x \in V} B_\epsilon(x)$. But X is separable, so it follows from Lindelöf's Theorem that there exist open balls $B_j := B_{\epsilon_j}(x_j)$ such that $V \subseteq \cup_{j=1}^\infty B_j$. On the other hand, $\cup_{j=1}^\infty B_j \subseteq V$ since each $B_j \subseteq V$. Therefore $V = \cup_{j=1}^\infty B_j$ as required.

10.4.6. Suppose that f is uniformly continuous on E. Thus given $\epsilon > 0$ there is a $\delta > 0$ such that $\rho(x, y) < \delta$ and $x, y \in E$ imply $\tau(f(x), f(y)) < \epsilon$.

Let $a \in \partial E$ and let $x_n \in E$ with $x_n \to a$ as $n \to \infty$. Clearly, x_n is Cauchy. By repeating the proof of Lemma 3.38 we can show that $f(x_n)$ is Cauchy. Since Y is complete, it follows that $f(x_n) \to y$ for some $y \in Y$. This

y does not depend on the sequence x_n. Indeed, if y_n also converges to a, then $\rho(x_n, y_n) < \delta$ for large n, so $\tau(f(x_n), f(y_n)) < \epsilon$ for large n. Thus let $g(a) := y$.

Do this for all $a \in \partial E$ to define g on ∂E. Define g on E by $g = f$. Since $\overline{E} = E \cup \partial E$, it follows from the sequential characterization of continuity and our construction that g is a continuous extension of f to $\overline{E}$.

Conversely, since $\overline{E}$ is closed and bounded, it follows from the Heine-Borel Theorem that $\overline{E}$ is compact. Thus by theorem 10.52, g is uniformly continuous on $\overline{E}$. We conclude that $f = g$ is uniformly continuous on the subset E.

Since X is complete and satisfies the Bolzano–Weierstrass Property

10.4.7. a) Since both sets are nonempty and $\rho(x, y)$ is bounded below by 0, the dist (A, B) exists and is finite. By the Approximation Property for Infima, choose $x_k \in A$ and $y_k \in B$ such that $\rho(x_k, y_k) \to$ dist (A, B). Since A and B are compact (hence bounded—see Theorem 10.46), use the Bolzano-Weierstrass Property to choose subsequences such that $x_{k_j} \to x_0 \in A$ and $y_{k_j} \to y_0 \in B$. Since $A \cap B = \emptyset$, $x_0 \neq y_0$. Hence dist $(A, B) = \rho(x_0, y_0) > 0$.

b) Let $A = \{(x, y) : y = 0\}$ and $B = \{(x, y) : y = 1/x\}$. Then A and B are closed, $A \cap B = \emptyset$, but dist $(A, B) = 0$ because $1/x \to 0$ as $x \to \infty$.

10.4.8. a) Suppose not, i.e., $\cap H_k = \emptyset$. Then by DeMorgan's Law, $\cup H_k^c = X$, in particular, $\{H_k^c\}$ is an open cover of H_1. Since H_1 is compact, choose N so large that $H_1^c, \dots, H_N^c$ covers H_1. But since these sets are nested, $H_1^c \subseteq \dots \subseteq H_N^c$, so the union of this covering is H_N^c. Since $H_N^c \supseteq H_1$, and $H_N \supseteq H_1$, it follows that $\emptyset = H_N^c \cap H_N \supseteq H_1$, which contradicts the hypothesis that H_1 is nonempty.

b) Every convergent sequence in $E := (\sqrt{2}, \sqrt{3}) \cap \mathbf{Q}$ must have a limit in $\mathbf{Q}$ and cannot converge to the irrational endpoints, so by Theorem 10.16, E is closed. Since it is contained in a bounded interval, E is bounded. It is not compact since we if we choose rationals $a_n \downarrow \sqrt{2}$ and $b_n \uparrow \sqrt{3}$, then $\{(a_n, b_n) \cap \mathbf{Q}$ is a countably infinite open covering of E which has no finite subcover.

c) Let x_n be irrational which satisfy $x_n \downarrow \sqrt{2}$. By the argument in part b), $(\sqrt{2}, x_n)$ are closed and bounded, but not compact. Moreover, it is obvious that they are nested. Of course, their intersection is $\emptyset$ because $\sqrt{2} \notin \mathbf{Q}$.

10.4.9. Since $\sup_{x \in [0,1]} |x^n| = 1$, $\|f_n\|_\infty = 1$ for all $n \in \mathbf{N}$. Suppose to the contrary there is a subsequence $\{n_k\}$ of integers such that $\|f_{n_k} - f\|_\infty \to 0$ as $k \to \infty$. Then $|f_{n_k}(x) - f(x)| \to 0$ as $k \to \infty$ for each $x \in [0, 1]$. But $x^{n_k} \to 0$ when $x \in [0, 1)$ and to 1 when $x = 1$. Hence $f(x)$ cannot be continuous at $x = 1$.

10.4.10. a) Suppose that H is compact and $x_k \in H$. There are two cases. Either there is an $a \in H$ such that for each $r > 0$, $B_r(a)$ contains x_k for infinitely many k's, or for each $a \in H$ there exists an $r_a > 0$ such that $B_{r_a}(a)$ contains x_k for only finitely many k's.

If the second case holds, then

$$H \subseteq \bigcup_{a \in H} B_{r_a}(a).$$

Since H is compact, there are points $a_1, a_2, \dots, a_N$ such that

$$H \subseteq \bigcup_{j=1}^{N} B_{r_{a_j}}(a_j).$$

Since each $B_{r_{a_j}}(a_j)$ contains x_k for only finitely many k's and $x_k \in H$ for all $k \in \mathbf{N}$, it follows that $\mathbf{N}$ is finite, a contradiction. Hence, the second case cannot hold.

Evidently, the first case holds. Let $x_{k_1} \in B_1(a)$. Since $B_{1/2}(a)$ contains x_k for infinitely many k's, choose $k_2 > k_1$ such that $x_{k_2} \in B_{1/2}(a)$. Continuing in this manner, we can choose integers $k_1 < k_2 < \dots$ such that $x_{k_j} \in B_{1/j}(a)$ for $j \in \mathbf{N}$. Since $\rho(x_{k_j}, a) < 1/j$, x_{k_j} converges to a.

b) Let E be sequentially compact. By Exercise 10.1.10, E is closed and bounded. Hence by the Heine–Borel Theorem, E is compact. Conversely, if E is compact then by part a), E is sequentially compact.

10.5 Connected Sets.

10.5.1. a) Let $R = [a, b] \times [c, d]$. Since $(x, y) \in R$ implies $\|(x, y)\| \leq |x| + |y| \leq \max\{|a|, |b|\} + \max\{|c|, |d|\}$, R is bounded. If $(x_k, y_k) \in R$ converges to some (x, y) then $a \leq x_k \leq b$ implies $a \leq x \leq b$ and similarly, $c \leq y \leq d$. Thus $(x, y) \in R$ and it follows from Theorem 10.16 that R is closed. Hence by the Heine- Borel Theorem, E is compact. It also is connected because it cannot be broken into disjoint open pieces.

b) The set is bounded, but not closed (since $(-2 + 1/n, 0)$ belongs to the set but its limit, $(-2, 0)$ does not). Hence by the Heine Borel Theorem, it is not compact. It is also not connected, because $\{(x, y) : x < 0\}$ and $\{(x, y) : x > 0\}$ separates the set.

10.5.2. a) It is relatively open in $\{(x,y) : y \geq 0\}$ because each of its points lies in a relative open ball which stays inside the set. It is relatively closed in $\{(x,y) : x^2 + 2y^2 < 6\}$ because the limit of any convergent sequence (in the SUBSPACE sense) in the set stays in the set.

b) It is relatively open in $\overline{B_1(0,0)}$ because each of its points lies in a relative open ball which stays inside the set. It is relatively closed in $B_{\sqrt{2}}(2,0)$ because the limit of any convergent sequence (in the SUBSPACE sense) in the set stays in the set.

10.5.3. a) Let I and J be connected in $\mathbf{R}$. Then I and J are intervals by Theorem 10.56. Hence $I \cap J$ is empty or an interval, hence connected by definition or Theorem 10.56.

Let $A = \{(x,y) : y = x^2\}$ and $B = \{(x,y) : y = 1\}$. Then A and B are connected in $\mathbf{R}^2$ but $A \cap B = \{(-1,1),(1,1)\}$ is not connected.

b) If $E = \bigcap_{\alpha \in A} E_\alpha$ is empty or contains a single point, then E is connected by definition. If E contains two points, say a,b, then $a,b \in E_\alpha$ for every $\alpha \in A$. But E_α is an interval, hence $(a,b) \subset E_\alpha$ for all $\alpha \in A$, i.e., $(a,b) \subset E$. Hence E is an interval, so connected by Theorem 10.56.

10.5.4. a) If E is connected in $\mathbf{R}$ then E is an interval, hence E^o is either empty or an interval, hence connected by definition or Theorem 10.56.

b) The set $E = B_1(0,0) \cup B_1(3,0) \cup \{(x,0) : 1 \leq x \leq 2\}$ is connected in $\mathbf{R}^2$, but $E^o = B_1(0,0) \cup B_1(3,0)$ is not.

10.5.5. Suppose A is not connected. Then there is a pair of open sets U,V which separates A. We claim that $E \cap U \neq \emptyset$. If $E \cap U = \emptyset$ then since $A \cap U \neq \emptyset$, there exists a point $x \in U \cap (A \setminus E)$. But $E \subseteq A \subseteq \overline{E}$ implies $A \setminus E \subseteq \overline{E} \setminus E = \partial E$. Thus $x \in \partial E \cap U$. Since U is open it follows that $E \cap U \neq \emptyset$, a contradiction. This verifies the claim. Similarly, $E \cap V \neq \emptyset$. Thus the pair U,V separates E, which contradicts the fact that E is connected.

10.5.6. For each $x \in X$, f is constant on B_x. Since X is compact and is covered by $\{B_x\}_{x \in X}$, there exist $x_1, \dots, x_N$ such that

$$X = \bigcup_{j=1}^{N} B_{x_j}.$$

Let $x \in X$. Then $x \in B_{x_j}$ for some $1 \leq j \leq N$, so $f(x) = f(x_j)$. It follows that $f(x) \in \{f(x_1), \dots, f(x_N)\}$ for all $x \in X$. In particular, $f(X) \subset \{f(x_1), \dots, f(x_N)\}$, so $f(X)$ is finite, say $X = \{y_1, \dots, y_M\}$. But a finite set is nonempty connected if and only if it is a single point. Indeed, if $M > 1$, then set $r = \min\{\rho(y_j, y_k) : j,k \in [1,M]\}$ and notice that the $B_r(y_j)$'s are open, nonempty, and disjoint, hence separate $f(X)$. Hence, $N = 1$, i.e., $f(x) = f(x_1)$ for all $x \in X$.

10.5.7. Suppose H is compact. Let $\mathcal{E} := \{U_\alpha\}_{\alpha \in A}$ be a relatively open covering of H. Choose V_α, open in $\mathbf{R}^n$, such that $U_\alpha = H \cap V_\alpha$. Then $\{V_\alpha\}_{\alpha \in A}$ is an open covering of H. Since H is compact, there exists a finite subset A_0 of A such that $\{V_\alpha\}_{\alpha \in A_0}$ covers H. In particular, $\{U_\alpha\}_{\alpha \in A_0}$ is a finite subcovering of $\mathcal{E}$ which covers H.

Conversely, suppose every relatively open covering of H has a finite subcover. If $\{V_\alpha\}_{\alpha \in A}$ is an open covering of H then $\{H \cap V_\alpha\}_{\alpha \in A}$ is a relatively open covering of H. Therefore, there exists a finite subset A_0 of A such that $\{H \cap V_\alpha\}_{\alpha \in A_0}$ covers H. In particular, $\{V_\alpha\}_{\alpha \in A_0}$ covers H and H is compact.

10.5.8. a) By Remark 10.11, $\emptyset$ and X are clopen.

b) Suppose E is clopen and $\emptyset \subset E \subset X$. Then $U = E$ and $V = X \setminus E$ are nonempty open sets, $U \cap V = \emptyset$, and $X = U \cup V$. Therefore, X is not connected.

Conversely, if X is not connected then there exist nonempty open sets U and V such that $U \cap V = \emptyset$ and $X = U \cup V$. Thus $E := U = X \setminus V$ is clopen and $\emptyset \subset E \subset X$. In particular, X contains more than two clopen sets.

10.5.9. Let E be a nonempty, proper subset of X. By Theorem 10.34, E has no boundary if and only if $\overline{E} \setminus E^o = \partial E = \emptyset$, i.e., if and only if $\overline{E} = E^o$. Thus E has no boundary if and only if E is clopen. This happens, by Exercise 10.5.8, if and only if X is not connected.

10.5.10. a) Suppose E is polygonally connected but some pair of open sets U,V separates E. Let $\mathbf{x}_1 \in E \cap U$, $\mathbf{x}_2 \in E \cap V$. Since E is polygonally connected, there is a continuous function $f : [0,1] \to E$ with $f(0) = \mathbf{x}_1$ and $f(1) = \mathbf{x}_2$. By Theorems 10.56 and 10.62, $f([0,1])$ is connected. But since $f([0,1]) \subseteq E$, U,V separates $f([0,1])$, a contradiction.

b) Let $\mathbf{x} \in U$. Since E is open, choose $r > 0$ such that $B_r(\mathbf{x}) \subset E$. Let $\mathbf{y} \in B_r(\mathbf{x})$ and let P be a polygonal path from $\mathbf{x}_0$ to $\mathbf{x}$ which lies in E. Then the path $P \cup L(\mathbf{x};\mathbf{y})$ goes from $\mathbf{x}_0$ to $\mathbf{y}$ and lies in E, i.e., $\mathbf{y} \in U$. Therefore, $B_r(\mathbf{x}) \subseteq U$ and U is open.

c) Suppose E is open and connected but not polygonally connected. By part b), given any $\mathbf{x} \in E$ the set $U_{\mathbf{x}}$ which can be polygonally connected to $\mathbf{x}$ through E is open. Since E is not polygonally connected, there exist points $\mathbf{x}_0 \neq \mathbf{y}_0$ in E such that $U_{\mathbf{x}_0} \cap U_{\mathbf{y}_0} = \emptyset$. Let $U := U_{\mathbf{x}_0}$ and $V := \cup\{U_{\mathbf{y}} : U_{\mathbf{y}} \cap U_{\mathbf{x}_0} = \emptyset\}$. Then U, V are nonempty open sets, $U \cap V = \emptyset$ and $U \cup V = E$. In particular, the pair U, V separates E, a contradiction.

10.5.11. Suppose E is not connected. Then there exists a pair of open sets U, V which separates E. Let $x \in \cap_{\alpha \in A} E_\alpha$. Since $E \subseteq U \cup V$ we may suppose $x \in U$. Choose $\alpha_0 \in A$ such that $V \cap E_{\alpha_0} \neq \emptyset$. Since $x \in E_{\alpha_0}$, we also have $U \cap E_{\alpha_0} \neq \emptyset$. Therefore, the pair U, V separates E_{α_0}, a contradiction.

10.6 Continuous Functions.

10.6.1. a) $f(0,\pi) = (0,1]$ is not open and we don't expect it to be; $f[0,\pi] = [0,1]$ is compact and connected as Theorems 10.61 and 10.62 say it should; $f(-1,1) = (-\sin 1, \sin 1)$ is open, big deal; $f[-1,1] = [-\sin 1, \sin 1]$ is compact and connected as Theorems 10.61 and 10.62 say it should.

$g(0,\pi) = \{1\}$ is connected as Theorem 10.62 says it should; $g[0,\pi] = \{0,1\}$ is compact but not connected–note that Theorem 10.61 does not apply since g is not continuous; $g(-1,1) = \{-1,0,1\}$ is not open and we don't expect it to be; $g[-1,1] = \{-1,0,1\}$ is compact but not connected–note that Theorem 10.61 does not apply since g is not continuous.

b) $f^{-1}(0,\pi) = \dots(0,\pi)\cup(2\pi,3\pi)\cup\dots$ is open as Theorem 10.58 says it should; $f^{-1}[0,\pi] = \dots[0,\pi]\cup[2\pi,3\pi]\cup\dots$ is closed as Exercise 10.6.3 says it should; $f^{-1}(-1,1) = \mathbf{R}\setminus\{x : x = (2k+1)\pi/2, k \in \mathbf{Z}\}$ is open as Theorem 10.58 says it should; $f^{-1}[-1,1] = \mathbf{R}$ is closed as Exercise 10.6.3 says it should.

$g^{-1}(0,\pi) = (0,\infty)$ is open, no big deal; $g^{-1}[0,\pi] = [0,\infty)$ is closed–note that Exercise 10.6.3 does not apply since g is not continuous; $g^{-1}(-1,1) = \{0\}$ is not open and we don't expect it to be; $g^{-1}[-1,1] = \mathbf{R}$ is closed–note that Exercise 10.6.3 does not apply since g is not continuous.

10.6.2. a) $f(0,1) = (0,1)$ is open, no big deal; $f[0,1) = [0,1)$ is neither open nor closed; $f[0,1] = [0,1]$ is compact and connected as Theorems 10.61 and 10.62 say it should.

$g(0,1) = (1,\infty)$ is connected as Theorem 10.62 says it should; $g[0,1) = \{0\}\cup(1,\infty)$ is neither compact nor connected–note that Theorems 10.61 and 10.62 do not apply since g is not continuous; $g[0,1] = \{0\}\cup[1,\infty)$ is neither compact nor connected–note that Theorems 10.61 and 10.62 do not apply since g is not continuous.

b) $f^{-1}(-1,1) = [0,1)$ is relatively open in $[0,\infty)$, the domain of f as Theorem 10.58 says it should; $f^{-1}[-1,1] = [0,1]$ is relatively closed in $[0,\infty)$ as Exercise 10.6.4 says it should.

$g^{-1}(-1,1) = (-\infty,-1)\cup(1,\infty)\cup\{0\}$ is not open and g is not continuous; $g^{-1}[-1,1] = (-\infty,-1]\cup[1,\infty)\cup\{0\}$ is closed, no big deal–note that Exercise 10.6.3 does not apply since g is not continuous.

10.6.3. Let C be closed in Y. Then $Y\setminus C$ is open in Y, so by Theorems 10.58 and 1.37, $f^{-1}(Y\setminus C) = X\setminus f^{-1}(C)$ is open in X, i.e., $f^{-1}(C)$ is closed in X.

10.6.4. a) First, notice by definition and the fact that every subspace is a metric space in its own right, a set is relatively open if and only if its complement is relatively closed.

Suppose f is continuous on E and A is closed in Y. Then $A^c := Y \setminus A$ is open in Y, so by Corollary 10.59, $A_0 := f^{-1}(A^c)\cap E$ is relatively open in E. This means that there is an open set V in X such that $A_0 = V \cap E$. Since

$$E \setminus A_0 = E \cap f^{-1}(A) = E \cap (X \setminus V)$$

and $X \setminus V$ is closed, it follows that $f^{-1}(A)\cap E$ is relatively closed in E. A similar argument proves that if $f^{-1}(A)\cap E$ is relatively closed in E for all closed sets A in Y, then f is continuous on E.

b) Let V be relatively open in $f(E)$, i.e., $V = U \cap f(E)$ for some U open in Y. By Corollary 10.59, $f^{-1}(U)\cap E$ is relatively open in E. But by Theorem 1.37,

$$f^{-1}(V)\cap E = f^{-1}(V\cap f(E)) = f^{-1}(U\cap f(E)) = f^{-1}(U)\cap E.$$

Hence $f^{-1}(V)\cap E$ is relatively open in E.

A similar proof, using part a) in place of Corollary 10.59, shows that $f^{-1}(A)\cap E$ is relatively closed in E for all relatively closed sets A in $f(E)$.

10.6.5. By Theorem 10.62, $f(E)$ is connected in $\mathbf{R}$. Hence by Theorem 10.56, $f(E)$ is an interval. Since $f(a)$ and $f(b)$ both belong to $f(E)$, the interval $(f(a), f(b))$ is a subset of $f(E)$. In particular, $y \in f(E)$ as required.

10.6.6. a) Since f is continuous, so is $\|f\|$ (modify the proof of Exercise 3.1.6). Therefore, $\|f\|_H$ is finite and attained by the Extreme Value Theorem.

b), c) Repeat the proof of Exercise 9.4.7.

10.6.7. a) Repeat the proof of Exercise 3.4.5a,b. Compactness was used to prove fg is uniformly continuous since both functions need to be bounded. Compactness is not needed to prove $f+g$ is uniformly continuous.

b) If g is not zero, then $1/g$ is continuous on E, hence bounded by the Extreme Value Theorem.

c) By part b) and the Extreme Value Theorem, $1/|g(x_0)| = \inf_{x\in E} 1/g(x)$ is positive. Hence by repeating the proof of Exercise 3.4.5d, we see that f/g is uniformly continuous on E.

10.6.8. a) By the proof of Lemma 3.38, if f is uniformly continuous, then f takes a Cauchy sequence in X to a Cauchy sequence in Y.

b) Let $x \in X$. Since D is dense, choose $x_n \in D$ such that $x_n \to x$. By part a, $f(x_n)$ is Cauchy, hence convergent since Y is complete. Define $g(x) := \lim_{n\to\infty} f(x_n)$. By the argument of Theorem 3.40, this definition is independent of the sequence $x_n \in D$ chosen to approximate x. Thus g is well defined on all of X. (Note: Because the boundary of $[a,b]$ contained only isolated points, this and the Sequential Characterization of Limits was enough to conclude that g was continuous on X. Here, we must also consider the possibility that the approximating sequence $x_n \to a$ approaches through ∂D. It's a little easier to resort to the ϵ-δ definition of continuity.)

Define $g(x) := f(x)$ when $x \in D$ and $g(x)$ as above when $x \in X \setminus D$. To show that g is continuous on X, let $\epsilon > 0$ and choose $\delta > 0$ such that

$$x, y \in D \quad \text{and} \quad \rho(x,y) < \delta \quad \text{imply} \quad \tau(f(x), f(y)) < \frac{\epsilon}{3}.$$

Fix $x_0, y_0 \in X$ with $\rho(x_0, y_0) < \delta/3$. If x_0 or y_0 belongs to D, set $x = x_0$ or $y = y_0$. Otherwise, use the density of D and the definition of g to choose $x, y \in D$ such that

$$\rho(x, x_0) < \frac{\delta}{3}, \quad \rho(y, y_0) < \frac{\delta}{3}, \quad \tau(f(x), g(x_0)) < \frac{\epsilon}{3}, \quad \text{and} \quad \tau(f(y), g(y_0)) < \frac{\epsilon}{3}.$$

Since $\rho(x,y) \le \rho(x,x_0) + \rho(x_0,y_0) + \rho(y_0,y) < \delta$ and $x, y \in D$, we have $\tau(f(x), f(y)) < \epsilon/3$. Therefore,

$$\tau(g(x_0), g(y_0)) \le \tau(g(x_0), f(x)) + \tau(f(x), f(y)) + \tau(f(y), g(y_0)) < \epsilon.$$

By definition, g is continuous on X.

10.6.9. Suppose that X is connected. By Theorems 10.62 and 10.56, $f(X)$ is an interval. Since f is nonconstant, $f(X)$ contains more than one point. In particular, $f(X)$ contains an open interval (a,b) for some $a < b$.

Now $g(t) = (t-a)/(b-a)$ is a 1–1 function from (a,b) onto $(0,1)$. Hence if X is countable, then so is $(0,1)$. This contradicts Remark 1.39.

10.7 Stone-Weierstrass Theorem.

10.7.1. a) Clearly, the collection, $\mathcal{P}$, of polynomials on $\mathbf{R}$ is an algebra in $\mathcal{C}[a,b]$ that contains the constants. If $x_1 \ne x_2$ and $f(x) = x$, then $f(x_1) \ne f(x_2)$. Thus $\mathcal{P}$ separates points of the compact set $[a,b]$. By the Stone-Weierstrass Theorem, then, there is a sequence of polynomials P_n such that $P_n \to f$ uniformly on $[a,b]$ as $n \to \infty$.

b) By part a) and the density of rationals, the polynomials with rational coefficients form a countable dense subset of $\mathcal{C}[a,b]$.

10.7.2. Clearly, the collection, $\mathcal{P}$, of polynomials on $\mathbf{R}^n$ is an algebra in $\mathcal{C}(A)$ that contains the constants. If $a,b) \ne (c,d)$, then either $a \ne c$ or $b \ne d$. If $f(x,y) = x$ and $g(x,y) = y$, then either $f(a,b) \ne f(c,d)$ or $g(a,b) \ne g(c,d)$. Thus $\mathcal{P}$ separates points of the compact set A. By the Stone-Weierstrass Theorem, then, there is a sequence of polynomials P_k on $\mathbf{R}^n$ such that $P_k \to f$ uniformly on A as $k \to \infty$.

10.7.3. Since collection of functions with separated variables is an algebra that contains the constants and separates points of the compact set R, the Stone-Weierstrass Theorem implies that there is a sequence of functions with separated variables, P_k, such that $P_k \to f$ uniformly on A as $k \to \infty$.

10.7.4. By Exercise 10.7.1, there exist polynomials P_n such that $P_n \to f$ uniformly on $[a,b]$. By hypothesis, $\int_a^b f(x)P_n(x)\,dx = 0$ for all n. Thus by Theorem 7.10,

$$\int_a^b f^2(x)\,dx = \lim_{n\to\infty} \int_a^b f(x)P(x)\,dx = 0.$$

Since $f^2 \geq 0$ and f is continuous, we conclude that $f = 0$ everywhere on $[a, b]$.

10.7.5. By Exercise 10.7.3, there exist functions P_n with separated variables such that $P_n \to f$ uniformly on $[a, b] \times [c, d]$. Notice that

$$\int_a^b \left(\int_c^d u(x)v(y)\,dy \right) dx = \int_a^b u(x)\,dx \int_c^d v(y)\,dy = \int_c^d \left(\int_a^b u(x)v(y)\,dx \right) dy.$$

Thus the result holds for any function with separated variables, e.g., for each P_n. Therefore, it follows from Theorem 7.10 that

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \lim_{n\to\infty} \int_a^b \left(\int_c^d P_n(x,y)\,dy \right) dx = \lim_{n\to\infty} \int_c^d \left(\int_a^b P_n(x,y)\,dx \right) dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

10.7.6. a) If $x, y \in T$ and $\Phi(x) = \Phi(y)$, then $\cos x = \cos y$ and $\sin x = \sin y$. If $x \neq y$, the first identity says $0 \leq x < \pi \leq y$; the second identity says $0 \leq x < \pi/2 < 3\pi/2 \leq y$ or $\pi/2 \leq x < y \leq 3\pi/2$. Since $\sin\theta$ and $\cos\theta$ are 1–1 on each of these "quarters" of T, it follows that $x = y$. Thus Φ is 1–1.

Φ takes T into $T_0 := \partial B_1(0,0)$. Since Φ is continuous, $\Phi(T)$ is connected in $\mathbf{R}^2$. But $T(0) = T(2\pi)$, so $\Phi(T) = T_0$. Thus Φ is onto.

b) Since $\|\ \|$ is a metric on $\mathbf{R}^2$ and Φ is 1–1, it is easy to see that $p(x, y)$ is a metric on T.

c) Since $p(x, y) < r$ means $\|\Phi(x) - \Phi(y)\| < r$, it is easy to check that $x \in (0, 2\pi)$ and $p(x_k, x) \to 0$ implies that $x_k \to x$ in $\mathbf{R}$, as $k \to \infty$. On the other hand, since $\Phi(0) = \Phi(2\pi)$, it is clear that $p(x_k, 0) < r < 1$ implies that $|x_k| < r$ or $|x_k - 2\pi| < r$. Thus if f is continuous on (T, p), then f is continuous on $[0, 2\pi)$ and $f(2\pi) := f(0)$ is a continuous extension of f from T to $[0, 2\pi]$. Conversely, if f is continuous and periodic on $[0, 2\pi]$, then f is continuous on (T, p).

d) $\partial B_1(0,0)$ is compact in $\mathbf{R}^2$ and Φ^{-1} is continuous from $\partial B_1(0,0)$ to (T, p), so its image T is a compact metric space by Theorem 10.61. The collection, $\mathcal{P}$, of trigonometric polynomials are an algebra on $\mathcal{C}(T)$. If $x \in [0, \pi/2] \cup (\pi, 3\pi/2]$, then $\sin x \neq \sin y$ and if $x \in (\pi/2, \pi] \cup (3\pi/2, 2\pi)$, then $\cos x \neq \cos y$. Therefore, $\mathcal{P}$ separate the points of the compact set T. By the Stone-Weierstrass Theorem, then, given $f \in \mathcal{C}(T)$, there is a sequence of trigonometric polynomials P_n such that $P_n \to f$ uniformly on T.

10.7.7. Repeat the proof of 10.7.4 with trigonometric polynomials replacing classical polynomials.

Chapter 11

11.1 Partial Derivatives and Partial Integrals.

11.1.1. a) $f_x = e^y$, $f_{xy} = e^y$, $f_y = xe^y$, and $f_{yx} = e^y$.
b) $f_x = -y\sin(xy)$, $f_{xy} = -\sin(xy) - xy\cos(xy)$, $f_y = -x\sin(xy)$, and $f_{yx} = -\sin(xy) - xy\cos(xy)$.
c) $f_x = (1-2xy-x^2)/(x^2+1)^2$, $f_{xy} = -2x/(x^2+1)^2$, $f_y = 1/(x^2+1)$, and $f_{yx} = -2x/(x^2+1)^2$.

11.1.2. a) If $(x,y)\neq(0,0)$, then $f_x = (2x^5+4x^3y^2-2xy^4)/(x^2+y^2)^2$. By definition,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^2-0}{h} = 0.$$

Since $|f_x(x,y)| \le 2|x|(x^2+y^2)^2/(x^2+y^2)^2 = 2|x| \to 0$ as $(x,y)\to(0,0)$, it follows that f_x is continuous everywhere on $\mathbf{R}^2$.

b) If $(x,y)\neq(0,0)$, then $f_x = (2x/3)\cdot(2x^2+4y^2)/(x^2+y^2)^{4/3}$. By definition,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^2/h^{2/3}-0}{h} = \lim_{h\to 0} h^{1/3} = 0.$$

Since $|f_x(x,y)| \le (2/3)\cdot 4(x^2+y^2)^{3/2}/(x^2+y^2)^{4/3} < 3(x^2+y^2)^{1/6} \to 0$ as $(x,y)\to(0,0)$, it follows that f_x is continuous everywhere on $\mathbf{R}^2$.

11.1.3. Let $\mathbf{x}\in B_r(\mathbf{a})$ and set $\mathbf{h} := \mathbf{x}-\mathbf{a}$. Then by the one-dimensional Mean Value Theorem, there exist c_k between x_k and a_k such that

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{a}) &= f(x_1,x_2,\dots,x_n) - f(a_1,x_2,\dots,x_n) \\ &\quad + f(a_1,x_2,\dots,x_n) - f(a_1,a_2,x_3,\dots,x_n) + \cdots \\ &\quad + f(a_1,\dots,a_{n-1},x_n) - f(a_1,\dots,a_n) \\ &= (x_1-a_1)f_{x_1}(c_1,x_2,\dots,x_n) + \cdots + (x_n-a_n)f_{x_n}(a_1,\dots,a_{n-1},c_n). \end{aligned}$$

For each $k=1,2,\dots,n$, set $\mathbf{d}_k = (a_1,\dots,a_{k-1},c_k,x_{k+1},\dots,x_n)$ and observe that $\mathbf{d}_k\in B_r(\mathbf{a})$. It follows from the calculation above and hypothesis that $f(\mathbf{x})-f(\mathbf{a}) = 0+\cdots+0 = 0$. In particular, $f(\mathbf{x}) = f(\mathbf{a})$ for all $\mathbf{x}\in B_r(\mathbf{a})$.

11.1.4. The integrable function g is bounded, so choose $M>0$ such that $|g(x)|\le M$ for all $x\in[a,b]$. Since f is continuous on the compact set H, f is uniformly continuous. Thus given $\epsilon>0$ there is a $\delta>0$ such that $y,w\in[c,d]$ and $|y-w|<\delta$ implies $|f(x,y)-f(x,w)| < \epsilon/(M(b-a))$. Therefore,

$$|F(y)-F(w)| = \left|\int_a^b g(x)(f(x,y)-f(x,w))\,dx\right| \le M\int_a^b |f(x,y)-f(x,w)|\,dx < \epsilon.$$

11.1.5. a) Since $e^{x^3y^2+x}$ is continuous on $[0,1]\times[-1,1]$, it follows from Theorem 11.4 that $\int_0^1 e^{x^3y^2+x}\,dx \to \int_0^1 e^x\,dx = e-1$ as $y\to 0$.

b) Notice that $\sin(e^xy-y^3+\pi-e^x)$ is $\mathcal{C}^\infty$ on $[0,1]\times[-1,1]$. It follows from Theorem 11.5 that

$$\frac{d}{dy}\int_0^1 \sin(e^xy-y^3+\pi-e^x)\,dx = \int_0^1 (e^x-3y^2)\cos(e^xy-y^3+\pi-e^x)\,dx.$$

At $y=1$ we obtain $\int_0^1 (e^x-3)\cos(\pi-1)\,dx = (e-4)\cos(\pi-1)$.

c) The partial with respect to x of $\sqrt{x^3+y^3+z^3-2}$ equals $3x^2(x^3+y^3+z^3-2)^{-1/2}/2$. Since $x,y,z\in[1,3]$ implies $x^3+y^3+z^3-2\ge 3-2 = 1>0$, this partial exists and is continuous on $[1,3]\times[1,3]\times[1,3]$. The same things happens for the partials with respect to y and z. Thus by Theorem 11.5,

$$\frac{\partial}{\partial x}\int_1^3 x^2\sqrt{x^3+y^3+z^3-2}\,dx = \frac{3}{2}\int_1^3 x^2(x^3+y^3+z^3-2)^{-1/2}\,dx.$$

At $(x,y)=(1,1)$ we obtain $1.5\int_1^3 z^{-3/2}\,dz = 3-\sqrt{3}$.

11.1.6. a) Since f and $|x-1|$ are continuous on $\mathbf{R}$, the intergand is continuous on $\mathbf{R}^2$. Hence by Theorem 11.4,

$$I = \int_0^2 f(|x-1|)e^0\,dx = \int_0^1 f(1-x)\,dx + \int_1^2 f(x-1)\,dx.$$

Changing variables ($u = 1-x$ and $w = x-1$), we have

$$I = -\int_1^0 f(u)\,du + \int_0^1 f(w)\,dw = 1+1 = 2.$$

b) By Theorem 11.4 and parts, the integral in question is

$$e + \int_0^\pi f(x)\cos x\,dx = e - \int_0^\pi f'(x)\sin x\,dx = e - e = 0.$$

c) By Theorem 11.5,

$$\frac{d}{dx}\int_0^1 f(y)\,e^{xy+y^2}\,dy = \int_0^1 yf(y)e^{xy+y^2}\,dy.$$

At $x=0$, we obtain (after a change of variables $y=\sqrt{x}$ and $2y\,dy = dx$)

$$\int_0^1 yf(y)e^{y^2}\,dy = \frac{1}{2}\int_0^1 f(\sqrt{x})e^x\,dx = 3.$$

11.1.7. a) For $x\in[0,1]$ and $y>0$, $|x\cos y/\sqrt[3]{1-x+y}| \le 1/\sqrt[3]{1-x+y} < 1/\sqrt[3]{1-x}$ and

$$\int_0^1 \frac{dx}{\sqrt[3]{1-x}} = -\frac{3}{2}(1-x)^{2/3}\Big|_0^1 = \frac{3}{2} < \infty.$$

Thus the original integral converges uniformly on $(0,\infty)$ by the Weierstrass-M Test. Hence it follows from Theorem 11.8 that

$$\lim_{y\to 0+}\int_0^1 \frac{x\cos y}{\sqrt[3]{1-x+y}}\,dx = \int_0^1 \frac{x}{\sqrt[3]{1-x}}\,dx = -\int_1^0 \frac{1-u}{\sqrt[3]{u}}\,du = \frac{9}{10}.$$

b) Since $|e^{-xy}\sin x/x| \le e^{-xy} \le e^{-x/2}$ for $y\in[1/2,3/2]$, $\int_\pi^\infty (e^{-xy}\sin x/x)\,dx$ converges uniformly on $[1/2,3/2]$ by the Weierstrass-M Test. Hence it follows from Theorem 11.9 that

$$\frac{d}{dy}\int_\pi^\infty \frac{e^{-xy}\sin x}{x}\,dx = -\int_\pi^\infty e^{-x}\sin x\,dx$$

at $y=1$. Integrating by parts twice, we obtain

$$\int_\pi^\infty e^{-x}\sin x\,dx = \int_\pi^\infty e^{-x}\cos x\,dx = e^{-\pi} - \int_\pi^\infty e^{-x}\sin x\,dx,$$

i.e., $\int_\pi^\infty e^{-x}\sin x\,dx = -e^{-\pi}/2$. Therefore, $\partial/\partial y(\int_\pi^\infty (e^{-xy}\sin x/x)\,dx) = e^{-\pi}/2$ when $y=1$.

11.1.8. a) Since $|\cos(x^2+y^2)| \le 1$ for any $y\in(-\infty,\infty)$ and $\int_0^1 dx/\sqrt{x} < \infty$, it follows from the Weierstrass-M Test that $\int_0^1 \cos(x^2+y^2)/\sqrt{x}\,dx$ converges uniformly on $(-\infty,\infty)$.

b) By definition, $\int_0^\infty e^{-xy}\,dx = -e^{-xy}/y\,\Big|_0^\infty = 1/y$. Given $\epsilon>0$ choose N so large that $e^{-N}<\epsilon$. Then $|1/y - \int_0^N e^{-xy}\,dx| = |e^{-Ny}/y| \le e^{-N} < \epsilon$ since $y\ge 1$. Thus $\int_0^\infty e^{-xy}\,dx$ converges uniformly on $[1,\infty)$ by definition.

c) By definition,

$$\int_0^\infty ye^{-xy}\,dx = \begin{cases} -e^{-xy}\,\Big|_0^\infty = 1 & y>0 \\ 0 & y=0. \end{cases}$$

If this integral converges uniformly on $[0,1]$ then

$$\left|1-\int_0^N ye^{-xy}\,dx\right| = e^{-Ny} \le \frac{1}{2}$$

uniformly for $y\in(0,1]$ for N large. This contradicts the fact that $e^{-Ny}\to 1$ as $y\to 0$. Therefore, $\int_0^\infty e^{-xy}\,dx$ does not converge uniformly on $[0,1]$.

On the other hand, if $[a,b]\subset(0,\infty)$ then

$$\left|1-\int_0^N ye^{-xy}\,dx\right| = e^{-Ny} \le e^{-Na}\to 0$$

independently of y. Hence $\int_0^\infty e^{-xy}\,dx$ converges uniformly on $[a,b]$.

11.1.9. a) $\int_0^\infty e^{-st}\,dt = -e^{-st}/s\,\big|_0^\infty = 1/s$ for $s>0$.

b) By parts and part a),

$$\int_0^\infty t^n e^{-st}\,dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt = \cdots = \frac{n!}{s^n}\int_0^\infty e^{-st}\,dt = \frac{n!}{s^{n+1}}$$

for $s>0$.

c) $\int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-t(s-a)}\,dt = 1/(s-a)$ for $s>a$.

d) Integrating by parts twice,

$$\int_0^\infty e^{-st}\cos bt\,dt = \frac{1}{s}-\frac{b}{s}\int_0^\infty e^{-st}\sin bt\,dt = \frac{1}{s}-\frac{b^2}{s^2}\int_0^\infty e^{-st}\cos bt\,dt$$

for $s>0$. Solving for the integral, we obtain $\int_0^\infty e^{-st}\cos bt\,dt = s/(s^2+b^2)$.

e) Integrating by parts twice,

$$\int_0^\infty e^{-st}\sin bt\,dt = \frac{b}{s}\int_0^\infty e^{-st}\cos bt\,dt = \frac{b}{s^2}-\frac{b^2}{s^2}\int_0^\infty e^{-st}\sin bt\,dt$$

for $s>0$. Solving for the integral, we obtain $\int_0^\infty e^{-st}\sin bt\,dt = b/(s^2+b^2)$.

11.1.10. a) By the Weierstrass M-Test, $\phi(t)$ converges uniformly on $(0,t]$ for each $t\in(0,\infty)$. Since $\phi(0)=0$, integration by parts yields

$$\int_0^N e^{-st}f(t)\,dt = e^{-(s-a)N}\phi(N) + (s-a)\int_0^N e^{-(s-a)t}\phi(t)\,dt.$$

b) Since f is bounded, $|\phi(t)|\le M<\infty$ for all $t\in(0,\infty)$. For $s\ge b>a$, $|e^{-(s-a)t}\phi(t)|\le Me^{-(b-a)t}$, hence by the Weierstrass M-Test, $\int_0^\infty e^{-(s-a)t}\phi(t)\,dt$ converges uniformly on $[b,\infty)$. Hence by part a), it remains to see that $e^{-(s-a)N}\phi(N)\to 0$ as $N\to\infty$. But this follows immediately from the Squeeze Theorem since $\phi(N)$ is bounded as $N\to\infty$ and $e^{-(s-a)N}\to 0$ as $N\to\infty$ for any $s\ge b>a$.

c) By part b) and Theorem 11.8, $\mathcal{L}\{f\}$ exists and is continuous on (a,∞). To show it vanishes at infinity, let $\epsilon>0$. Choose $\delta>0$ such that $|\phi(t)|<\epsilon$ for $0\le t<\delta$. Then

$$\mathcal{L}\{f\}(s) \le \epsilon(s-a)\int_0^\delta e^{-(s-a)t}\,dt + (s-a)e^{-\delta(s-a-1)}\int_\delta^\infty e^{-t}|\phi(t)|\,dt =: I_1+I_2.$$

Now $(s-a)\int_0^\delta e^{-(s-a)t}\,dt = 1-e^{-(s-a)\delta}\to 1$ as $s\to\infty$. Therefore, $\limsup_{s\to\infty} I_1\le\epsilon$. On the other hand, by part b), $I_2\le M(s-a)/e^{\delta(s-a-1)}\to 0$ as $s\to\infty$. Therefore, $\limsup_{s\to\infty}\mathcal{L}\{f\}(s)\le\epsilon$, i.e., $\mathcal{L}\{f\}(s)\to 0$ as $s\to\infty$.

d) By part b) $\mathcal{L}\{f\}(s) = (s-a)\int_0^\infty e^{-(s-a)t}\phi(t)\,dt$ so by Theorem 11.9,

$$\begin{aligned}\mathcal{L}\{f\}'(s) &= \int_0^\infty e^{-(s-a)t}\phi(t)\,dt - (s-a)\int_0^\infty e^{-(s-a)t}t\phi(t)\,dt\\ &= \int_0^\infty e^{-(s-a)t}\phi(t)(1-(s-a)t)\,dt.\end{aligned}$$

On the other hand, since $d/dt(te^{-(s-a)t}) = e^{-(s-a)t}(1-(s-a)t)$, integration by parts yields

$$\mathcal{L}\{tf(t)\}(s) = \int_0^\infty te^{-t(s-a)}f(t)e^{-at}\,dt = -\int_0^\infty e^{-(s-a)t}\phi(t)(1-(s-a)t)\,dt.$$

Therefore, $\mathcal{L}\{f\}'(s) = -\mathcal{L}\{tf(t)\}(s)$.

e) Integrating by parts, we have

$$\begin{aligned}\mathcal{L}(f')(s) &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0) + s\mathcal{L}(f)(s).\end{aligned}$$

11.1.11. a) By Exercises 11.1.8 and 11.1.9, $\mathcal{L}\{te^t\} = -\mathcal{L}\{e^t\}'(s) = -(1/(s-1))' = 1/(s-1)^2$.

b) By Exercises 11.1.8 and 11.1.9, $\mathcal{L}\{t\sin\pi t\} = -\mathcal{L}\{\sin\pi t\}'(s) = -(\pi/(s^2+\pi^2))' = 2s\pi/(s^2+\pi^2)^2$.

c) By Exercises 11.1.8 and 11.1.9, $\mathcal{L}\{t^2\cos t\} = \mathcal{L}\{\cos t\}''(s) = (s/(s^2+1))'' = ((1-s^2)/(s^2+1)^2)' = 2(s^3-3s)/(s^2+1)^3$.

11.2 The Definition of Differentiability.

11.2.1. Let V denote the open cube $(-1,1)\times\cdots\times(-1,1)$. Clearly,

$$\frac{\partial g}{\partial x_j}(\mathbf{x}) = f_1(x_1)\cdots\frac{\partial f_j}{\partial x_j}(x_j)\cdots f_n(x_n).$$

Therefore, by Exercise 9.3.6 and hypothesis, g and g_{x_j} are all continuous on V. Hence by Theorem 11.15, g is differentiable on V.

11.2.2. Since f has a scalar domain and is differentiable at a, we have

$$0 = \lim_{x\to a}\frac{f(x)-f(a)-Df(a)\cdot(x-a)}{|x-a|} = \lim_{x\to a}\frac{f(x)-f(a)}{|x-a|} + \frac{(x-a)}{|x-a|}Df(a).$$

Thus $(f(x)-f(a))/|x-a| \to Df(a)$ as $x\to a+$ and $(f(x)-f(a))/|x-a| \to -Df(a)$ as $x\to a-$, i.e., $\|f(x)-f(a)\|/|x-a| \to \|Df(a)\|$ as $x\to a$. Thus it follows from the hypothesis $f(a)=g(a)=0$ that

$$\frac{\|f(x)\|}{\|g(x)\|} = \frac{\|f(x)-f(a)\|/|x-a|}{\|g(x)-g(a)\|/|x-a|} \to \frac{\|Df(a)\|}{\|Dg(a)\|}$$

as $x\to a$.

11.2.3. By definition,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{|h\cdot 0|}}{h} = 0.$$

Similarly, $f_y(0,0)=0$. Thus $Df(0,0)=(0,0)$. Consequently,

$$\frac{f(h,k)-f(0,0)-Df(0,0)\cdot(h,k)}{\|(h,k)\|} = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}.$$

Along the path $H=0$, this expression is 0 and along the path $h=k$, this expression is $1/\sqrt{2}$. Therefore, the limit of this expression does not exist, i.e., f is not differentiable at $(0,0)$.

11.2.4. By definition,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h}{\sin|h|} = \begin{cases} 1 & \text{as } h\to 0+ \\ -1 & \text{as } h\to 0-. \end{cases}$$

Thus the first partials of f do not exist at $(0,0)$ and f cannot be differentiable at $(0,0)$ by Theorem 11.14.

11.2.5. At any $(x, y) \neq (0,0)$, f has continuous first partials , hence is differentiable by Theorem 11.15. To examine the case when $(x, y) = (0,0)$, notice first that $f_x(0,0) = \lim_{h\to 0}(f(h,0) - f(0,0))/h = \lim_{h\to 0} h^{3-2\alpha} = 0$ because $3 - 2\alpha > 0$. Similarly, $f_y(0,0) = 0$. Since

$$\frac{f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\|(h,k)\|} = \frac{h^4 + k^4}{(h^2+k^2)^{\alpha+1/2}} \leq 2(h^2+k^2)^{3/2-\alpha} \to 0$$

as $(h,k) \to (0,0)$, f is also differentiable at $(0,0)$. We conclude that f is differentiable on $\mathbf{R}^2$.

11.2.6. The function has continuous first partials at any $(x, y) \neq (0,0)$, hence is differentiable there by Theorem 11.15. To examine the case when $(x, y) = (0,0)$, notice first that $f_x(0,0) = \lim_{h\to 0}(f(h,0) - f(0,0))/h = \lim_{h\to 0} 0 = 0$. Similarly, $f_y(0,0) = 0$. Consequently,

$$\left|\frac{f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\|(h,k)\|}\right| = \left|\frac{(hk)^\alpha \log(h^2+k^2)}{(h^2+k^2)^{1/2}}\right|$$
$$\leq \frac{1}{2^\alpha}(h^2+k^2)^{\alpha-1/2}\log\left(\frac{1}{h^2+k^2}\right).$$

Now by l'Hôpital's Rule, $u^\epsilon \log(1/u) \to 0$ as $u \to 0$ for any $\epsilon > 0$. Therefore, $(f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k))/\|(h,k)\| \to 0$ as $(h,k) \to (0,0)$. In particular, f is differentiable at $(0,0)$. We conclude that f is differentiable on $\mathbf{R}^2$.

11.2.7. Clearly, f is continuous and has first-order partial derivatives at every point $(x, y) \neq (0,0)$. What happens at $(0,0)$? Since

$$|f(x,y)| = \frac{|x||x^2-y^2|}{x^2+y^2} \leq |x|,$$

it follows from the Squeeze Theorem that f is continuous at $(0,0)$ with $f(0,0) = 0$. Moreover, the function f has first-order partial derivatives at $(0,0)$, since

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{h^3}{h^3} = 1,$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h^3} = 0.$$

Finally, if f were differentiable at $(0,0)$, then

$$0 = \lim_{(h,k)\to(0,0)}\frac{f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{-2hk^2}{(h^2+k^2)^{3/2}}.$$

But the path $h = k$ gives a limit of $-1/\sqrt{2} \neq 0$ as $h \to 0+$. Thus f is not differentiable at $(0,0)$.

11.2.8. Since T is linear,

$$\frac{\|T(\mathbf{a}+\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = \frac{\|T(\mathbf{a}) + T(\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} \equiv \frac{0}{\|\mathbf{h}\|} = 0.$$

Thus by definition, T is differentiable at $\mathbf{a}$ and $DT(\mathbf{a}) = T$.

11.2.9. By the Squeeze Theorem, $|f(\mathbf{x})| \leq \|\mathbf{x}\| \to 0$ as $\mathbf{x} \to 0$ so $f(0) = 0$. Hence

$$\left|\frac{\partial f}{\partial x_j}(0)\right| = \lim_{h\to 0}\left|\frac{f(0,0,\dots,h,\dots,0) - f(0)}{h}\right| \leq \lim_{h\to 0}|h|^{\alpha-1} = 0$$

as $h \to 0$ since $\alpha > 1$. It follows that $f_{x_j}(0) = 0$ for $j = 1, \dots, m$, i.e., $\nabla f(0) = 0$. Since

$$\left|\frac{f(0+\mathbf{h}) - f(0) - \nabla f(0)\cdot\mathbf{h}}{\|\mathbf{h}\|}\right| = \frac{|f(\mathbf{h})|}{\|\mathbf{h}\|} \leq \|\mathbf{h}\|^{\alpha-1} \to 0$$

as $h \to 0$, we conclude that f is differentiable at 0.

Let $f(\mathbf{x}) = \|\mathbf{x}\|$. Then f satisfies the condition for $\alpha = 1$ but $f_x(0,0) = \lim_{h\to 0} |h|/h$ does not exist. Therefore, f is not differentiable at $(0,0)$.

11.2.10. a) By definition,

$$D_{\mathbf{e}_k} f(\mathbf{a}) = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_k) - f(\mathbf{a})}{t} = \frac{\partial f}{\partial x_k}(\mathbf{a}).$$

b) By part a), if f has directional derivative in all directions, then all first partials of f exist. To show the converse is not true, let f be given by Example 11.11. Clearly, $f_x(0,0) = f_y(0,0) = 1$ both exist. However, if $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$ then

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{f(t/\sqrt{2}, t/\sqrt{2}) - f(0,0)}{t} = \lim_{t\to 0} \frac{1}{t}$$

does not exist.

c) Let (u,v) be a unit vector. By definition,

$$D_{(u,v)} f(0,0) = \lim_{t\to 0} \frac{f(tu,tv) - f(0,0)}{t} = \lim_{t\to 0} \frac{u^2 v}{u^4 t^2 + v^2} = \begin{cases} u^2/v & v \neq 0 \\ 0 & v = 0. \end{cases}$$

Thus the directional derivatives of f exist. On the other hand, f is not continuous because along the path $x = 0$ the limit is zero, but along the path $y = x^2$ the limit is $1/2$.

11.2.11. a) By the one-dimensional Mean Value Theorem, $\Delta(h) = hf_y(a+h, b+th) - hf_y(a, b+th)$ for some $t \in (0,1)$, and

$$(*) \qquad \Delta(h) = hf_x(a+uh, b+h) - hf_x(a+uh, b)$$

for some $u \in (0,1)$. Since $\nabla f_y(a,b)\cdot(h,th) - \nabla f_y(a,b)\cdot(0,th) = hf_{yx}(a,b)$ we can write

$$\begin{aligned}\frac{\Delta(h)}{h} &= f_y(a+h, b+th) - f_y(a, b+th) \\ &= f_y(a+h, b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th) \\ &\qquad - (f_y(a, b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th)) + hf_{yx}(a,b).\end{aligned}$$

b) Since f_y is differentiable at (a,b), we have

$$\lim_{(u,v)\to(0,0)} \frac{f_y(a+u, b+v) - f_y(a,b) - \nabla f_y(a,b)\cdot(u,v)}{\|(u,v)\|} = 0.$$

Therefore, it follows from part a) that

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = \lim_{h\to 0} \frac{f_y(a+h, b+th) - f_y(a, b+th)}{h} = f_{yx}(a,b).$$

c) If we start with (*) and reverse the roles of x and y, we have

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{xy}(a,b).$$

Combining this with part b), we conclude that $f_{yx}(a,b) = f_{xy}(a,b)$.

11.3 Derivatives, Differentials, and Tangent Planes.

11.3.1. Since they are all $\mathcal{C}^1$ on their domains, they are all differentiable on their domains.

a) $Df(x,y) = [\,1 \quad -1\,]$ and $Dg(x,y) = [\,2x \quad 2y\,]$. Hence

$$D(f+g)(x,y) = [\,2x+1 \quad 2y-1\,] \quad \text{and} \quad D(f\cdot g)(x,y) = [\,3x^2 - 2xy + y^2 \quad 2xy - x^2 - 3y^2\,].$$

b) $Df(x,y) = [\,y \quad x\,]$ and $Dg(x,y) = [\,x\cos x + \sin x \quad \sin y\,]$. Hence

$$D(f+g)(x,y) = [\,x\cos x + \sin x + y \quad x + \sin y\,]$$

and

$$D(f\cdot g)(x,y) = [\, x^y\cos x + 2xy\sin x - y\cos y \quad xy\sin y + x^2\sin x - x\cos y\,].$$

c) Since

$$Df(x,y) = \begin{bmatrix} -y\sin(xy) & -x\sin(xy) \\ \log y & x/y \end{bmatrix} \quad \text{and} \quad Dg(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have

$$D(f+g)(x,y) = \begin{bmatrix} -y\sin(xy) & 1 - x\sin(xy) \\ 1+\log y & x/y \end{bmatrix}$$

and

$$D(f\cdot g)(x,y) = [\, 2x\log y - y^2\sin(xy) \quad \cos(xy) - xy\sin(xy) + x^2/y\,].$$

d) Since

$$Df(x,y,z) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad Dg(x,y,z) = \begin{bmatrix} yz & xz & xy \\ 0 & 2y & 0 \end{bmatrix},$$

we have

$$D(f+g)(x,y,z) = \begin{bmatrix} yz & xz+1 & xy \\ 1 & 2y & -1 \end{bmatrix} \quad \text{and} \quad D(f\cdot g)(x,y,z) = [\, y^2(z+1) \quad 2xyz + 2xy - 2yz \quad y^2(x-1)\,].$$

11.3.2. a) Since $\nabla f = (2x, 2y) = (2,-2)$ at $(1,-1)$, and the equation of the tangent plane is $z = f(1,-1) + \nabla f(1,-1)\cdot(x-1, y+1)$, we have $z = 2x - 2y - 2$.

b) Since $\nabla f = (3x^2y - y^3, x^3 - 3xy^2) = (2,-2)$ at $(1,1)$, and the equation of the tangent plane is $z = f(1,1) + \nabla(1,1)\cdot(x-1,y-1)$, we have $z = 2x - 2y$.

c) Since $\nabla f = (y, x, \cos z) = (0,1,0)$ at $(1,0,\pi/2)$, and the equation of the tangent plane is $w = f(1,0,\pi/2) + \nabla f(1,0,\pi/2)\cdot(x-1, y, z-\pi/2)$, we have $w = y+1$.

11.3.3. By Theorem 11.22, the normal direction is given by $(-2x,-2y,1)$. This vector is parallel to $(1,1,1)$, the normal of the plane $x+y+z=1$, if and only if $x = y = -1/2$. Thus the point on the paraboloid where the tangent plane is parallel to $x+y+z=1$ is $(-1/2,-1/2,1/2)$ and an equation of this tangent plane is $2x+2y+2z=-1$. A portion of the plane $x+y+z=1$ lies above the first quadrant of the xy plane and slants back toward the z axis, so the point (x_0,y_0,z_0) where the tangent plane is parallel should be on the "back" side of the paraboloid, i.e., (x_0,y_0) should lie in the fourth quadrant.

11.3.4. a) If $(x,y)\neq(0,0)$, then by the Chain Rule, a normal to $\mathcal{K}$ at (x,y,z) is given by $(x/z, y/z, -1)$. But $(a/c, b/c, -1)\cdot(1,0,1) = 0$ implies that $a = c$. If (a,b,c) belongs to the cone, then $c^2 = a^2+b^2 = c^2+b^2$, i.e., $b = 0$. Thus an equation of the plane tangent to $\mathcal{K}$ perpendicular to $x+z=5$ at a point (a,b,c) is

$$(1,0,1)\cdot(x-a, y, z-a) = 0,$$

i.e., $x - z = 0$.

b) If $(a,b,c)\in\mathcal{K}$ and $(a/c, b/c, -1) = t(1,-1,1)$, then $t=-1$, so $a = -c$ and $b = c$. Since (a,b,c) lies on the cone, it follows that $c^2 = a^2+b^2 = 2c^2$, i.e., $c = 0$. Since $c = 0$ implies $a = b = 0$ and $\mathcal{K}$ has no tangent plane at the origin, there are no tangent planes to this cone which are parallel to $x - y + z = 5$.

11.3.5. a) Let $T = Df(a)$ and $S = Dg(a)$. By the Triangle Inequality,

$$\frac{\|f(a+h)+g(a+h)-f(a)-g(a)-T(h)-S(h)\|}{\|h\|} \le \frac{\|f(a+h)-f(a)-T(h)\|}{\|h\|} + \frac{\|g(a+h)-g(a)-S(h)\|}{\|h\|}.$$

Since these last two terms converge to zero as $\|h\|\to 0$, it follows that $f+g$ is differentiable, and $D(f+g)(a) = T+S$. On the other hand, by homogeneity,

$$\frac{\|(\alpha f)(a+h)-(\alpha f)(a)-(\alpha T)(h)\|}{\|h\|} = |\alpha|\frac{\|f(a+h)-f(a)-T(h)\|}{\|h\|}$$

Since this last term converge to $\alpha\cdot 0 = 0$ as $\|h\|\to 0$, it follows that αf is differentiable, and $D(\alpha f)(a) = \alpha T$.

11.3.6. a) By modifying the proof of Lemma 3.28, we can prove that if $f(a) \neq 0$, then $|f(a+h)| > |f(a)|/2 > 0$ for h small.

b) By the definition of the operator norm,

$$\left\| \frac{Df(a)(h)}{\|h\|} \right\| = \frac{\|Df(a)(h)\|}{\|h\|} \leq \frac{\|Df(a)\| \, \|h\|\|}{\|h\|} = \|Df(a)\|$$

for all $h \neq 0$.

c) Choose $h \neq 0$ small enough so that $f(a+h) \neq 0$. Using $f(a)f(a+h)$ as a common denominator and the definition of T, we have

$$\begin{aligned}\frac{1}{f(a+h)} - \frac{1}{f(a)} - T(h) &= \frac{f(a) - f(a+h) - f(a)f(a+h)T(h)}{f(a)f(a+h)} \\ &= \frac{f(a) - f(a+h) + Df(a)(h)f(a+h)/f(a)}{f(a)f(a+h)} \\ &= \frac{f(a) - f(a+h) + Df(a)(h)}{f(a)f(a+h)} + \frac{(f(a+h) - f(a))Df(a)(h)}{f^2(a)f(a+h)} \\ &=: I_1 + I_2.\end{aligned}$$

d) Since f is differentiable at a, f is continuous at a and $I_1/\|h\| \to 0/f^2(a) = 0$ as $h \to 0$. Similarly, by part b) and the Squeeze Theorem, $I_2/\|h\| \to 0/f^3(a) \equiv 0$ as $h \to 0$. It follows that $1/f$ is differentiable at a and its derivative is T. This is sometimes called the Reciprocal Rule.

e) By the Product and Reciprocal Rules, $D(f/g)(\mathbf{a}) = D(f \cdot (1/g))(\mathbf{a}) = (1/g(\mathbf{a}))Df(\mathbf{a}) + f(\mathbf{a})(-Dg(\mathbf{a})/g^2(\mathbf{a}) = (g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a}))/g^2(\mathbf{a})$.

11.3.7. Define $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ by $= T(\mathbf{y}) := f(\mathbf{a}) \times (Dg(\mathbf{a})(\mathbf{y})) - g(\mathbf{a}) \times (Df(\mathbf{a})(\mathbf{y}))$. Notice that $Df(\mathbf{a})(\mathbf{y}) \in \mathbf{R}^3$ so this cross- product makes sense under the identification of 3×1 matrices with vectors in $\mathbf{R}^3$. Fix $\mathbf{h}$ with norm so small that $f(\mathbf{a}+\mathbf{h})$ is defined, and observe by the distributive law that

$$\begin{aligned}f(\mathbf{a}+\mathbf{h}) \times g(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \times g(\mathbf{a}) - T(\mathbf{h}) &= f(\mathbf{a}+\mathbf{h}) \times (g(\mathbf{a}+\mathbf{h}) - g(\mathbf{a})) \\ &\quad + (f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})) \times g(\mathbf{a}) \\ &\quad\quad - f(\mathbf{a}) \times Dg(\mathbf{a})(\mathbf{h}) - Df(\mathbf{a})(\mathbf{h}) \times g(\mathbf{a}) \\ &= f(\mathbf{a}+\mathbf{h}) \times (g(\mathbf{a}+\mathbf{h}) - g(\mathbf{a}) - Dg(\mathbf{a})(\mathbf{h})) \\ &\quad + (f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})) \times Dg(\mathbf{a})(\mathbf{h}) \\ &\quad\quad (f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{h})) \times g(\mathbf{a}) \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Since g is differentiable at $\mathbf{a}$, $I_1/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to 0$. Since f is differentiable at $\mathbf{a}$, $I_3/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to 0$. Finally, since f is continuous and (by the definition of the operator norm) $\|Dg(\mathbf{a})(\mathbf{h})\| \leq \|Dg(\mathbf{a})\| \, \|\mathbf{h}\|$,

$$\|I_2\|/\|\mathbf{h}\| \leq \|Dg(\mathbf{a})\| \, \|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})\| \to 0$$

as $\mathbf{h} \to 0$. It follows from the Squeeze Theorem that $f \times g$ is differentiable and its total derivative is T.

11.3.8. a) $dz = 2x\,dx + 2y\,dy$.

b) $dz = y\cos(xy)\,dx + x\cos(xy)\,dy$.

c)
$$dz = \frac{(1 - x^2 + y^2)y}{(1 + x^2 + y^2)^2}\,dx + \frac{(1 + x^2 - y^2)x}{(1 + x^2 + y^2)^2}\,dy.$$

11.3.9. $dw = 2xy\,dx + x^2\,dy + dz$ so $\Delta w \approx 4(.01) + 1^2(-.02) + .03 = .05$. The actual value is $\Delta w = f(1.01, 1.98, 1.03) - f(1, 2, 1) \approx 3.049798 - 3 = 0.049798$.

11.3.10. By definition,

$$dT = 2\pi\frac{1}{2}\sqrt{\frac{g}{L}}\left(\frac{g\,dL - L\,dg}{g^2}\right) = \frac{T}{2}\left(\frac{dL}{L} - \frac{dg}{g}\right).$$

Thus $dL/L = 2dT/T + dg/g$. The worst possible scenario is $dT/T = \pm 0.02$ and $dg/g = \mp 0.01$ so $dL/L = \pm 0.03$. Hence, allow no more than an error of 3%.

Note: $dL/L = \pm 0.05$ does not work because then $dT/T = (dL/L - dg/g)/2$ might equal $(0.05+0.01)/2 = 0.03$, outside the 2% error allowed for T.

11.3.11. Since

$$-\frac{1}{w^2}\,dw = -\frac{1}{x^2}\,dx - \frac{1}{y^2}\,dy - \frac{1}{z^2}\,dz$$

we have $dw/w = \pm p(w/x + w/y + w/z) = \pm p$.

11.4 The Chain Rule.

11.4.1. By the Chain Rule,

$$\frac{\partial w}{\partial p} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial p}, \qquad \frac{\partial w}{\partial q} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial q}.$$

Hence by the Product Rule and Theorem 11.2,

$$\begin{aligned}\frac{\partial^2 w}{\partial p^2} &= \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial x}\right)\frac{\partial x}{\partial p} + \frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial p^2} + \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial y}\right)\frac{\partial y}{\partial p} + \frac{\partial F}{\partial y}\frac{\partial^2 y}{\partial p^2} + \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial z}\right)\frac{\partial z}{\partial p} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial p^2}\\ &= \frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial p^2} + \frac{\partial F}{\partial y}\frac{\partial^2 y}{\partial p^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial p^2}\\ &\quad + \frac{\partial F^2}{\partial x^2}\left(\frac{\partial x}{\partial p}\right)^2 + \frac{\partial^2 F}{\partial y^2}\left(\frac{\partial y}{\partial p}\right)^2 + \frac{\partial^2 F}{\partial z^2}\left(\frac{\partial z}{\partial p}\right)^2\\ &\qquad + 2\frac{\partial^2 F}{\partial x\partial y}\left(\frac{\partial x}{\partial p}\right)\left(\frac{\partial y}{\partial p}\right) + 2\frac{\partial^2 F}{\partial x\partial z}\left(\frac{\partial x}{\partial p}\right)\left(\frac{\partial z}{\partial p}\right) + 2\frac{\partial^2 F}{\partial y\partial z}\left(\frac{\partial y}{\partial p}\right)\left(\frac{\partial z}{\partial p}\right).\end{aligned}$$

11.4.2. a) By the Chain Rule

$$[\,\partial h/\partial x_1(\mathbf{a}) \quad \dots \quad \partial h/\partial x_n(\mathbf{a})\,] = \nabla f(g(\mathbf{a}))\begin{bmatrix}\partial g_1/\partial x_1(\mathbf{a}) & \dots & \partial g_1/\partial x_n(\mathbf{a})\\ \vdots & \dots & \vdots\\ \partial g_m/\partial x_1(\mathbf{a}) & \dots & \partial g_m/\partial x_n(\mathbf{a})\end{bmatrix}.$$

Therefore, $\partial h/\partial x_j(\mathbf{a}) = \nabla f(g(\mathbf{a}))\cdot \partial g/\partial x_j(\mathbf{a})$.

b) Since $D(f\circ g)(\mathbf{a}) = Df(g(\mathbf{a}))Dg(\mathbf{a})$ and the determinant of a product is the product of the determinants, we have $\Delta_{f\circ g}(\mathbf{a}) = \Delta_f(g(\mathbf{a}))\Delta_g(\mathbf{a})$.

11.4.3. If f is homogeneous of order k, then $f(0) = f(0\cdot\mathbf{x}) = 0^k f(\mathbf{x}) = 0$. Thus the formula holds when $\mathbf{x} = 0$. If $\mathbf{x} \neq 0$, then for any $\rho \neq 0$ we have

$$f_{x_j}(\rho\mathbf{x}) = \lim_{h\to 0}\frac{f(\rho\mathbf{x} + \rho h\mathbf{e}_j) - f(\rho\mathbf{x})}{\rho h} = \frac{\rho^k}{\rho}\lim_{h\to 0}\frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} = \rho^{k-1}f_{x_j}(\mathbf{x}).$$

Thus by the Chain Rule, homogeneity, and the Power Rule,

$$\rho^{k-1}\sum_{j=1}^{n} x_j f_{x_j}(\mathbf{x}) = \sum_{j=1}^{n} x_j f_{x_j}(\rho\mathbf{x}) = \frac{d}{d\rho}(f(\rho\mathbf{x})) = \frac{d}{d\rho}(\rho^k f(\mathbf{x})) = k\rho^{k-1}f(\mathbf{x}).$$

11.4.4. By the Chain Rule, $u_x = yf'(xy)$ and $u_y = xf'(xy)$. Hence $xu_x - yu_y = xyf'(xy) - xyf'(xy) = 0$. Similarly, $v_x = f'(x-y) + g'(x+y)$, $v_y = -f'(x-y) + g'(x+y)$, and $v_{xx} - v_{yy} = f''(x-y) + g''(x+y) - (f''(x-y) + g''(x+y)) = 0$.

11.4.5. By the Chain Rule, $u_r = f_x\cos\theta + f_y\sin\theta$, $v_r = g_x\cos\theta + g_y\sin\theta$, $u_\theta = -f_x\,r\sin\theta + f_y\,r\cos\theta$, and $v_\theta = -g_x r\sin\theta + g_y\,r\cos\theta$. Therefore, $u_r = g_y\cos\theta - g_x\sin\theta = v_\theta/r$ and $v_r = -f_y\cos\theta + f_x\sin\theta = -u_\theta/r$.

11.4.6. $u_{rr} = f_{xx}\cos^2\theta + f_{xy}\sin\theta\cos\theta + f_{yx}\sin\theta\cos\theta + f_{yy}\sin^2\theta$, and $u_{\theta\theta} = -f_x\, r\cos\theta + f_{xx}\, r^2\sin^2\theta - f_{xy}\, r^2\sin\theta\cos\theta - f_y\, r\sin\theta - f_{yx}r^2\sin\theta\cos\theta + f_{yy}\, r^2\cos^2\theta$. Therefore,

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = f_{xx} + f_{yy} = 0.$$

11.4.7. a) By the Product Rule,

$$u_{xx}(x,t) = \frac{\partial}{\partial x}\left(\frac{-2x}{4t}u(x,t)\right) = \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)u(x,t) = u_t(x,t).$$

b) If $x \ge a$ then $u(x,t) \le e^{-a^2/4t}/\sqrt{4\pi t} \to 0$ as $t \to 0+$ independently of x.

11.4.8. Let $w = \sqrt{x^2+y^2+z^2}$. By the Chain Rule, $F_x = u'(w)x/w$, $F_y = u'(w)y/w$, and $F_z = u'(w)z/w$. Therefore,

$$\sqrt{F_x^2 + F_y^2 + F_z^2} = |u'(w)|\sqrt{x^2/w^2 + y^2/w^2 + z^2/w^2} = |u'(w)|\sqrt{1} = |u'(\sqrt{x^2+y^2+z^2})|.$$

11.4.9. Let $y = f(x)$ and take the derivative of $F(x, f(x)) = 0$ with respect to x. By the Chain Rule, $0 = F_x(a,b) + F_y(a,b)(dy/dx)$, hence $dy/dx = -F_x(a,b)/F_y(a,b)$.

11.4.10. By hypothesis, $(f\cdot f)(t) = r^2$ is constant on I, hence by the Dot Product Rule, $0 = (f\cdot f)'(t) = f'(t)\cdot f(t) + f(t)\cdot f'(t) = 2f(t)\cdot f'(t)$ for all $t \in I$. In particular, $f(t)$ is orthogonal to $f'(t)$.

11.4.11. a) Let $g(t) = \mathbf{a} + t\mathbf{u}$ and $h(t) = f\circ g(t)$. By Exercise 11.2.8, $Dg(t) = \mathbf{u}$ for all t and by definition,

$$h'(0) = \lim_{t\to 0}\frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t} = D_{\mathbf{u}}f(\mathbf{a}).$$

Hence by the Chain Rule, $D_{\mathbf{u}}f(\mathbf{a}) = (f\circ g)'(0) = Df(g(0))\cdot g'(0) = \nabla f(\mathbf{a})\cdot\mathbf{u}$.

b) By (2) in 5.1 and part a), $\cos\theta = \nabla f(\mathbf{a})\cdot\mathbf{u}/(\|\nabla f(\mathbf{a})\|\,\|\mathbf{u}\|) = D_{\mathbf{u}}f(\mathbf{a})/\|\nabla f(\mathbf{a})\|$. Therefore, $D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\|\cos\theta$.

c) If $\nabla f(\mathbf{a}) = 0$ then $D_{\mathbf{u}}f(\mathbf{a}) = 0$ and there is nothing to prove. Otherwise, $D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\|\cos\theta$ ranges from $-\|\nabla f(\mathbf{a})\|$ (when $\theta = \pi$) to $\|\nabla f(\mathbf{a})\|$ (when $\theta = 0$), with maximum value of $\|\nabla f(\mathbf{a})\|$ when $\theta = 0$, i.e., when $\mathbf{u}$ is parallel to $\nabla f(\mathbf{a})$.

11.5 The Mean Value Theorem and Taylor's Formula.

11.5.1. a) Clearly, $f_x = 2x + y$, $f_y = x + 2y$, $f_{xx} = 2$, $f_{xy} = 1$, and $f_{yy} = 2$. Hence

$$f(x,y) = 1 - (x+1) + (y-1) + (x+1)^2 + (x+1)(y-1) + (y-1)^2$$

by Taylor's Formula.

b) Clearly, $f_x = 1/(2\sqrt{x})$, $f_y = 1/(2\sqrt{y})$, $f_{xx} = -1/(4x^{3/2})$, $f_{xy} = 0$, $f_{yy} = -1/(4y^{3/2})$, $f_{xxx} = 3/(8x^{5/2})$, $f_{yyy} = 3/(8y^{5/2})$, and all mixed third order partials are zero. Thus by Taylor's Formula,

$$\sqrt{x} + \sqrt{y} = 3 + \frac{x-1}{2} + \frac{y-4}{4} - \frac{(x-1)^2}{8} - \frac{(y-4)^2}{64} + \frac{(x-1)^3}{16\sqrt{c^5}} + \frac{(y-4)^3}{16\sqrt{d^5}}$$

for some $(c,d) \in L((x,y);(1,4))$.

c) Clearly, $f_x = ye^{xy}$, $f_y = xe^{xy}$, $f_{xx} = y^2e^{xy}$, $f_{xy} = (xy+1)e^{xy}$, $f_{yy} = x^2e^{xy}$, $\dots$, $f_{xxxx} = y^4e^{xy}$, $f_{xxxy} = (3y^2 + xy^3)e^{xy}$, $f_{xxyy} = (2 + 4xy + x^2y^2)e^{xy}$, $f_{xyyy} = (3x^2 + x^3y)e^{xy}$, and $f_{yyyy} = x^4e^{xy}$. Thus by Taylor's Formula,

$$e^{xy} = 1 + xy + \frac{e^{cd}}{4!}((dx+cy)^4 + 12(dx+cy)^2xy + 12x^2y^2)$$

for some $(c,d) \in L((x,y);(0,0))$.

11.5.2. We must show

$$D^{(\ell)}f((a,b);(h,k)) = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\partial^\ell f}{\partial x^j \partial y^{\ell-j}}(a,b)h^j k^{\ell-j}$$

for $\ell \in \mathbf{N}$.

This formula holds for $\ell = 1$. Suppose it holds for $\ell - 1$. Then by definition,

$$\begin{aligned}
D^{(\ell)}f((a,b);(h,k)) &= \frac{\partial D^{(\ell-1)}f}{\partial x}(a,b)h + \frac{\partial D^{(\ell-1)}f}{\partial y}(a,b)k \\
&= \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} \frac{\partial^\ell f}{\partial x^{j+1} \partial y^{\ell-1-j}}(a,b)h^{j+1}k^{\ell-1-j} \\
&\qquad + \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} \frac{\partial^\ell f}{\partial x^j \partial y^{\ell-j}}(a,b)h^j k^{\ell-j} \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\partial^\ell f}{\partial x^j \partial y^{\ell-j}}(a,b)h^j k^{\ell-j}.
\end{aligned}$$

Thus by induction, the formula holds for all $\ell \in \mathbf{N}$.

11.5.3. By the Mean Value Theorem and the assumption about Dg,

$$f(g(\mathbf{x})) - f(g(\mathbf{a})) = Df(g(\mathbf{c}))Dg(\mathbf{c})(\mathbf{x}-\mathbf{a}) = Df(g(\mathbf{c}))(\mathbf{x}-\mathbf{a})$$

for some $\mathbf{c}$ on the line segment from $\mathbf{a}$ to $\mathbf{x}$. It follows from the definition of the operator norm that

$$|f(g(\mathbf{x})) - f(g(\mathbf{a}))| \le \|Df(g(\mathbf{c}))\| \, \|\mathbf{x}-\mathbf{a}\|.$$

Thus set $h(\mathbf{x}) = \mathbf{c}$.

11.5.4. Let $B = [b_{ij}]$ be the $n \times n$ matrix that represents $Df(\mathbf{a})$ and set $S(\mathbf{x}) = B(\mathbf{x})$. By Remark 8.14, $S \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^n)$. Moreover, by Exercise 11.2.8 and hypothesis, $DS(\mathbf{x}) = B = Df(\mathbf{x})$ for all $\mathbf{x} \in V$. It follows from the proof of Corollary 11.34, applied to $f - S$, that

$$\|f(\mathbf{x})-S(\mathbf{x})-f(\mathbf{a})+S(\mathbf{a})\|^2 \le \|f(\mathbf{x})-S(\mathbf{x})-f(\mathbf{a})+S(\mathbf{a})\| \, \|D(f-S)(\mathbf{c})(\mathbf{x}-\mathbf{a})\| = \|f(\mathbf{x})-S(\mathbf{x})-f(\mathbf{a})+S(\mathbf{a})\| \cdot 0 = 0$$

for all $\mathbf{x} \in V$. Thus $f(\mathbf{x}) = S(\mathbf{x}) + f(\mathbf{a}) - S(\mathbf{a})$ on V, so set $\mathbf{c} = f(\mathbf{a}) - S(\mathbf{a})$.

11.5.5. Let F be defined as in the proof of Theorem 11.35. Using Lagrange's integral form of the remainder term for the one-dimensional Taylor's Formula, we obtain

$$f(\mathbf{x}) - f(\mathbf{a}) = F(1) - F(0) = \sum_{j=1}^{p-1} \frac{1}{j!} F^{(j)}(0) + \frac{1}{(p-1)!} \int_0^1 (1-t)^{p-1} F^{(p)}(t)\,dt.$$

Since $F^{(p)}(t) = D^{(p)}f(\mathbf{a} + t(\mathbf{x}-\mathbf{a});(\mathbf{x}-\mathbf{a}))$, the result follows at once.

11.5.6. a) By the Chain Rule, $g'(t) = f_x(tx+(1-t)a, y)(x-a) + f_y(a, ty+(1-t)b)(y-b)$.

b) By the Mean Value Theorem,

$$f(x,y) - f(a,b) = g(1) - g(0) = g'(t) = f_x(c,y)(x-a) + f_y(a,d)(y-b)$$

for some $t \in (0,1)$, where $c = tx + (1-t)a$ and $d = ty + (1-t)b$.

11.5.7. Set $E = B_r(0)$. If $\mathbf{x}_k \in E$ satisfies $\mathbf{x}_k \to 0$ as $k \to \infty$, then by the continuity of f,

$$|f(0)| = \lim_{k\to\infty} |f(\mathbf{x}_k)| \le \lim_{k\to\infty} \|\mathbf{x}_k\|^\alpha = 0.$$

Moreover, $\overline{E}$ is convex, closed, and bounded, hence compact. Hence by Corollary 11.34, there is a constant $M > 0$ such that $|f(\mathbf{x})| = |f(\mathbf{x}) - f(0)| \le M\|\mathbf{x} - 0\| = M\|\mathbf{x}\|$ for all $\mathbf{x} \in \overline{E}$.

11.5.8. Let $\mathbf{x} \in H$. Since H is convex, use Taylor's Formula to write $f(\mathbf{x}) = f(\mathbf{a}) + D^{(2)} f(\mathbf{c}; \mathbf{x} - \mathbf{a})/2!$. By definition,

$$D^{(2)} f(\mathbf{c}; \mathbf{x} - \mathbf{a}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_1 \partial x_j}(\mathbf{c})(x_i - a_i)(x_j - a_j).$$

Since H is compact and all these partial derivatives are continuous on H, it follows that there is a $C > 0$ such that

$$|D^{(2)} f(\mathbf{c}; \mathbf{x} - \mathbf{a})| \le C \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - a_i|\,|x_j - a_j| \le n^2 C \|\mathbf{x} - \mathbf{a}\|^2.$$

Therefore, $|f(\mathbf{x}) - f(\mathbf{a})| = |D^{(2)} f(\mathbf{c}; \mathbf{x} - \mathbf{a})|/2! \le M\|\mathbf{x} - \mathbf{a}\|^2$ for $M = n^2 C/2$.

11.5.9. Let $F(t) = f(\mathbf{a} + t\mathbf{u})$ and observe by definition that

$$F'(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = \lim_{h \to 0} \frac{f(\mathbf{a} + t\mathbf{u} + h\mathbf{u}) - f(\mathbf{a} + t\mathbf{u})}{h} = D_{\mathbf{u}} f(\mathbf{a} + t\mathbf{u}).$$

Thus F is a differentiable real function on $[0, 1]$, and it follows from the one-dimensional Mean Value Theorem that

$$f(\mathbf{a} + \mathbf{u}) - f(\mathbf{a}) = F(1) - F(0) = F'(t) = f(\mathbf{a} + t\mathbf{u})$$

for some $t \in (0, 1)$.

11.5.10. By Taylor's Formula,

$$f(a+x, b+y) = f(a,b) + f_x(a,b)x + f_y(a,b)y + f_{xx}(a,b)\frac{x^2}{2!} + f_{xy}(a,b)xy + f_{yy}(a,b)\frac{y^2}{2!}$$
$$+ f_{xxx}(c,d)\frac{x^3}{3!} + f_{xxy}(c,d)\frac{x^2 y}{2!} + f_{xyy}(c,d)\frac{xy^2}{2!} + f_{yyy}(c,d)y^3$$

for some $(c, d) \in L((a, b); (a + x, b + y))$. Thus

$$\int_0^{2\pi} f(a + r\cos\theta, b + r\sin\theta)\cos(2\theta)\,d\theta = f_{xx}(a,b)\frac{\pi r^2}{4} - f_{yy}(a,b)\frac{\pi r^2}{4} + R$$

where

$$R := f_{xxx}(c,d)\int_0^{2\pi} \frac{(r\cos\theta)^3}{6}\cos(2\theta)\,d\theta + f_{xxy}(c,d)\int_0^{2\pi} \frac{(r\cos\theta)^2(r\sin\theta)}{2}\cos(2\theta)\,d\theta$$
$$+ f_{xyy}(c,d)\int_0^{2\pi} \frac{(r\cos\theta)(r\sin\theta)^2}{2}\cos(2\theta)\,d\theta$$
$$+ f_{yyy}(c,d)\int_0^{2\pi} \frac{(r\sin\theta)^3}{6}\cos(2\theta)\,d\theta.$$

Hence it suffices to prove that $R/r^2 \to 0$ as $r \to 0$. Since f is $\mathcal{C}^3$, its third partial derivatives are all bounded on $B_r(a, b)$. Therefore,

$$|R| \le \frac{M|r|^3}{6}\left|\int_0^{2\pi} (\cos\theta + \sin\theta)^3 \cos(2\theta)\,d\theta\right| \le \frac{M|r|^3}{6} 16\pi,$$

i.e., $|R|/r^2 \to 0$ as $r \to 0$.

11.5.11. a) Let $\epsilon > 0$. Given $(x_0, t_0) \in \partial H$, choose $\delta > 0$ such that $u(x, y) \ge -\epsilon$ for $(x, y) \in B_\delta(x_0, t_0)$. Since ∂H is compact, it can be covered by finitely many such balls, say $B_1, \dots, B_N$, where $u(x, t) \ge -\epsilon$ on $U := \cup_{j=1}^{N} B_j$. Since the complement $K = H \setminus U$ is a finite intersection of closed sets disjoint from the boundary of H, K is a compact subset of H^0 and $u(x, t) \ge -\epsilon$ for $(x, t) \in H \setminus K$.

b) Suppose $u(x_1, t_1) = -\ell < 0$ for some $(x_1, t_1) \in H^0$. Let $r > 0$ be so small that $rt_1 < \ell/2$ and set $w(x, t) = u(x, t) + r(t - t_1)$. Apply part a) to $\epsilon := \ell/2 - rt_1$ to choose a compact set $K \subset H^0$ such that $u(x, t) \ge -\epsilon$ on $H \setminus K$. Then $w(x, t) \ge -\epsilon - rt_1 = -\ell/2$ for every $(x, t) \in H \setminus K$, i.e., is greater than the value of w at (x_1, t_1). Thus the minimum of w on H must be less than or equal to $-\ell$ and must occur on the compact set K.

c) Suppose u is not nonnegative on H. Since it is nonnegative on ∂H, there is a point $(x_1, t_1) \in H^0$ such that $u(x_1, t_1) < 0$. Hence by part b), there is a point $(x_2, t_2) \in K$ where the absolute minimum of w occurs.

If u satisfies the heat equation, then $w_{xx} - w_t = -r < 0$ on V. We shall obtain a contradiction by showing that $w_{xx}(x_2, t_2) - w_t(x_2, t_2) \ge 0$. First observe by a one- dimensional result that $w_t(x_2, t_2) = 0$. Hence by Taylor's Formula,

$$w(x_2 + h, t_2) = w(x_2, t_2) + w_{xx}(x_2, t_2)\frac{h^2}{2} + (w_{xx}(c, t_2) - w_{xx}(x_2, t_2))\frac{h^2}{2}$$

for some c between x_2 and $x_2 + h$. Since $w_{xx}(c, t_2) - w_{xx}(x_2, t_2) \to 0$ as $h \to 0$, and $w(x_2, t_2) \le w(x_2 + h, t_2)$ (this point is a local minimum), it follows that $w_{xx}(x_2, t_2) \ge 0$. Therefore, $w_{xx}(x_2, t_2) - w_t(x_2, t_2) = w_{xx}(x_2, t_2) \ge 0$, a contradiction.

11.5.12. a) Suppose E is convex but not connected. Then there is a pair of open sets U, V which separates E. Let $\mathbf{x} \in E \cap U$ and $\mathbf{y} \in E \cap V$. Let $t_0 = \sup\{t \in (0,1) : (1-t)\mathbf{x} + t\mathbf{y} \in U\}$ and set $\mathbf{x}_0 = (1 - t_0)\mathbf{x} + t_0\mathbf{y}$. Since E is convex, $\mathbf{x}_0 \in L(\mathbf{x}; \mathbf{y}) \subseteq E$. Thus either $\mathbf{x}_0 \in U$ or $\mathbf{x}_0 \in V$. If $\mathbf{x}_0 \in U$ then $B_\epsilon(\mathbf{x}_0) \subseteq U$ for some $\epsilon > 0$, which contradicts the choice of t_0. Similarly, $\mathbf{x}_0 \in V$ also leads to a contradiction. Therefore, E is connected.

b) The converse is false. Indeed, the set $E := \overline{B_1(0,0)} \cup \overline{B_1(1,0)}$ is connected but not convex. Indeed, $L((0,1);(1,1))$ intersects E at only two points.

c) Suppose f is convex. Let $E := \{(x, y) : y \ge f(x)\}$ and suppose $(x_1, y_1), (x_2, y_2) \in E$. Let (x, y) be a point on the line segment between (x_1, y_1) and (x_2, y_2), and (x, y^*) be a point on the chord from $(x_1, f(x_1))$ to $(x_2, f(x_2))$. Since f is convex, $f(x) \le y^*$. Since $y_1 \ge f(x_1)$ and $y_2 \ge f(x_2)$, we also have $y \ge y^*$. Thus $f(x) \le y^* \le y$, i.e., $(x, y) \in E$.

Conversely, if E is convex and $(x_1, y_1), (x_2, y_2) \in E$, then $L((x_1, f(x_1)); (x_2, f(x_2))) \subseteq E$. In particular, the chord from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ lies on or above the graph of $y = f(x)$, i.e., f is convex.

11.6 The Inverse Function Theorem.

11.6.1. a) Since

$$Df(u, v) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix},$$

we have

$$D^{-1}f(a, b) = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5/17 & 1/17 \\ -2/17 & 3/17 \end{bmatrix}.$$

b) Since $f(u, v) = (0, 1)$ implies $u = (2k+1)\pi/2$ or $u = 2k\pi$, $k \in \mathbf{Z}$ and

$$Df(u, v) = \begin{bmatrix} 1 & 1 \\ \cos u & -\sin v \end{bmatrix},$$

we have

$$D^{-1}f(0, 1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},$$

or

$$D^{-1}f(0, 1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

c) Since $f(u, v) = (2, 5)$ implies $u = \pm 2$, $v = \pm 1$ or $u = \pm 1$, $v = \pm 2$ and

$$Df(u, v) = \begin{bmatrix} v & u \\ 2u & 2v \end{bmatrix},$$

we have

$$D^{-1}f(2, 5) = \begin{bmatrix} \pm 1 & \pm 2 \\ \pm 4 & \pm 2 \end{bmatrix}^{-1} = \begin{bmatrix} \mp 1/3 & \pm 1/3 \\ \pm 2/3 & \mp 1/6 \end{bmatrix},$$

or

$$D^{-1}f(2, 5) = \begin{bmatrix} \pm 2 & \pm 1 \\ \pm 2 & \pm 4 \end{bmatrix}^{-1} = \begin{bmatrix} \pm 2/3 & \mp 1/6 \\ \mp 1/3 & \pm 1/3 \end{bmatrix}.$$

d) Since $f(0,1)=(-1,0)$ and

$$Df(u,v)=\begin{bmatrix} 3u^2 & -2v \\ \cos u & -1/v \end{bmatrix},$$

we have $\Delta_f(0,1)=2\neq 0$, and it follows from the Inverse Function Theorem that

$$D(f^{-1})(-1,0)=(Df(0,1))^{-1}=\begin{bmatrix} -1/2 & 1 \\ -1/2 & 0 \end{bmatrix}.$$

11.6.2. a) Set $F(x,y,z)=xyz+\sin(x+y+z)$. Since $F(0,0,0)=0$ and

$$\frac{\partial F}{\partial z}=xy+\cos(x+y+z)$$

equals $1\neq 0$ at $(0,0,0)$, the expression has a differentiable solution near $(0,0,0)$ by the Implicit Function Theorem.

b) Set $F(x,y,z)=x^2+y^2+z^2+\sqrt{\sin(x^2+y^2)+3z+4}-2$. Since $F(0,0,0)=0$ and

$$\frac{\partial F}{\partial z}=2z+\frac{3}{2\sqrt{\sin(x^2+y^2)+3z+4}}$$

equals $3/4\neq 0$ at $(0,0,0)$, the expression has a differentiable solution near $(0,0,0)$ by the Implicit Function Theorem.

c) Set $F(x,y,z)=xyz(2\cos y-\cos z)+z\cos x-x\cos y$. Since $F(0,0,0)=0$ and

$$\frac{\partial F}{\partial z}=xy(2\cos y-\cos z)+xyz\sin z+\cos x$$

equals $1\neq 0$ at $(0,0,0)$, the expression has a differentiable solution near $(0,0,0)$ by the Implicit Function Theorem.

d) Set $F(x,y,z)=x+y+z+g(x,y)$. Since $F(0,0,0)=0$ and

$$\frac{\partial F}{\partial z}(0,0,0)=1+g_z(0,0,0)>1$$

is nonzero, the expression has a differentiable solution near $(0,0,0)$ by the Implicit Function Theorem.

11.6.3. Let $F(x,y,u,v,w)=(u^5+xv^2-y+w, v^5+yu^2-x+w, w^4+y^5-x^4-1)$ and observe that $F(1,1,1,1,-1)=(0,0,0)$. We want to solve for u,v,w, so we must compute

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}=\det\begin{pmatrix} 5u^4 & 2xv & 1 \\ 2uy & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{pmatrix}=4w^3(25u^4v^4-4uvxy).$$

Since this determinant is nonzero at $(1,1,1,-1)$, we can apply the Implicit Function Theorem to verify such functions u,v,w exist.

11.6.4. Let $F(x,y,u,v)=(xu^2+yv^2+xy-9, xv^2+yu^2-xy-7)$ and observe that

$$\frac{\partial(F_1,F_2)}{\partial(u,v)}=\det\begin{pmatrix} 2ux & 2vy \\ 2uy & 2vx \end{pmatrix}=4uvx^2-4uvy^2=4uv(x^2-y^2).$$

Thus by the Implicit Function Theorem, if $F(x_0,y_0,u_0,v_0)=(0,0)$, $x_0^2\neq y_0^2$, and $u_0\neq 0\neq v_0$, then such solutions u,v exist. Moreover, adding the two given identities, we have $x(u^2+v^2)+y(u^2+v^2)=9+7$, i.e., $(x+y)(u^2+v^2)=16$.

11.6.5. Let $F(x,y,u,v,s,t)=(u^2+sx+ty, v^2+tx+sy, 2s^2x+2t^2y-1, s^2x-t^2y)$ and observe that

$$\frac{\partial(F_1,F_2,F_3,F_4)}{\partial(u,v,s,t)}=\det\begin{pmatrix} 2u & 0 & x & y \\ 0 & 2v & y & x \\ 0 & 0 & 4sx & 4ty \\ 0 & 0 & 2sx & -2ty \end{pmatrix}=-64uvsxty.$$

Since each of the numbers $x_0, y_0, u_0, v_0, s_0, t_0$ is nonzero, this determinant is nonzero. Hence by the Implicit Function Theorem such functions u, v, s, t exist.

11.6.6. a) Notice that $s = x + y$, $t = xy$, and $(x, y) \in E$ imply $s > 0$, $t > 0$, $x = s - y$, and $t = sy - y^2$. In particular, $y = (s \pm \sqrt{s^2 - 4t})/2$ and $x = (s \mp \sqrt{s^2 - 4t})/2$. The condition $0 < y < x$ cannot be satisfied by the pair which eventuates when x takes the minus sign and y the plus sign. However, $0 < y < x$ is satisfied by the other pair because $s > 2\sqrt{t} > 0$. Thus

$$f^{-1}(s,t) = \left(\frac{s + \sqrt{s^2 - 4t}}{2}, \frac{s - \sqrt{s^2 - 4t}}{2} \right).$$

b) Since

$$Df(x,y) = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

is invertible when $0 \neq \Delta_f(x, y) = x - y$, the inverse exists for $0 < y < x$ and by the Inverse Function Theorem,

$$D(f^{-1})(f(x,y)) = (Df(x,y))^{-1} = \frac{1}{x-y} \begin{pmatrix} x & -1 \\ -y & 1 \end{pmatrix} = \begin{pmatrix} x/(x-y) & 1/(y-x) \\ y/(y-x) & 1/(x-y) \end{pmatrix}.$$

c) By part a), $x - y = \sqrt{s^2 - 4t}$ so $x/(x - y) = s/\sqrt{s^2 - 4t} + 1/2$ which is the partial of the first component of f^{-1} with respect to s, i.e., the first entries in the matrices $D(f^{-1})(s, t)$ and $D(f^{-1})(f(x, y))$ coincide. Similarly, the other three entries also coincide.

11.6.7. By the Implicit Function Theorem, solutions $g_j(\mathbf{u}^{(j)})$ exist for each j. Moreover, by the Chain Rule (see Exercise 11.4.9), $\partial g_j/\partial x_k = -F_{x_k}/F_{x_j}$. Therefore,

$$\frac{\partial g_1}{\partial x_n}\frac{\partial g_2}{\partial x_1} \cdots \frac{\partial g_n}{\partial x_{n-1}} = \frac{-F_{x_n}}{F_{x_1}}\frac{-F_{x_1}}{F_{x_2}} \cdots \frac{-F_{x_{n-1}}}{F_{x_n}} = (-1)^n.$$

11.6.8. By Theorem C.5,

$$Df = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} \quad \text{implies} \quad (Df)^{-1} = \frac{1}{\Delta_f} \begin{bmatrix} \partial f_2/\partial y & -\partial f_1/\partial y \\ -\partial f_2/\partial x & \partial f_1/\partial x \end{bmatrix}.$$

Hence by the Inverse Function Theorem, $\partial f_1^{-1}/\partial x = \partial f_2/\partial y/\Delta_f$, etc.

11.6.9. Suppose $F_z(a, b, c) \neq 0$. Then by the Implicit Function Theorem there is an open set $V \subset \mathbf{R}^2$ containing (a, b) and a continuously differentiable $f : V \to \mathbf{R}$ such that $z = f(x, y)$ satisfies $F(x, y, z) = 0$ for $(x, y) \in V$. Thus $\mathcal{G}$ has a tangent plane at (a, b, c) by Theorem 11.22.

Let $w = F(x, y, f(x, y))$. By the Chain Rule, $0 = w_x = F_x + F_z z_x$ and $0 = w_y = F_y + F_z z_y$ on $V \times f(V)$, hence $z_x = -F_x/F_z$ and $z_y = -F_y/F_z$. Hence by Theorem 11.22, a normal to the tangent plane at (a, b, c) is given by $\widetilde{\mathbf{n}} = (F_x(a, b, c)/F_z(a, b, c), F_y(a, b, c)/F_z(a, b, c), 1)$. In particular, $\mathbf{n} = (F_x(a, b, c), F_y(a, b, c), F_z(a, b, c))$ is a normal to the tangent plane at (a, b, c).

If $F_z(a, b, c) = 0$ then one of the other partials of F is nonzero, say $F_x(a, b, c) = 0$. Repeating the argument above, we find a normal of the form $(1, F_y/F_x, F_z/F_x)$ which again is parallel to $\mathbf{n} = (F_x(a, b, c), F_y(a, b, c), F_z(a, b, c))$.

11.6.10. If $\nabla f(t_0) \neq 0$, then either $u'(t_0) \neq 0$ or $v'(t_0) \neq 0$. Without loss of generality, we suppose the former. Consider $F(x, t) := u(t) - x$. Since F is $\mathcal{C}^1$ and $F_t(x_0, t_0) = u'(t_0) \neq 0$, it follows from the Implicit Function Theorem that there is an open interval I_0 containing x_0 and a $\mathcal{C}^1$ function $g : I_0 \to \mathbf{R}$ such that $g(x_0) = t_0$ and $0 = F(x, g(x)) = u(g(x)) - x$ for all $x \in I_0$.

11.6.11. a) By Exercise 11.6.9, the normal of $\mathcal{H}$ at (a, b, c) is parallel to $\nabla F = (2x, 2y, -2z)$. Hence we can use $(-a, -b, c)$ for a normal at the point (a, b, c).

b) If $(a, b, c) \in \mathcal{H}$ and $(0, 0, 1) \cdot (-a, -b, c) = 0$ then $c = 0$ and $a^2 + b^2 = 1$. Thus $\mathbf{n} = (a, b, 0)$ and an equation of the tangent plane is $ax + by = a^2 + b^2 = 1$.

c) If $(a, b, c) \in \mathcal{H}$ and $t(1, 1, -1) = (-a, -b, c)$ then $a = b = c = t$ hence $a^2 = a^2 + b^2 - c^2 = 1$, i.e., $a = \pm 1$. Hence $(1, 1, 1)$ and $(-1, -1, -1)$ are the only points where the tangent plane of $\mathcal{H}$ is parallel to $x + y - z = 1$. Corresponding equations these tangent planes are $x + y - z = 1$ and $x + y - z = -1$. A portion of the plane $x + y - z = 1$ lies above the first quadrant of the xy plane and slants away from the z axis, so there are two points where the tangent plane to $\mathcal{H}$ is parallel to $x + y - z = 1$, one on the "front" side of $\mathcal{H}$ lying above the xy plane, and one on the "back" side of $\mathcal{H}$ lying below the xy plane.

11.7 Optimization.

11.7.1. a) $0 = f_x = 2x - y$ and $0 = f_y = -x + 3y^2 - 1$ imply $y = 2x$ and $12x^2 - x - 1 = 0$, i.e., $x = 1/3, -1/4$. Since $D = 12y - 1$, we see that $f(1/3, 2/3) = -13/27$ is a local minimum and $(-1/4, -1/2)$ is a saddle point.

b) $0 = f_x = \cos x$ and $0 = f_y = -\sin y$ imply $x = (2k+1)\pi/2$ and $y = j\pi$, $k, j \in \mathbf{Z}$. Since $D = \sin x \cos y$, $f((2k+1)\pi/2, j\pi) = 2$ is a local maximum if k and j are even, $f((2k+1)\pi/2, j\pi) = -2$ is a local minimum if k and j are odd, and $((2k+1)\pi/2, j\pi)$ is a saddle point if $k + j$ is odd.

c) $0 = f_x = e^{x+y}\cos z$, $0 = f_y = e^{x+y}\cos z$, and $0 = -e^{x+y}\sin z$ imply $\cos z = \sin z = 0$. Since these functions have no common zero, this function has no local extrema.

d) $0 = f_x = 2ax + by$ and $0 = f_y = bx + 2cy$ imply $(b^2 - 4ac)y = 0$, i.e., $x = y = 0$. Since $D = 4ac - b^2$, $f(0,0) = 0$ is a local minimum if $a > 0$ and $b^2 - 4ac < 0$, a local maximum if $a < 0$ and $b^2 - 4ac < 0$, and $(0,0)$ is a saddle point if $b^2 - 4ac > 0$.

11.7.2. a) $0 = f_x = 2x + 2$ and $0 = f_y = -2y$ implies $x = -1$ and $y = 0$. Note $f(-1, 0) = -1$. For the boundary, let $x = 2\cos\theta$ and $y = \sin\theta$. Then $f(x,y) = x^2 + 2x - y^2 = 5\cos^2\theta + 4\cos\theta - 1 =: h(\theta)$. Since $h'(\theta) = 0$ implies $\theta = 0$ or π, or $\cos\theta = -2/5$. Since $\sin^2\theta + \cos^2\theta = 1$, it follows that the critical points are $(2,0)$, $(-2,0)$, and $(-4/5, \pm\sqrt{21}/5$. Thus the absolute maximum of f on H is $f(2,0) = 8$ and the absolute minimum of f on H is $f(-4/5, \pm\sqrt{21}/5) = -9/5$.

b) $0 = f_x = 2x + 2y$ and $0 = f_y = 2x + 6y$ imply $x = y = 0$. This point is outside H so can be disregarded. We check the boundary in three pieces. If $x = 1$, $0 \le y \le 2$, then $f(x,y) = 1 + 2y + 3y^2$ which takes its minimum at $y = -1/3$ which is out of range. If $y = 0$, $1 \le x \le 3$, then $f(x,y) = x^2$ takes its minimum at $x = 0$. Since $(0,0)$ lies outside of H, we can disregard it. Finally, if $y = 3 - x$, $1 \le x \le 3$, then $f(x,y) = 2x^2 - 12x + 27$ takes its minimum at $x = 3$, an extreme point of H. Checking the extreme points of H, $f(1,0) = 1$, $f(3,0) = 9$, and $f(1,2) = 17$. Thus the absolute minimum of f on H is $f(1,0) = 1$ and the absolute maximum of f on H is $f(1,2) = 17$.

c) $0 = f_x = 3x^2 + 3y$ and $0 = f_y = 3x - 3y^2$ imply $y = 0$ or $y = -1$, which correspond to the points $(0,0)$ and $(1,-1)$. We check the boundary in four pieces. If $x = 1$ then $f(y) = 1 + 3y - y^3$ has critical points $y = \pm 1$, which correspond to extreme points of H. If $y = 1$ then $f(x,y) = x^3 + 3x - 1$ which has no critical points. If $x = -1$ then $f(x,y) = -1 - 3y - y^3$ which has no critical points. And, if $y = -1$ then $f(x,y) = x^3 - 3x + 1$ has critical points $x = \pm 1$ which correspond to extreme points of H. Checking the critical point $f(0,0) = 0$, and extreme points of H, $f(1,1) = 3$, $f(1,-1) = -1$, $f(-1,1) = -5$, and $f(-1,-1) = 3$, we conclude that the absolute maximum of f on H of f on H is $f(1,1) = f(-1,-1) = 3$, and the absolute minimum of f on H of f on H is $f(-1,1) = -5$.

11.7.3. a) The Lagrange equations are $1 = 2x\lambda$ and $2y = 2y\lambda$. If $y = 0$ then the constraint implies $x = \pm 2$. If $y \ne 0$ then $\lambda = 1$ so $x = 1/2$. The constraint implies $y^2 = 15/4$, i.e., $y = \pm\sqrt{15}/2$. We conclude that $f(-2,0) = -2$ is the minimum, $f(1/2, \pm\sqrt{15}/2) = 17/4$ is the maximum (and $f(2,0) = 2$ is a saddle point).

b) The Lagrange equations are $2x - 4y = 2x\lambda$ and $-4x + 8y = 2y\lambda$, i.e., $(2x + y)\lambda = 0$. If $\lambda = 0$ then $x = 2y$ and the constraint implies $y = \pm 1/\sqrt{5}$. If $\lambda \ne 0$, then $y = -2x$ and constraint implies $x = \pm 1/\sqrt{5}$. We conclude that $f(\pm 2/\sqrt{5}, \pm 1/\sqrt{5}) = 0$ is the minimum and $f(\pm 1/\sqrt{5}, \mp 2/\sqrt{5}) = 5$ is the maximum.

c) The Lagrange equations are $y = 2x\lambda + \mu$, $x = 2y\lambda + \mu$, and $0 = 2z\lambda + \mu$. Multiplying the first by x, the second by y, the third by z, adding and using both constraints, we see that $xy = \lambda$. By adding the Lagrange equations and using the second constraint, we see that $\mu = (x+y)/3$. Substituting these values for λ and μ into the first two Lagrange equations, we obtain $2y = 6x^2y + x$ and $2x = 6y^2x + y$, i.e., $y = \pm x$.

If $y = x$ then $2x = 6x^3 + x$, i.e., $x = 0$ or $x = \pm 1/\sqrt{6}$. If $y = -x$ then $-2x = -6x^3 + x$, i.e., $x = 0$ or $x = \pm 1/\sqrt{2}$. We conclude that $f(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}, 0) = -1/2$ is the minimum and $f(\pm 1/\sqrt{6}, \pm 1/\sqrt{6}, \mp 2/\sqrt{6}) = 1/6$ is the maximum.

d) The Lagrange equations are $3 = 6\lambda x - 3\mu x^2$, $1 = \lambda$, $0 = 12z^2\lambda + 12z^3\mu$, $1 = \mu$. Plugging $\lambda = \mu = 1$ into the first and third of these equations, we have $3x^2 - 6x + 3 = 0$ and $12z^2 + 12z^3 = 0$, i.e., $x = 1$ and $z = 0, -1$. If $x = 1$ and $z = 0$, then $3 + y = 1$ and $-1 + w = 0$, i.e., $y = -2$, $w = 1$. If $x = 1$ and $z = -1$, then $3 + y - 4 = 1$ and $-1 + 3 + w = 0$, i.e., $y = 2$, $w = -2$. We conclude that $f(1, -2, 0, 1) = 2$ is the minimum and $f(1, 2, -1, -2) = 3$ is the maximum.

11.7.4. By Remark 11.51, $\nabla g(\mathbf{b}) = 0$, hence by the Chain Rule,

$$\nabla(g \circ f)(\mathbf{a}) = \nabla g(\mathbf{b}) Df(\mathbf{a}) = 0.$$

11.7.5. If $f_{xy}(a,b) \ne 0$ then $f_{xx}(a,b) = f_{yy}(a,b) = 0$ and it follows that $D^{(2)}f(a,b) = f_{xy}(a,b)hk$ takes both positive and negative values as h, k range over $\mathbf{R}$. Thus by Theorem 11.58, (a,b) is a saddle point. On the other hand, if $f_{xy}(a,b) = 0$ then either $f_{xx}(a,b) \neq 0$ or $f_{yy}(a,b) \neq 0$. We may suppose $f_{xx}(a,b) \neq 0$. Then $D^{(2)}f(a,b) = f_{xx}(a,b)h^2$ always has the same sign, and it follows from Theorem 11.58 that $f(a,b)$ is either a local maximum or a local minimum, depending on the sign of $f_{xx}(a,b)$. In particular, (a,b) is not a saddle point.

11.7.6. Suppose $D^{(2)}f(\mathbf{a})(\mathbf{h}_0) < 0$ for some $\mathbf{h}_0 \in \mathbf{R}^2$. Since f is $\mathcal{C}^2$ on V and

$$D^{(2)}f(\mathbf{c})(\mathbf{h}) = D^{(2)}f(\mathbf{a})(\mathbf{h}) + \sum_{j,k=1}^{2} \left(\frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{c}) - \frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{a}) \right) h_j h_k,$$

choose $\delta > 0$ such that

$$(*) \qquad D^{(2)}f(\mathbf{c})(\mathbf{h}_0) < D^{(2)}f(\mathbf{a})(\mathbf{h}_0) - \frac{1}{2}D^{(2)}f(\mathbf{a})(\mathbf{h}_0) = \frac{1}{2}D^{(2)}f(\mathbf{a})(\mathbf{h}_0) < 0$$

for $\mathbf{c} \in B_\delta(\mathbf{a})$. On the other hand, since $f(\mathbf{a})$ is a local minimum we see by Taylor's Formula that

$$() \qquad 0 \leq f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = D^{(2)}f(\mathbf{c})(\mathbf{h})$$

for $\mathbf{c} \in L(\mathbf{a}; \mathbf{a}+\mathbf{h})$ and $\|\mathbf{h}\|$ sufficiently small. Let $\mathbf{h} = \alpha \mathbf{h}_0$, $\alpha \neq 0$, where $\|\mathbf{h}\| < \delta$ and let $\mathbf{c} \in L(\mathbf{a}; \mathbf{a}+\mathbf{h})$. Then by (*) and (),

$$0 \leq D^{(2)}f(\mathbf{c})(\mathbf{h}) = \alpha^2 D^{(2)}f(\mathbf{c})(\mathbf{h}_0) < \frac{\alpha^2}{2}D^{(2)}f(\mathbf{a})(\mathbf{h}_0) < 0,$$

a contradiction.

11.7.7. a) The Lagrange equations are $a = -2Dx\lambda$, $b = -2Ey\lambda$, and $c = \lambda$. Thus

$$x = -\frac{a}{2cD}, \quad y = -\frac{b}{2cE}, \quad \text{and} \quad z = \frac{1}{4c^2}\left(\frac{a^2}{D} + \frac{b^2}{E}\right).$$

To determine whether this is a maximum or a minimum, notice that the discriminant of $F(x,y) := ax + by + cDx^2 + cEy^2$ is $4c^2DE$. Since $DE > 0$ and $c \neq 0$, it follows from Theorem 11.61 that the point (x,y,z) identified above is a maximum when $F_{xx}/2 = cD < 0$ and a minimum when $cD > 0$.

b) If $DE < 0$ then by part a), the discriminant is negative. Thus the point (x,y,z) is a saddle point and $ax + by + cz$ has no extrema subject to the constraint $z = Dx^2 + Ey^2$.

11.7.8. a) If $g_x(a,b,c) = g_y(a,b,c) = g_z(a,b,c) = 0$, then the equations obviously hold. If one of these partial derivatives is nonzero, then by Lagrange's Theorem there is a scalar λ such that $\nabla f(a,b,c) = \lambda \nabla g(a,b,c)$. Thus $\nabla f(a,b,c)$ and $\nabla g(a,b,c)$ are parallel, i.e.,

$$(0,0,0) = \nabla f(a,b,c) \times \nabla g(a,b,c) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x(a,b,c) & f_y(a,b,c) & f_z(a,b,c) \\ g_x(a,b,c) & g_y(a,b,c) & g_z(a,b,c) \end{pmatrix}.$$

In particular,

$$f_x(a,b,c)g_z(a,b,c) - f_z(a,b,c)g_x(a,b,c) = 0 = f_y(a,b,c)g_z(a,b,c) - f_z(a,b,c)g_y(a,b,c).$$

b) By part a), if $f(x,y,z)$ is an extremum then $4y^2x - 2y^2z = 0$ and $4x^2y - 2x^2z = 0$, i.e., $y^2(4x - 2z) = 0 = x^2(4y - 2z)$. Since $xyz = 16$, neither x nor y is zero. Hence $x = z/2$ and $y = z/2$. Plugging this into the constraint, we obtain $z^3/4 = 16$, i.e., $z = 4$, $x = y = 2$. Thus $f(2,2,4) = 48$ is the minimum.

11.7.9. a) By symmetry, we may suppose that each $x_j \geq 0$. Since $(|t|^p)' = p|t|^{p-1}$ exists for all $t \in \mathbf{R}$ and $p > 1$, it follows from Lagrange's Theorem that if $f(\mathbf{x})$ is an extremum subject to the constraint $\sum_{k=1}^n |x_k|^p = 1$ then $2x_j = p|x_j|^{p-1}\lambda$. Since we assumed $x_j \geq 0$, this equation can be rewritten as $2x_j^2 = p|x_j|^p\lambda$. Summing over all j, we obtain

$$2\sum_{j=1}^n x_j^2 = p\lambda \sum_{j=1}^n |x_j|^p = p\lambda,$$

i.e., $\lambda = (2/p)\sum_{j=1}^n x_j^2$. If $x_j \neq 0$, notice that $|x_j|^{p-2} = 2/(p\lambda)$ so $x_j^2 = (2/(p\lambda))^{2/(p-2)}$. If m is the number of nonzero components of the vector $\mathbf{x}$, then

$$\sum_{j=1}^n x_j^2 = m\left(\frac{2}{p\lambda}\right)^{2/(p-2)} = m\left(\frac{1}{\sum_{j=1}^n x_j^2}\right)^{2/(p-2)}.$$

Hence $\sum_{j=1}^n x_j^2 = m^{(p-2)/p}$. In particular, if $p \geq 2$ then the maximum of f is $n^{(p-2)/p}$ and the minimum is 1. And, if $1 < p \leq 2$, then the maximum is 1 and the minimum is $n^{(p-2)/p}$.

b) Let $a = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$. If $a = 0$ then the inequalities are trivial. If $a \neq 0$, then $\sum_{j=1}^n |x_j/a|^p = \sum_{j=1}^n |x_j|^p/a^p = 1$. Hence by part a),

$$\frac{1}{n^{(2-p)/p}} \leq \sum_{j=1}^n \left|\frac{x_j}{a}\right|^2 \leq 1$$

for $1 < p \leq 2$. Taking the limit of these inequalities as $p \to 1+$ we see that this inequality holds for $p = 1$ too. Thus

$$\frac{a^2}{n^{(2-p)/p}} \leq \sum_{j=1}^n |x_j|^2 \leq a^2 \quad \text{hence} \quad \frac{a}{n^{(2-p)/(2p)}} \leq \left(\sum_{j=1}^n |x_j|^2\right)^{1/2} \leq a$$

for all $1 \leq p \leq 2$.

c) Suppose $\sum_{k=1}^\infty |x_j| < \infty$. Then by part b) (with $p = 1$) and the Comparison Theorem,

$$\left(\sum_{k=1}^\infty x_j^2\right)^{1/2} \leq \sum_{k=1}^\infty |x_j| < \infty.$$

11.7.10. a) Since

$$F(a,b) = \sum_{k=1}^n y_k^2 - 2a\sum_{k=1}^n x_k y_k - 2b\sum_{k=1}^n y_k + \sum_{k=1}^n (ax_k + b)^2,$$

it is clear that

$$F_a = -2\sum_{k=1}^n x_k y_k + 2a\sum_{k=1}^n x_k^2 + 2b\sum_{k=1}^n x_k,$$

and

$$F_b = -2\sum_{k=1}^n y_k + 2a\sum_{k=1}^n x_k + 2nb.$$

This yields two equations in the two unknowns a, b:

$$(\sum_{k=1}^n x_k^2)a + (\sum_{k=1}^n x_k)b = \sum_{k=1}^n x_k y_k$$
$$(\sum_{k=1}^n x_k)a + nb = \sum_{k=1}^n y_k$$

so the matrix of coefficients has determinant d_0. Thus this system can be solved by Cramer's Rule as indicated.

b) To minimize the function $F(a,b)$, as a and b vary, we first find the critical points by setting $\nabla F = 0$. From part a), we see that this function has only one critical point: (a_0, b_0).

Since $F_{aa} = 2\sum_{k=1}^n x_k^2 > 0$, this critical point is either a minimum or a saddle point. To decide which, look at the discriminant. By algebra,

$$F_{aa}F_{bb} - F_{ab}^2 = 4n\sum_{k=1}^n x_k^2 - 4\left(\sum_{k=1}^n x_k\right)^2 = 4\sum_{j<k}(x_j - x_k)^2 > 0.$$

Thus by Theorem 11.59, (a_0, b_0) is a local minimum. Since there are no other critical points, it is an absolute minimum.

12.1 Jordan regions.

12.1.1. a) $m = 1$: Each rectangle has area $1/4$ and there are 3 rectangles which intersect E. Hence $V(E;\mathcal{G}_1) = 3/4$ and $v(E;\mathcal{G}_1) = 0$. $m = 2$: Each rectangle has area $1/16$ and there are 7 rectangles which intersect E. Hence $V(E;\mathcal{G}_2) = 7/16$ and $v(E;\mathcal{G}_2) = 0$. $m = 3$: Each rectangle has area $1/64$ and there are 15 rectangles which intersect E. Hence $V(E;\mathcal{G}_2) = 15/64$ and $v(E;\mathcal{G}_2) = 0$.

b) $m = 1$: Each rectangle has area $1/4$ and there are 4 rectangles which intersect E but none in the interior of E. Hence $V(E;\mathcal{G}_1) = 1$ and $v(E;\mathcal{G}_1) = 0$. $m = 2$: Each rectangle has area $1/16$ and there are 13 rectangles which intersect E but none lie in the interior of E. Hence $V(E;\mathcal{G}_2) = 13/16$ and $v(E;\mathcal{G}_2) = 0$. $m = 3$: Each rectangle has area $1/64$ and there are 43 rectangles which intersect E but only 10 in the interior of E. Hence $V(E;\mathcal{G}_2) = 43/64$ and $v(E;\mathcal{G}_2) = 5/32$.

c) $m = 1$: Each rectangle has area $1/4$ and there are 4 rectangles which intersect E but none in the interior of E. Hence $V(E;\mathcal{G}_1) = 1$ and $v(E;\mathcal{G}_1) = 0$. $m = 2$: Each rectangle has area $1/16$ and there are 16 rectangles which intersect E but only 4 lie in the interior of E. Hence $V(E;\mathcal{G}_2) = 1$ and $v(E;\mathcal{G}_2) = 1/4$. $m = 3$: Each rectangle has area $1/64$ and there are 60 rectangles which intersect E but only 32 in the interior of E. Hence $V(E;\mathcal{G}_2) = 15/16$ and $v(E;\mathcal{G}_2) = 1/2$.

12.1.2. a) Let $E = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ and $\epsilon > 0$. Choose s so small that $Ns^n < \epsilon$. If Q_k is a cube of side s which contains $\mathbf{x}_k$ then

$$\sum_{k=1}^{N} |Q_k| = Ns^n < \epsilon.$$

Hence by Theorem 12.4, E is a Jordan region of volume zero.

b) The set A in Example 12.2 is countable but not a Jordan region.

c) We may suppose that $E = \{(x, c) : a \le x \le b\}$. Let $\varepsilon > 0$ and let $\mathcal{G}_m$ be the dyadic grid of Exercise 12.1.1 with m so large that $2^{-m} < \varepsilon$. Since the only rectangles which intersect E lie on the x-axis, there are only 2^m of these. Thus $V(E;\mathcal{G}_m) = 2^m \cdot 2^{2m} = 2^{-m} < \varepsilon$. It follows from Theorem 12.4 that E is a Jordan region and it has area zero.

12.1.3. By Remark 12.6, every rectangle is a Jordan region. Thus it suffices to show that if $R = [a, b] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ and $Q = [b, c] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, then $\text{Vol}\,(R \cap Q) = 0$.

Let $H = [b - \epsilon, b + \epsilon] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ and observe that H covers $R \cap Q = \{b\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Since $|H| = 2\epsilon(b_2 - a_2) \dots (b_n - a_n)$ it follows from Theorem 12.4 that $\text{Vol}\,(R \cap Q) = 0$.

12.1.4. a) Since $\partial B_r = \overline{B}_r \setminus B_r^o$, it suffices to show

$$B_r^o(\mathbf{a}) = B_r(\mathbf{a}) \quad \text{and} \quad \overline{B_r(\mathbf{a})} = E := \{\mathbf{x} : \|\mathbf{x} - \mathbf{a}\| \le r\}.$$

Since B_r is open, the first identity is trivial. Since E is closed, it is clear that $\overline{B}_r \subseteq E$. On the other hand, given $\mathbf{x} \in E$, there is a sequence $\mathbf{x}_j \in B_r$ such that $\mathbf{x}_j \to \mathbf{x}$ as $j \to \infty$. Thus $E \subseteq \overline{B}_r$.

b) See the proof of Theorem 12.39.

12.1.5. a) Notice by definition that $\overline{E}^0 = E^0$ and $\overline{E^0} = \overline{E}$. Hence by Theorem 10.39, $\partial E = \overline{E} \setminus E^0 = \partial(E^0) = \partial(\overline{E})$. Therefore, E is a Jordan region if and only if $\overline{E}$ and E^0 are Jordan regions by Definition 12.5.

b) By Theorems 12.7 and 12.4,

$$\text{Vol}\,(\overline{E}) = \text{Vol}\,(E^0 \cup \partial E) \le \text{Vol}\,(E^0) + \text{Vol}\,(\partial E) = \text{Vol}\,(E^0).$$

On the other hand, $\text{Vol}\,(E^0) \le \text{Vol}\,(E) \le \text{Vol}\,(\overline{E})$ by Exercise 12.1.6a. Hence $\text{Vol}\,(E^0) = \text{Vol}\,(E) = \text{Vol}\,(\overline{E})$.

c) If $\text{Vol}\,(E) > 0$ then $\text{Vol}\,(E^0) > 0$ by part b), hence E^0 cannot be empty. Conversely, if $E^0 \ne \emptyset$ then since E^0 is open it must contain a ball, hence a rectangle R. Thus $\text{Vol}\,(E) = \inf_{\mathcal{G}} V(E;\mathcal{G}) \ge |R| > 0$.

d) Let $\mathcal{G}(f)$ represent the graph of $y = f(x)$ as x varies over $[a, b]$. Given $\epsilon > 0$ choose $\delta > 0$ such that $x, y \in [a, b]$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon/(2(b - a))$. Let $\{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$ whose norm is $< \delta$ and set

$$R_j = [x_{j-1}, x_j] \times \left[f(x_j) - \frac{\epsilon}{2(b - a)}, f(x_j) + \frac{\epsilon}{2(b - a)} \right].$$

If $(x,y) \in \mathcal{G}(f)$ then $x \in [x_{j-1}, x_j]$ for some j and $|x - x_j| < \delta$. Therefore, $|f(x) - f(x_j)| < \epsilon/(2(b-a))$, i.e., $(x,y) \in R_j$. It follows that $\mathcal{G}(f)$ is covered by the R_j's. Since

$$\sum_{j=1}^{N} |R_j| = \frac{\epsilon}{b-a} \sum_{j=1}^{N} |x_j - x_{j-1}| = \epsilon,$$

we conclude by Theorem 12.4 that $\mathrm{Vol}\,(\mathcal{G}(f)) = 0$.

e) Given any partition $\mathcal{P}$ of $[a,b]$, $S(f;\mathcal{P}) - s(f;\mathcal{P})$ is the area of a collection of rectangles which covers $\mathcal{G}(f)$. Hence if f is integrable, we can choose $\mathcal{P}$ so that $V(\mathcal{G}(f),\mathcal{G}) \le S(f;\mathcal{P}) - s(f;\mathcal{P}) < \epsilon$. It follows that $\mathcal{G}(f)$ is of volume zero.

The result does not hold for bounded functions. Let $f(x) = 1$ for all dyadic rationals $x \in (0,1)$, $f(x) = 1/2$ for all triadic rationals $x \in (0,1)$, i.e., rationals of the form $x = p/3^q$, $p, q \in \mathbf{N}$, $f(x) = 3/4$ for all rationals $x \in (0,1)$ of the form $x = p/5^q$, $p, q \in \mathbf{N}$, $f(x) = 1/4$ for all rationals $x \in (0,1)$ of the form $x = p/7^q$, $p, q \in \mathbf{N}$, etc. Let $f(x) = 0$ for all other points $x \in [0,1]$. Then f is bounded, but the graph of $y = f(x)$ intersects any rectangle R in the unit square $[0,1] \times [0,1]$. Thus $V(\mathcal{G}(f),\mathcal{G}) \ge 1 \ne 0$ for all grids $\mathcal{G}$.

12.1.6. a) If $R_j \cap E_1 \ne \emptyset$ then $R_j \cap E_2 \ne \emptyset$. Hence $V(E_1;\mathcal{G}) \le V(E_2,\mathcal{G})$ for every grid $\mathcal{G}$. Taking the infimum of this inequality, we obtain $\mathrm{Vol}\,(E_1) \le \mathrm{Vol}\,(E_2)$.

b) By Theorem 8.37 or 10.40, $\partial(E_1 \cap E_2) \subseteq \partial E_1 \cup \partial E_2$. Thus $E_1 \cap E_2$ is a Jordan region by Theorem 12.4. Since $E_1 \setminus E_2 = E_1 \cap E_2^c$ and $\partial(E_2^c) = \partial E_2$ imply $\partial(E_1 \setminus E_2) = \partial(E_1 \cap E_2^c) \subseteq \partial E_1 \cup \partial E_2$, the set $E_1 \setminus E_2$ is also a Jordan region.

c) By Theorem 12.7 we must show that $\mathrm{Vol}\,(E_1 \cup E_2) \ge \mathrm{Vol}\,(E_1) + \mathrm{Vol}\,(E_2)$. Let $\epsilon > 0$ and choose a grid $\mathcal{G}$ such that $\mathrm{Vol}\,(E_1 \cup E_2) + \epsilon > V(E_1 \cup E_2;\mathcal{G})$ and $V(E_1 \cap E_2;\mathcal{G}) < \epsilon$. Then by Theorem 8.37

$$\begin{aligned}
\mathrm{Vol}\,(E_1 \cup E_2) &> V(E_1 \cup E_2;\mathcal{G}) - \epsilon \\
&\ge \sum_{R_j \cap \overline{E}_1 \ne \emptyset} |R_j| + \sum_{R_j \cap \overline{E}_2 \ne \emptyset} |R_j| - \sum_{R_j \cap \overline{E_1 \cap E_2} \ne \emptyset} |R_j| - \epsilon \\
&\ge \mathrm{Vol}\,(E_1) + \mathrm{Vol}\,(E_2) - V(E_1 \cap E_2;\mathcal{G}) - \epsilon \\
&> \mathrm{Vol}\,(E_1) + \mathrm{Vol}\,(E_2) - 2\epsilon.
\end{aligned}$$

It follows that $\mathrm{Vol}\,(E_1 \cup E_2) \ge \mathrm{Vol}\,(E_1) + \mathrm{Vol}\,(E_2) - 2\epsilon$. Taking the limit of this inequality as $\epsilon \to 0$, we conclude that $\mathrm{Vol}\,(E_1 \cup E_2) \ge \mathrm{Vol}\,(E_1) + \mathrm{Vol}\,(E_2)$.

d) By part c), $\mathrm{Vol}\,(E_1) = \mathrm{Vol}\,((E_1 \setminus E_2) \cup E_2) = \mathrm{Vol}\,(E_1 \setminus E_2) + \mathrm{Vol}\,(E_2)$.

e) By parts c) and d),

$$\begin{aligned}
\mathrm{Vol}\,(E_1 \cup E_2) &= \mathrm{Vol}\,((E_1 \setminus (E_1 \cap E_2)) \cup (E_2 \setminus (E_1 \cap E_2)) \cup (E_1 \cap E_2)) \\
&= \mathrm{Vol}\,(E_1) + \mathrm{Vol}\,(E_2) - 2\mathrm{Vol}\,(E_1 \cap E_2) + \mathrm{Vol}\,(E_1 \cap E_2).
\end{aligned}$$

12.1.7. a) Fix $\mathbf{x} \in \mathbf{R}^n$. Since $(\mathbf{x} + E)^c = \mathbf{x} + E^c$, it is easy to see that $\partial(\mathbf{x} + E) = \mathbf{x} + \partial E$. Since $|R| = |\mathbf{x} + R|$ for any rectangle R, it follows from Theorem 12.4 that A is of volume zero if and only if $\mathbf{x} + A$ is of volume zero. Therefore, $\mathbf{x} + E$ is a Jordan region if and only if E is. It is also easy to check that a rectangle R_j satisfies $R_j \cap \overline{E} \ne \emptyset$ if and only if $\mathbf{x} + R_j \cap \overline{\mathbf{x} + E} \ne \emptyset$). Therefore,

$$\mathrm{Vol}\,(\mathbf{x} + E) := \inf_{\mathcal{G}} \sum_{R_j \cap E \ne \emptyset} |\mathbf{x} + R_j| = \inf_{\mathcal{G}} \sum_{R_j \cap E \ne \emptyset} |R_j| = \mathrm{Vol}\,(E)$$

b) Since $\phi(\mathbf{x}) := \alpha\mathbf{x}$ is $\mathcal{C}^1$ and $\Delta_\phi = \alpha^n \ne 0$, it is clear by Theorem 12.10 that αE is a Jordan region if and only if E is. Since $\alpha > 0$, we also have $R_j \cap (\alpha E) \ne \emptyset$ if and only if $(1/\alpha)R_j \cap E \ne \emptyset$. Since $|(1/\alpha)R_j| = (1/\alpha)^n |R_j|$, it follows that

$$\mathrm{Vol}\,(\alpha E) = \inf_{\mathcal{G}} \sum_{R_j \cap E \ne \emptyset} |\alpha R_j| = \alpha^n \sup_{\mathcal{G}} \sum_{R_j \cap E \ne \emptyset} |R_j| = \alpha^n \mathrm{Vol}\,(E).$$

12.1.8. a) If E is of volume zero, then by definition there is a finite collection of rectangles $\{R_j : j = 1, \dots, N\}$ which covers E such that $\sum_{j=1}^{N} |R_j| < \epsilon$. Hence E is of measure zero.

b) If $E = \{\mathbf{x}_1, \dots\}$ then let R_j be a rectangle which contains $\mathbf{x}_j$ such that $|R_j| < \epsilon/2^j$. Since $\sum_{k=1}^{\infty} |R_k| < \epsilon$, it follows that E is of measure zero.

c) The set A given in Example 12.2 is countable, hence of Lebesgue measure zero, but not a Jordan region.

12.1.9. By Exercise 12.1.5, we may suppose E is closed. Let $\delta > 0$ and let $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ be the cluster points of E. Then

$$A := E \setminus \bigcup_{j=1}^{N} B_\delta(\mathbf{x}_j)$$

is closed and bounded, hence compact. Since any cluster point of A is a cluster point of E, A has no cluster points. Hence by the Bolzano–Weierstrass Theorem, A is finite.

Let $\epsilon > 0$. For each j choose a rectangle R_j such that $\mathbf{x}_j \in R_j^0$ and $|R_j| < \epsilon/(2N)$. Let $\delta > 0$ be so small that $B_\delta(\mathbf{x}_j) \subset R_j^0$ and define A as above. Then A is finite, so we can choose a finite collection of rectangles $\{R_j : j = N+1, \dots, M\}$ which covers A such that $\sum_{j=N+1}^{M} |R_j| < \epsilon/2$. It follows that $\{R_j : j = 1, \dots, M\}$ covers E and $\sum_{j=1}^{M} |R_j| < \epsilon$. In particular, E is a Jordan region of volume zero.

12.2 Riemann Integration on Jordan Regions.

12.2.1. Clearly, $M_{jk} = jk/2^{2m}$ and $m_{jk} = (j-1)(k-1)/2^{2m}$. Hence

$$S(f; \mathcal{G}_m) = \frac{1}{2^{4m}} \sum_{j=1}^{2^m} \sum_{k=1}^{2^m} jk = \frac{2^{2m}(2^m+1)^2}{2^{4m}} = \frac{2^{4m} + 2^{3m+1} + 2^{2m}}{2^{4m}}.$$

Similarly, $s(f; \mathcal{G}_m) = (2^{4m} - 2^{3m+1} + 2^{2m})/2^{4m}$ Consequently, $S(f; \mathcal{G}_m) - s(f; \mathcal{G}_m) = 2^{3m+2}/2^{4m} = 4/2^m \to 0$ as $m \to \infty$.

12.2.2. If $\mathbf{x} \in [0,1] \times \cdots \times [0,1]$, then $0 \le x_j^2 \le 1$. Thus by Theorem 12.26 and the Intermediate Value Theorem, there is an $t_j \in [0,1]$ such that

$$\iint_E x^2 (f(x,y) - g(x,y))\, dA = c_j^2 \iint_E (f(x,y) - g(x,y))\, dA = t_j(1 - (-1)) = 2t_j =: c_j.$$

12.2.3. Let $\epsilon > 0$ and choose $r > 0$ so small that $\mathbf{x} \in B_r(\mathbf{x}_0)$ implies $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$. Then

$$\begin{aligned}\left| \frac{1}{\mathrm{Vol}\,(B_r(\mathbf{x}_0))} \int_{B_r(\mathbf{x}_0)} f(\mathbf{x})\, d\mathbf{x} - f(\mathbf{x}_0) \right| &= \left| \frac{1}{\mathrm{Vol}\,(B_r(\mathbf{x}_0))} \int_{B_r(\mathbf{x}_0)} (f(\mathbf{x}) - f(\mathbf{x}_0))\, d\mathbf{x} \right| \\ &\le \frac{1}{\mathrm{Vol}\,(B_r(\mathbf{x}_0))} \int_{B_r(\mathbf{x}_0)} |f(\mathbf{x}) - f(\mathbf{x}_0)|\, d\mathbf{x} \\ &< \frac{\epsilon}{\mathrm{Vol}\,(B_r(\mathbf{x}_0))} \int_{B_r(\mathbf{x}_0)} d\mathbf{x} = \epsilon.\end{aligned}$$

12.2.4. a) Since $U(f, \mathcal{G}) - L(f, \mathcal{G}) \le U(f - f_N, \mathcal{G}) - L(f - f_N, \mathcal{G}) + U(f_N, \mathcal{G}) - L(f_N, \mathcal{G}) =: I_1 + I_2$ and f_N is integrable, we can show that f is integrable if we show I_1 is small. We may suppose that $\mathrm{Vol}\,(E) \ne 0$. By hypothesis, given $\epsilon > 0$ choose $N \in \mathbf{N}$ such that $|f_k(\mathbf{x}) - f(\mathbf{x})| < \epsilon/(2\mathrm{Vol}\,(E) + 2)$ for $k \ge N$ and $\mathbf{x} \in E$. Hence, $M_j(f - f_N)$ and $m_j(f - f_N)$ are both less than ε, and it follows that

$$I_1 \le 2 \frac{\varepsilon}{2\mathrm{Vol}\,(E) + 2} \sum_{R_j \cap E \ne \emptyset} |R_j| \le \frac{\varepsilon}{\mathrm{Vol}\,(E) + 1} V(E, \mathcal{G}).$$

Since we can choose $\mathcal{G}$ so that $V(E, \mathcal{G}) < \mathrm{Vol}\,(E) + 1$, it follows that I_1 is small, hence f is integrable. Finally, by the Comparison Theorem, if $k \ge N$, then

$$\int_E |f_k - f|\, dV \le \frac{\epsilon}{\mathrm{Vol}\,(E) + 1} \mathrm{Vol}\,(E) < \epsilon.$$

b) Since E is bounded, choose M so large that $|x| < M$ and $|y| < M$ for all $(x,y) \in E$. Since $1 - \cos\theta$ has a minimum at $\theta = 0$ and is even, we have

$$|1 - \cos(x/k)| \le 1 - \cos(M/k)$$

for $(x,y) \in E$ and k sufficiently large. Since $0 \le e^{y/k} \le e^{M/k}$, it follows that

$$|e^{y/k} - e^{y/k}\cos(x/k)| = |e^{y/k}|\,|1 - \cos(x/k)| \le e^{M/k}(1 - \cos(M/k)) \to 0$$

uniformly on E as $k \to \infty$. On the other hand, if $E^+ = \{(x,y) : y > 0\}$ and $E^- = \{(x,y) : y < 0\}$, then on E^+,

$$|1 - e^{y/k}| = e^{y/k} - 1 \le e^{M/k} - 1 \to 0$$

uniformly as $k \to \infty$, and on E^-,

$$|1 - e^{y/k}| = 1 - e^{y/k} \le 1 - e^{-M/k} \to 0$$

uniformly as $k \to \infty$. Hence $e^{y/k}\cos(y/k) \to 1$ uniformly on E as $k \to \infty$. We conclude that

$$\lim_{k\to\infty} \iint_E e^{y/k}\cos(x/k)\,dA = \iint_E 1\,dA = \text{Area}\,(E).$$

12.2.5. Let $\epsilon > 0$ and choose $M > 0$ such that $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in E$. Since f is integrable on E and E_1 is a Jordan region, we can choose a grid $\mathcal{G} = \{R_1, \dots, R_N\}$ such that

$$\sum_{R_j \cap E \ne \emptyset} (M_j - m_j)|R_j| < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{R_j \cap \partial E \ne \emptyset} |R_j| < \frac{\epsilon}{4M}.$$

Now the upper and lower sums on E_1 can be estimated as follows:

$$\begin{aligned} U(f,\mathcal{G}) - L(f,\mathcal{G}) &= \sum_{R_j \cap E_1 \ne \emptyset} (M_j - m_j)|R_j| \\ &= \sum_{R_j \subset E_1^0} (M_j - m_j)|R_j| + \sum_{R_j \cap \partial E_1 \ne \emptyset} (M_j - m_j)|R_j| \\ &\le \sum_{R_j \cap E \ne \emptyset} (M_j - m_j)|R_j| + 2M \sum_{R_j \cap \partial E_1 \ne \emptyset} |R_j| \\ &< \frac{\epsilon}{2} + M\frac{\epsilon}{4M} = \epsilon. \end{aligned}$$

It follows from Definition 12.17 that f is integrable on E_1.

12.2.6. Let $m = \inf_{\mathbf{x}\in H} f(\mathbf{x})$ and $M = \sup_{\mathbf{x}\in H} f(\mathbf{x})$. By Theorem 12.26, there is a $c \in [m, M]$ such that

$$c \int_H g(\mathbf{x})\,d\mathbf{x} = \int_H f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}.$$

By Theorems 9.29 and 9.30, or 10.58 and 10.61, $f(H)$ is compact and connected in $\mathbf{R}$, hence a closed bounded interval. In particular, $f(H) = [m, M]$, i.e., $c = f(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in H$.

12.2.7. By the one-dimensional Mean Value Theorem, for each $u, v \in \mathbf{R}$, there is a c between u and v such that

$$f(u,y) - f(v,y) = f_x(c,y)(u - v).$$

Since $|f_x(c,y)| \le 1$ it follows that

$$|F(x,y)| \le \frac{1}{x^3} \iint_{B_x(0,0)} |u - v|\,d(u,v) = \frac{1}{x^3} \int_0^{2\pi} \int_0^x |r\cos\theta - r\sin\theta| r\,dr\,d\theta.$$

In particular, $F(x,y)$ is bounded by

$$\frac{1}{3}\int_0^{2\pi} |\cos\theta - \sin\theta|\,d\theta.$$

12.2.8. By repeating the proof of Corollary 5.23, we can prove that the square of any integrable function is integrable, in particular, $(f+g)^2$ is integrable on E. It follows that $fg = ((f+g)^2 - f^2 - g^2)/2$, $f \vee g = (f+g+|f-g|)/2$, and $f \wedge g = (f+g-|f-g|)/2$ are integrable on E.

12.2.9. Let $\mathbf{x}_0 \in V$. Then $B_r(\mathbf{x}_0)$ is a subset of V for r sufficiently small. Hence by Exercise 12.2.3 and hypothesis,

$$f(\mathbf{x}_0) = \lim_{r\to 0+} \frac{1}{\mathrm{Vol}\,(B_r(\mathbf{x}_0))}\int_{B_r(\mathbf{x}_0)} f(\mathbf{x})\,d\mathbf{x} = 0.$$

12.2.10. a) Let $\varepsilon > 0$. By the Extreme Value Theorem, choose $M > 0$ such that $|\phi(\mathbf{x})| \le M$ for all $\mathbf{x} \in H$. Since ϕ is uniformly continuous on H, choose $0 < \delta < \varepsilon/(4M)$ such that

$$s, t \in H \text{ and } |s-t| < \delta \text{ implies } |\phi(s) - \phi(t)| < \varepsilon/(2\mathrm{Vol}\,(E)).$$

Since f is integrable on E, choose a grid $\mathcal{G}$ on E^o such that $U(f,\mathcal{G}) - L(f,\mathcal{G}) < \delta^2$ (see Theorem 12.20).

Let $A = \{j : M_j(f) - m_j(f) < \delta\}$ and $B = \{j : M_j(f) - m_j(f) \ge \delta\}$. Clearly,

$$\begin{aligned} U(\phi\circ f,\mathcal{G}) - L(\phi\circ f,\mathcal{G}) &= \sum_{j\in A}(M_j(\phi\circ f) - m_j(\phi\circ f))|R_j| + \sum_{j\in B}(M_j(\phi\circ f) - m_j(\phi\circ f))|R_j| \\ &=: I_1 + I_2. \end{aligned}$$

To estimate I_1, notice that if $j \in A$, then $M_j(\phi\circ f) - m_j(\phi\circ f) \le \varepsilon/(2\mathrm{Vol}\,(E))$. Thus

$$I_1 \le \frac{\varepsilon}{2\mathrm{Vol}\,(E)}\sum_{j\in A}|R_j| \le \varepsilon/2.$$

On the other hand, if $j \in B$, then $1 \le (M_j(f) - m_j(f))/\delta$, so

$$\begin{aligned} I_2 &\le 2M\sum_{j\in B}|R_j| \\ &\le \frac{2M}{\delta}\sum_{j\in B}(M_j(f) - m_j(f))|R_j| \\ &\le \frac{2M}{\delta}(U(f,\mathcal{G}) - L(f,\mathcal{G})) \\ &\le \frac{2M}{\delta}\cdot\delta^2 = 2M\delta \end{aligned}$$

by the choice of $\mathcal{G}$. Since $\delta < \varepsilon/(4M)$, it follows that $I_2 \le \varepsilon/2$. Thus $I_1 + I_2 < \varepsilon$, i.e., $\phi\circ f$ is integrable on E.

b) The function f in Example 3.34 is integrable (since the set of points of discontinuity of f form a countable, hence is of Lebesgue measure zero). The function $\phi := g$ in Example 3.34 is discontinuous only at one point. But the functions $\phi\circ g$ is nowhere continuous, hence cannot be integrable by Lebesgue's Theorem.

12.2.11. Let $\epsilon > 0$. Since ∂E and E_0 are both of volume zero, choose cubes $Q_1, \dots, Q_N$ such that

$$\partial E \cup E_0 \subset U := \left(\bigcup_{k=1}^N Q_k\right)^0$$

and $\sum_{k=1}^N |Q_k| < \epsilon/(4C)$, where $|f(\mathbf{x}) \le C$ for $\mathbf{x} \in E$. Since $\overline{E}\setminus U \subset E^0$ is compact and f is continuous on E^0, use uniform continuity to choose a $\delta > 0$ such that $\|\mathbf{x}-\mathbf{y}\| < \delta$ and $\mathbf{x},\mathbf{y} \in E^0$ imply $|f(\mathbf{x}) - f(\mathbf{y}| < \epsilon/(2\mathrm{Vol}\,(E))$. Finally, let $\mathcal{G} = \{R_1, \dots, R_M\}$ be a grid such that $\mathbf{x},\mathbf{y} \in R_j$ implies $\|\mathbf{x}-\mathbf{y}\| < \delta$ and each Q_k is a union of R_j's.

By construction, $V(\partial E \cup E_0) < \epsilon$. Moreover, if R_j is not a rectangle that intersects ∂E or E_0, then $M_j - m_j < \epsilon$. It follows that

$$\sum_{R_j\cap E\ne\emptyset}(M_j - m_j)|R_j| \le 2MV(\partial E\cup E_0) + \frac{\epsilon}{2\mathrm{Vol}\,(E)}\sum_{R_j\subseteq E\setminus E_0}|R_j| < \epsilon.$$

Therefore, f is integrable on E.

12.3 Iterated Integrals.

12.3.1. a) $$\int_0^1 \int_0^1 (x+y)\,dx\,dy = \int_0^1 (1/2+y)\,dy = 1.$$

b) $$\int_0^3 \int_0^1 \sqrt{xy+x}\,dx\,dy = \frac{2}{3}\int_0^3 \sqrt{y+1}\,dy = \frac{28}{9}.$$

c) $$\int_0^\pi \int_0^\pi y\cos(xy)\,dy\,dx = \int_0^\pi \int_0^\pi y\cos(xy)\,dx\,dy = \int_0^\pi \sin(\pi y)\,dy = \frac{1}{\pi}(1-\cos(\pi^2)).$$

12.3.2. a) $E = \{(x,y) : 0 \le x \le 1, x \le y \le x^2+1\}$ and

$$\int_0^1 \int_x^{x^2+1} (x+1)\,dy\,dx = \int_0^1 (x^3+1)\,dx = \frac{5}{4}.$$

b) $E = \{(x,y) : 0 \le y \le 1, y \le x \le 1\} = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$, hence by Fubini's Theorem,

$$\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 x\sin(x^2)\,dx = \frac{1-\cos(1)}{2}.$$

c) $E = \{(x,y,z) : 0 \le y \le 1, \sqrt{y} \le x \le 1, 0 \le z \le x^2+y^2\}$ and

$$\int_0^1 \int_{\sqrt{y}}^1 (x^2+y^2)\,dx\,dy = \frac{1}{3}\int_0^1 (1+3y^2-y^{3/2}-3y^{5/2})\,dy = \frac{26}{105}.$$

d) $E = \{(x,y,z) : 0 \le y \le 1, \sqrt{y} \le x \le 1, x^3 \le z \le 1\}$, hence by Fubini's Theorem and the substitution $u = x^3+1$, we have

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 \int_{x^3}^1 \sqrt{x^3+z}\,dz\,dx\,dy &= \frac{2}{3}\int_0^1 \int_0^{x^2} ((x^3+1)^{3/2}-(2x^3)^{3/2})\,dy\,dx \\ &= \frac{2}{3}\int_0^1 (x^2(x^3+1)^{3/2}-x^2(2x^3)^{3/2})\,dx \\ &= \frac{2}{9}\int_1^2 u^{3/2}\,du - \frac{2^{7/2}}{45} = \frac{4(2\sqrt{2}-1)}{45}.\end{aligned}$$

12.3.3. a) $$\int_0^1 \int_0^{x^2} \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \frac{x^3}{1+x^2}\,dx = \frac{1}{2}\int_1^2 (1-\frac{1}{u})\,du = \frac{1-\log 2}{2}.$$

b) $$\int_0^2 \int_0^{1-x/2} (x+y)\,dy\,dx = \frac{1}{8}\int_0^2 (4+4x-3x^2)\,dx = 1.$$

c) $$\int_0^1 \int_0^x x^2 e^{xy}\,dx\,dy = \int_0^1 x(e^{x^2}-1)\,dx = \frac{e-2}{2}.$$

d) $$\int_0^1 \int_0^{1-x^2} \int_0^{x^2+z^2} x\,dy\,dz\,dx = \int_0^1 \int_0^{1-x^2} x(x^2+z^2)\,dz\,dx = \frac{1}{3}\int_0^1 (x-x^7)\,dx = \frac{1}{8}.$$

12.3.4. a) Since $x+y=3$ and $x^2+y^2=1$ do not intersect,

$$\text{Vol}\,(E)=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3-x-y)\,dy\,dx=2\int_{-1}^{1}(3\sqrt{1-x^2}-x\sqrt{1-x^2})\,dx=3\pi.$$

b) The curves $x=\sqrt{y/2}$ and $y=x^2/4$ intersect at $(0,0)$, $y=3-x$ and $y=x^2/4$ at $(2,1)$, $x=\sqrt{y/2}$ and $y=3-x$ at $(1,2)$. Therefore,

$$\text{Vol}\,(E)=\int_0^1\int_{x^2/4}^{2x^2}(x+y)\,dy\,dx+\int_1^2\int_{x^2/4}^{3-x}(x+y)\,dy\,dx=\int_0^1(\frac{7x^3}{4}+\frac{63x^4}{32})\,dx=\frac{91}{30}.$$

c) This region is the set of points "under" the paraboloid $x=y^2+z^2$ which lies "over" the region in the yz plane bounded by $z=y^2$ and $z=4$. Therefore,

$$\text{Vol}\,(E)=\int_{-1}^{1}\int_{y^2}^{1}\int_0^{y^2+z^2}dx\,dz\,dy=\frac{1}{3}\int_{-1}^{1}(1+3y^2-3y^4-y^6)\,dy=\frac{88}{105}.$$

d) This region is the set of points "under" the cubical cylinder $y=x^3$ which lies "over" the region in the xz plane bounded by $z=x^2$ and $z=\sqrt{x}$. Therefore,

$$\text{Vol}\,(E)=\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x^3}dy\,dz\,dx=\int_0^1 x^3(\sqrt{x}-x^2)\,dx=\frac{1}{18}.$$

12.3.5. a) If f is continuous on R then f is integrable on R by Theorem 12.21, $f(\cdot,y)$ is integrable on $[a,b]$ by Theorem 5.10, and $f(x,\cdot)$ is integrable on $[c,d]$ (also by Theorem 5.10).

b) The proof of Remark 12.33 depends only on three properties satisfied by the function f: i) $f(x,y_0)$ is zero off $[2^{-n-1},2^{-n+1})$, ii) $\int_0^1 f(x,y_0)\,dx=0$, and iii) $\int_0^1\int_0^1 f(x,y)\,dy\,dx=1$. Therefore, we need only show that there is a continuous function f which satisfies these three properties.

Let ϕ_k be defined to be zero off the interval $I_k:=[2^{-k},2^{-k+1})$ and be defined on I_k so its graph forms a triangle with base I_k and height 2^{k+1}. Then ϕ_k is zero off I_k and $\int_0^1\phi_k(t)\,dt=1$.

Set $f(x,y)=\sum_{k=1}^{\infty}(\phi_k(x)-\phi_{k+1}(x))\,\phi_k(y)$, and note that f is continuous in each variable. Its iterated integrals, however, are not equal. Indeed,

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)\,dx\,dy&=\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\left(\int_0^1 f(x,y)\,dx\right)\,dy\\&=\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\phi_k(y)\left(\int_0^1(\phi_k(x)-\phi_{k+1}(x))\,dx\right)\,dy=0,\end{aligned}$$

but

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)\,dy\,dx&=\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\left(\int_0^1 f(x,y)\,dy\right)\,dx\\&=\int_{1/2}^{1}\int_0^1\phi_1(x)\phi_1(y)\,dy\,dx+\sum_{k=2}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\phi_k(x)\left(\int_0^1(\phi_k(y)-\phi_{k+1}(y))\,dy\right)\,dx\\&=1+0=1.\end{aligned}$$

12.3.6. a) We may suppose that $n=2$. Suppose f is integrable on $[a,b]$, g is integrable on $[c,d]$, $h(x,y):=f(x)g(y)$, and $R:=[a,b]\times[c,d]$. Let $\epsilon>0$ and choose a grid $\mathcal{G}=\mathcal{P}\times\mathcal{Q}$ on R, where $\mathcal{P}=\{a_0,\dots,a_N\}$ is a partition of $[a,b]$ and $\mathcal{Q}=\{c_0,\dots,c_M\}$ is a partition of $[c,d]$, such that

$$S(f;\mathcal{P})<\int_a^b f(x)\,dx+\epsilon\quad\text{and}\quad S(g;\mathcal{Q})<\int_c^d g(y)\,dy+\epsilon.$$

Choose $(x_j, y_k) \in R_{jk} := [a_{j-1}, a_j] \times [c_{k-1}, c_k]$ such that $M_{j,k}(h) < f(x_j)g(y_k) + \epsilon$. Then

$$\begin{aligned} U(h;\mathcal{G}) &= \sum_{j=1}^{N}\sum_{k=1}^{M} M_{jk}(h)|R_{jk}| \\ &< \sum_{j=1}^{N} f(x_j)(a_j - a_{j-1}) \sum_{k=1}^{M} g(y_k)(c_k - c_{k-1}) + \epsilon|R| = S(f;\mathcal{P})S(g;\mathcal{Q}) + \epsilon|R| \\ &< \left(\int_a^b f(x)\,dx + \epsilon\right)\left(\int_c^d g(y)\,dy + \epsilon\right) + \epsilon|R|. \end{aligned}$$

Hence

$$\begin{aligned} U(h;\mathcal{G}) - \int_a^b f(x)\,dx \int_c^d g(y)\,dy &< \epsilon\left(\int_a^b f(x)\,dx + \int_c^d g(y)\,dy + \epsilon + |R|\right) \\ &\le \epsilon C_0, \end{aligned}$$

where C_0 is a bounded constant when $\epsilon \le 1$, say. It follows that $(U)\iint_R h\,dA \le \int_a^b f(x)\,dx \int_c^d g(y)\,dy$. A similar argument proves $(L)\iint_R h dA \ge \int_a^b f(x)\,dx \int_c^d g(y)\,dy$. Therefore, h is integrable on R and the integral of h is the product of the integrals of f and g.

b) $$\int_Q e^{-\mathbf{x}\cdot\mathbf{y}}\,d\mathbf{x} = \int_Q e^{-x_1}\dots e^{-x_n}\,dV = \left(\int_0^1 e^{-t}\,dt\right)^n = \left(\frac{e-1}{e}\right)^n.$$

12.3.7. a) Let n and m be integers which satisfy $n \le a < n+1$ and $m \le b < m+1$. Since

$$\int_k^{k+1} (x - k - 1/2)\,dx = \frac{2k+1}{2} - \left(k + \frac{1}{2}\right) = 0$$

for all integers k, we have

$$\int_a^b \phi(x)\,dx = \int_a^{n+1} \phi(x)\,dx + \int_m^b \phi(x)\,dx = 0.$$

By elementary integration, then,

$$2\int_a^b \phi(x)\,dx = 1 - (a-n)^2 - (1 - a + n) + (b-m)^2 - (b-m) = ((b-m) - (a-n))((b-m) + (a-n) - 1).$$

Since R is $\mathbf{Z}$-asymmetric, $\gamma := a + b - n - m - 1 \ne 0$. Thus

(*) $$\frac{2}{\gamma}\int_a^b \phi(x)\,dx = b - a - (m-n).$$

Let $R = [a,b] \times [c,d]$. By Exercise 12.3.6, $\iint_R \phi\,dA = 0$ if and only if $\int_a^b \phi(x)\,dx = 0$ or $\int_c^d \phi(y)\,dy = 0$. Without loss of generality, we suppose the first one. Hence, by (*), $\iint_R \psi\,dA = 0$ if and only if $b - a - (m-n) = 0$. But $m - n - 1 < b - a < m - n + 1$. Thus $b - a = m - n$ if and only if $b - a$ is an integer.

b) If $R = \cup_{j=1}^N R_j$ is nonoverlapping and each R_j has at least one integer side, then by the proof of part a),

$$\iint_R \psi\,dA = \sum_{j=1}^N \iint_{R_j} \psi\,dA = 0.$$

Hence, by the converse of part a), R has at least one integer side.

12.3.8. By Exercise 12.1.5b and Theorem 12.24ii, we may suppose that E is closed.

The easiest way to prove this result is to modify the proof of Theorem 12.39. First notice that except for continuity of the bounding functions, Ω is a type I region with $\phi = 0$ and $\psi = f$.

The first issue to look at: Is Ω a Jordan region? As in the proof of Theorem 12.39, $\partial\Omega$ has a top T, a bottom B, and sides S, and both B and S are of volume zero. We cannot yet conclude that T is of volume zero because $\psi = f$ is not continuous this time. However, since f is integrable, choose a grid $\mathcal{G} = \{R_1, \dots, R_N\}$ on E such that $U(f,\mathcal{G}) - L(f,\mathcal{G}) < \epsilon$. Set $M_j = \sup f(R_j)$ and $m_j = \inf f(R_j)$ and observe that if $H_j := R_j \times [m_j, M_j]$, then $|H_j| = (M_j - m_j)|R_j|$ and $\bigcup_{j=1}^N H_j$ contains T. Let $\mathcal{H}$ represent the grid generated by the H_j's, i.e., the partitions $\mathcal{P}_k(\mathcal{H})$, $k = 1, 2$, are the same as those of $\mathcal{G}$, and the partition $\mathcal{P}_3(\mathcal{H}) := \{M_j, m_j : j = 1, 2, \dots, N\}$ arranged in increasing order. Then $\mathcal{H}$ is a grid on T, and

$$V(T;\mathcal{H}) = \sum_{H_j \cap T \neq \emptyset} |H_j| = \sum_{R_j \cap E \neq \emptyset} (M_j - m_j)|R_j| = U(f,\mathcal{G}) - L(f,\mathcal{G}) < \epsilon.$$

We conclude by Theorem 12.4ii that $\partial\Omega$ is of volume zero, hence Ω is a Jordan region.

Now repeat the proof of the second half of Theorem 12.39. Since continuity of ϕ and ψ were not used there, we conclude that

$$\text{Vol}\,(\Omega) = \iiint_\Omega dV = \iint_E \int_0^{f(x,y)} dz\, d(x,y) = \iint_E f\, dV.$$

12.3.9. a) These inequalities can be verified by repeating the proof of Lemma 12.30 with $(X)\int_c^d f(x,y)\,dy$ in place of $\int_c^d f(x,y)\,dy$.

b) If f is integrable on R then the inequalities of part a) become equalities for both $X = L$ and $X = U$. This proves part b)

c) Since

$$(L)\int_0^1 f(x,y)\,dy = x(1-0) = x \quad \text{and} \quad (U)\int_0^1 f(x,y)\,dy = 1(1-0) = 1,$$

we have

$$\int_0^1 \left((L)\int_0^1 f(x,y)\,dy \right) dx = \frac{1}{2} \neq 1 = \int_0^1 \left((U)\int_0^1 f(x,y)\,dy \right) dx.$$

Hence by part b), f cannot be integrable on $[0,1]\times[0,1]$.

12.3.10. Let $\epsilon > 0$ and choose $a < A < B < b$ such that $|F(y) - \int_\alpha^\beta f(x,y)\,dx| < \epsilon$ for all $a < \alpha < A$ and $B < \beta < b$. Then

$$\left| \int_c^d \left(F(y) - \int_\alpha^\beta f(x,y)\,dx \right) dy \right| < \epsilon(d-c),$$

i.e., $\int_c^d \int_\alpha^\beta f(x,y)\,dx\,dy$ converges to $\int_c^d F(y)\,dy$ as $\alpha \to a+$ and $\beta \to b-$. Hence it follows from Fubini's Theorem that

$$\begin{aligned}\int_c^d \int_a^b f(x,y)\,dx\,dy &= \lim_{\alpha\to a+,\beta\to b-} \int_c^d \int_\alpha^\beta f(x,y)\,dx\,dy \\ &= \lim_{\alpha\to a+,\beta\to b-} \int_\alpha^\beta \int_c^d f(x,y)\,dy\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx.\end{aligned}$$

12.4 Change of Variables.

12.4.1. a)

$$\int_0^{\pi/2} \int_0^2 \sin(r^2) r\,dr\,d\theta = \frac{\pi}{4}(1 - \cos 4).$$

b) By Fubini's Theorem, and the substitution $u = 2y - y^2$, $du = (2-2y)\,dy$, we have

$$\int_0^1 \int_0^x \sqrt[3]{(2y-y^2)^2}\,dy\,dx = \int_0^1 \int_y^1 \sqrt[3]{(2y-y^2)^2}\,dx\,dy = \frac{1}{2}\int_0^1 u^{2/3}\,du = \frac{3}{10}.$$

c) Since $x = b$ implies $r = b\sec\theta$, the integral can be written as

$$\int_0^{\pi/4}\int_{a\sec\theta}^{b\sec\theta} r^2\,dr\,d\theta = \frac{b^3-a^3}{3}\int_0^{\pi/4}\sec^3\theta\,d\theta.$$

Integrating by parts, and using the fact that $\int\sec\theta\,d\theta = \log|\sec\theta+\tan\theta|$, we have

$$\int_0^{\pi/4}\sec^3\theta\,d\theta = \frac{1}{2}\left(\sec\theta\tan\theta\,\Big|_0^{\pi/4} + \log|\sec\theta+\tan\theta|\,\Big|_0^{\pi/4}\right) = \frac{1}{2}(\sqrt{2}+\log(1+\sqrt{2})).$$

Therefore, the value of the original integral is $(b^3-a^3)(\sqrt{2}+\log(1+\sqrt{2}))/6$.

12.4.2. a) If $x = r\cos\theta$ and $y = \sqrt{3}r\sin\theta$, then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sqrt{3}\sin\theta & \sqrt{3}r\cos\theta\end{bmatrix} = \sqrt{3}r.$$

Therefore, by Theorem 12.66,

$$\iint_E \cos(3x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 \cos(3r^2)\sqrt{3}r\,dr\,d\theta = \frac{\pi\sqrt{3}}{3}\sin 3.$$

b) If $u = x-2y$ and $v = y$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix} = 1.$$

Since $y = x/2$ implies $u = x-2y = 0$, $x = 4$ implies $u = 4-2v$, and $y = 0$ implies $v = 0$, E can be described in the uv plane by $0 \le u \le 4-2v$, $0 \le v \le 2$. Hence by Theorem 12.46 and the one- dimensional change of variables $w = 4-2v$, $dw = -2\,dv$, we have

$$\iint_E y\sqrt{x-2y}\,dA = \int_0^2\int_0^{4-2v} v\sqrt{u}\,du\,dv = \frac{1}{6}\int_0^4 (4-w)w^{3/2}\,dw = \frac{16^2}{3\cdot 5\cdot 7}.$$

12.4.3. a) These surfaces intersect when $6-z^2 = z$, i.e., $z = 2,-3$. Thus the projection E_3 is the circle $\{(x,y) : x^2+y^2 \le 2\}$. Using cylindrical coordinates, we have

$$\begin{aligned}\iiint_E z^2\,dV &= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} z^2\,r\,dz\,dr\,d\theta\\ &= \frac{1}{3}\int_0^{2\pi}\int_0^{\sqrt{2}}((6-r^2)^{3/2}r - r^7)\,dr\,d\theta = \frac{4\pi}{5}(6\sqrt{6}-7).\end{aligned}$$

b) Using cylindrical coordinates, the one-dimensional change of variables $u = 9-r^2$, $du = -2r\,dr$, and then integrating by parts, we have

$$\begin{aligned}\iiint_E e^z\,dV &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{9-r^2}} e^z r\,dz\,dr\,d\theta\\ &= 2\pi\int_0^1 (e^{\sqrt{9-r^2}}r - 1)\,dr = \pi\left(\int_8^9 e^{\sqrt{u}}\,du - 1\right)\\ &= \pi(2\sqrt{u}e^{\sqrt{u}} - 2e^{\sqrt{u}}\,\Big|_8^9 - 1) = \pi(4e^3 - 1 - 2(\sqrt{8}-1)e^{\sqrt{8}}).\end{aligned}$$

c) Since E is bounded by a sphere and a cone, we use spherical coordinates:

$$\begin{aligned}\iiint_E (x-y)z\,dV &= \int_{-\pi/2}^{\pi/2}\int_0^{\pi/4}\int_0^2 (\rho\cos\theta\sin\varphi - \rho\sin\theta\sin\varphi)\rho\cos\varphi\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= \frac{2^5}{5}\int_{-\pi/2}^{\pi/2}\int_0^{\pi/4}(\cos\theta-\sin\theta)\sin^2\varphi\cos\varphi\,d\varphi\,d\theta\\ &= \frac{32}{5}\frac{\sin^3\varphi}{3}\,\Big|_0^{\pi/4}\,(\sin\theta+\cos\theta\,\Big|_{-\pi/2}^{\pi/2} = \frac{16}{15}\sqrt{2}.\end{aligned}$$

12.4.4. a) The Jacobian of the change of variables $x = a\rho\sin\varphi\cos\theta$, $y = b\rho\cos\varphi\cos\theta$, $z = c\rho\cos\varphi$ is $abc\rho^2\sin\varphi$. Thus the volume of the ellipsoid $E = \{(x,y,z) : x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1\}$ is

$$\mathrm{Vol}\,(E) = \iiint_E 1\,dV = abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{4\pi}{3}abc.$$

b) The projections of these surfaces in the yz plane are the curves $by + cz = d$ and $y^2 + z^2 = r^2$. Suppose these curves intersect. Solving for y we have $(d - cz/b)^2 = r^2 - z^2$, i.e., $(b^2 + c^2)z^2 - 2cd\,z + (d^2 - b^2r^2) = 0$. This quadratic has discriminant $4b^2(r^2(b^2 + c^2) - d^2)$ which is negative since $r^2 < d^2/(b^2 + c^2)$. Thus these two curves do not intersect. It follows that the region E bounded by $y^2 + z^2 = r^2$ and $ax + by + cz = d$ is the cylinder $y^2 + z^2 = r^2$ with back $x = 0$ and front $x = (d - by - cz)/a$. Using the change of variables $y = \rho\cos\theta$, $z = \rho\sin\theta$, we have

$$\begin{aligned}\mathrm{Vol}\,(E) &= \int_0^{2\pi}\int_0^{r}\int_0^{(d-b\rho\cos\theta-c\rho\sin\theta)/a} \rho\,dx\,d\rho\,d\theta \\ &= \frac{1}{a}\int_0^{2\pi}\int_0^{r}(d - b\rho\cos\theta - c\rho\sin\theta)\rho\,d\rho\,d\theta = \frac{\pi d r^2}{a}.\end{aligned}$$

c) Using only the portion of this region which lies in the first octant, we have

$$\mathrm{Vol}\,(E) = 8\int_0^{a}\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-z^2}} dx\,dy\,dz = 8\int_0^a (a^2 - z^2)\,dz = \frac{16}{3}a^3.$$

12.4.5. a) If $u = x - y$ and $v = x + 2y$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix} = 3.$$

Since $y = x$ implies $u = x - y = 0$, $y = -x/2$ implies $v = x + 2y = 0$, $x - y = 1$ implies $u = 1$, and $x + 2y = 1$ implies $v = 1$, this change of variables takes the parallelogram E to the square $[0,1] \times [0,1]$ in the uv plane. Therefore,

$$\iint_E \sqrt{x-y}\sqrt{x+2y}\,dA = \frac{1}{3}\int_0^1\int_0^1 \sqrt{u}\sqrt{v}\,du\,dv = \frac{4}{27}.$$

b) Notice that $2x^2 - 5xy - 3y^2 = (x - 3y)(2x + y)$. If $u = x - 3y$ and $v = 2x + y$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix}1 & -3\\ 2 & 1\end{bmatrix} = 7.$$

Since $y = x/3$ implies $u = x - 3y = 0$, $y = (x - 1)/3$ implies $u = x - 3y = 1$, $y = -2x$ implies $v = 2x + y = 0$, and $y = 1 - 2x$ implies $v = 2x + y = 1$, this change of variables takes the parallelogram E to the square $[0,1] \times [0,1]$ in the uv plane. Therefore,

$$\iint_E \sqrt[3]{2x^2 - 5xy - 3y^2}\,dA = \frac{1}{7}\int_0^1\int_0^1 \sqrt[3]{u}\sqrt[3]{v}\,du\,dv = \frac{9}{112}.$$

c) If $u = x + y$ and $v = y - x$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} = 2.$$

Since $x + y = 2$ implies $u = 2$, $y = 0$ implies $v = -u$, $x + y = 4$ implies $u = 4$, and $y = x$ implies $v = 0$, the trapezoid E can be described in the uv plane by $-u \le v \le 0$, $2 \le u \le 4$. Therefore,

$$\iint_E e^{(y-x)/(x+y)}\,dA = \frac{1}{2}\int_2^4\int_{-u}^{0} e^{v/u}\,dv\,du = \frac{1}{2}\int_2^4 u(1 - e^{-1})\,du = 3\left(\frac{e-1}{e}\right).$$

d) Let $u = x - y$ and $v = x$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = 1.$$

Since $y = x$ implies $u = 0$, $y = 0$ implies $u = v$, and $x = 1$ implies $v = 1$, the region $0 \le y \le x$, $0 \le y \le 1$ can be described in the uv plane by $u \le v \le 1$, $0 \le u \le 1$. Therefore, by Theorem 12.46, we have

$$\int_0^1 \int_0^x f(x-y)\,dy\,dx = \int_0^1 \int_u^1 f(u)\,dv\,du = \int_0^1 (1-u)f(u)\,du = 5.$$

12.4.6. Fix $\mathbf{x}_0 \in V$. By the Inverse Function Theorem, there is an $r_0 > 0$ such that f is 1–1 on $B_{r_0}(\mathbf{x}_0)$. By Theorem 12.46,

$$\text{Vol}\,(f(B_r(\mathbf{x}_0))) = \iiint_{f(B_r(\mathbf{x}_0))} 1\,dV = \iiint_{B_r(\mathbf{x}_0)} |\Delta_f|\,dV$$

for $0 < r < r_0$. Therefore, by Exercise 12.2.3,

$$\lim_{r\to 0} \frac{\text{Vol}\,(f(B_r(\mathbf{x}_0)))}{\text{Vol}\,(B_r(\mathbf{x}_0))} = \lim_{r\to 0} \frac{1}{\text{Vol}\,(B_r(\mathbf{x}_0))} \iiint_{B_r(\mathbf{x}_0)} |\Delta_f|\,dV = |\Delta_f(\mathbf{x}_0)|.$$

12.4.7. If

$$\phi(x,y) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

then

$$\Delta_\phi = \det \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = 1.$$

Hence

$$\text{Vol}\,(\phi(E)) = \int_{\phi(E)} 1\,dV = \int_E |\Delta_\phi|\,dV = \text{Vol}\,(E).$$

12.4.8. a) Let $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$. Then

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} &= \det \begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix} \\ &= \cos\varphi(\rho^2\cos\varphi\sin\varphi) + \rho\sin\varphi(\rho\sin^2\varphi) = \rho^2\sin\varphi. \end{aligned}$$

b) Since Vol is translation and rotation invariant, we can compute the volume of a typical sphere and a typical cone in any position. We choose these positions so that the calculations are simplest. Namely, let B represent the sphere centered at the origin of radius r and C represent the cone centered around the z axis with vertex at the origin of radius r and altitude h. Then

$$\text{Vol}\,(B) = \iiint_B dV = \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{4\pi r^3}{3}.$$

Moreover, since the top of C is given by $z = h$, i.e., $\rho = h\sec\varphi$, we have

$$\begin{aligned} \text{Vol}\,(C) &= \int_0^{2\pi} \int_0^{\arctan(r/h)} \int_0^{h\sec\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= \frac{2\pi h^3}{3} \int_0^{\arctan(r/h)} \tan\varphi\sec^2\varphi\,d\varphi = \frac{\pi r^2 h}{3}. \end{aligned}$$

12.4.9. Let $\phi(t_1,\dots,t_n) = t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n$. Then ϕ takes $[0,1]^n$ onto $\mathcal{P}(\mathbf{v}_1,\dots,\mathbf{v}_n)$ and $\Delta_\phi = \det(\mathbf{v}_1,\dots,\mathbf{v}_n)$. Hence

$$\text{Vol}\,(\mathcal{P}(\mathbf{v}_1,\dots,\mathbf{v}_n)) = \int_{\mathcal{P}(\mathbf{v}_1,\dots,\mathbf{v}_n)} 1\,dV = \int_{[0,1]^n} |\Delta_\phi|\,dV = |\det(\mathbf{v}_1,\dots,\mathbf{v}_n)|.$$

Suppose $n=2$. Then $\det((a,b),(c,d)) = ad-bc$, hence by Exercise 8.2.7,

$$\begin{aligned}\text{Area}\,(\mathcal{P}((a,b),(c,d))) &= \|(a,b,0)\times(c,d,0)\| \\ &= \|(0,0,ad-bc)\| = |ad-bc| = |\det((a,b),(c,d))|.\end{aligned}$$

Similarly, by Exercise 8.2.7 and Theorem 8.9,

$$\text{Vol}\,(\mathcal{P}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)) = |\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)| = |\det(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)|.$$

12.4.10. a) It is clear that $\int_0^1 e^{-x^2}\,dx$ is finite. Since $x\ge 1$ implies $x^2\ge x$, we also have

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx < \infty.$$

Therefore, $\int_0^\infty e^{-x^2}\,dx$ converges to a finite real number I.
b) By definition,

$$\begin{aligned}I^2 &= \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)\\ &= \lim_{N\to\infty}\left(\int_0^N e^{-x^2}\,dx\right)\left(\int_0^N e^{-y^2}\,dy\right) = \lim_{N\to\infty}\int_0^N\int_0^N e^{-x^2-y^2}\,dx\,dy.\end{aligned}$$

Let $Q_N=[0,N]\times[0,N]$ and B_N represent the quarter circle formed by intersecting $B_N(0,0)$ with the first quadrant in the xy plane. Notice that

$$0\le \iint_{Q_N\setminus B_N} e^{-x^2-y^2}\,dx\,dy \le e^{-N^2}\text{Vol}\,(Q_N\setminus B_N)\le N^2e^{-N^2}\to 0$$

as $N\to\infty$. Therefore,

$$\begin{aligned}I^2 &= \lim_{N\to\infty}\iint_{Q_N} e^{-x^2-y^2}\,dA = \lim_{N\to\infty}\left(\iint_{Q_N\setminus B_N} e^{-x^2-y^2}\,dA + \iint_{B_N} e^{-x^2-y^2}\,dA\right)\\ &= \lim_{N\to\infty}\iint_{B_N} e^{-x^2-y^2}\,dA = \lim_{N\to\infty}\int_0^{\pi/2}\int_0^N re^{-r^2}\,dr\,d\theta.\end{aligned}$$

c) By part b),

$$I^2 = \lim_{N\to\infty}\int_0^{\pi/2}\int_0^N re^{-r^2}\,dr\,d\theta = -\frac{\pi}{4}\lim_{N\to\infty} e^{-r^2}\Big|_0^N = \frac{\pi}{4}.$$

Therefore, $I=\sqrt{\pi}/2$.
d) By part c) and Exercise 12.3.6,

$$\lim_{k\to\infty}\int_{Q_k} e^{-\|\mathbf{x}\|^2}\,d\mathbf{x} = \left(\int_{-\infty}^\infty e^{-t^2}\,dt\right)^n = \pi^{n/2}.$$

12.4.11. a) Let $\epsilon>0$. Since E is a compact subset of H^o, choose $\delta_0>0$ such that $\mathbf{x}\in E$ and $\|\mathbf{h}\|<\delta_0$ imply $\mathbf{x}+\mathbf{h}\in H^o$. Since $D\phi$ is uniformly continuous on H, there is a $0<\delta<\min\{1,\delta_0\}$ such that $\|\mathbf{h}\|<\delta$ and $\mathbf{x},\mathbf{x}+\mathbf{h}\in H$ imply $\|D\phi(\mathbf{x}+\mathbf{h})-D\phi(\mathbf{x})\|<\epsilon$. Since H is convex, it follows that if $\mathbf{x}\in E$ and $\|\mathbf{h}\|<\delta$, then any $\mathbf{c}$ on the line segment between $\mathbf{x}+\mathbf{h}$ and $\mathbf{x}$ satisfies $\|D\phi(\mathbf{c})-D\phi(\mathbf{x})\|<\epsilon$.
Let $\mathbf{x}\in E$ and $\|\mathbf{h}\|<\delta$. By Theorem 11.30, there is a $\mathbf{c}$ between $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$ such that

$$\epsilon_{\mathbf{h}}(\mathbf{x}) := \phi(\mathbf{x}+\mathbf{h})-\phi(\mathbf{x})-Df(\mathbf{x})(\mathbf{h}) = (D\phi(\mathbf{c})-D\phi(\mathbf{x}))(\mathbf{h}).$$

By the choice of δ, it follows that $|\epsilon_{\mathbf{h}}(\mathbf{x})|\le\epsilon\|\mathbf{h}\|$.

b) Fix $\mathbf{x} \in H^o$ and choose a rectangle R such that $\mathbf{x} \in R^o \subset H$. By part a), there is a $\delta > 0$ such that $|\epsilon_{\mathbf{x}-\mathbf{y}}(\mathbf{x})| < \epsilon \cdot \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in R$ which satisfy $\|\mathbf{x} - \mathbf{y}\| < \delta$. Apply $S := S_{\mathbf{x}} := (D\phi(\mathbf{x}))^{-1}$ to the identity $\phi(\mathbf{x}) - \phi(\mathbf{y}) = D\phi(\mathbf{x})(\mathbf{x} - \mathbf{y}) + \epsilon_{\mathbf{x}-\mathbf{y}}(\mathbf{x})$ to get

$$S \circ \phi(\mathbf{x}) - S \circ \phi(\mathbf{y}) = \mathbf{x} - \mathbf{y} + T(\mathbf{x}, \mathbf{y})$$

where $T(\mathbf{x}, \mathbf{y}) := S(\epsilon_{\mathbf{x}-\mathbf{y}}(\mathbf{x}))$. Since $D\phi$ is continuous on the compact set R, there is an $M > 0$ (which depends only on ϕ and R) such that $\|S_{\mathbf{x}}\| \leq M$ for all $\mathbf{x} \in R$. Hence, if $\mathbf{x}, \mathbf{y} \in R$ and $\|\mathbf{x} - \mathbf{y}\| < \delta$, then

(*) $$\|T(\mathbf{x}, \mathbf{y})\| \leq M\epsilon \cdot \|\mathbf{x} - \mathbf{y}\|.$$

c) Let Q be a cube contained in R with sides less than $\delta/n^{n/2}$, and notice that $\|\mathbf{x} - \mathbf{y}\| < \delta$ for all $\mathbf{x}, \mathbf{y} \in Q$. Let $\mathbf{y}_0 \in Q$ and observe by the definition T that

$$S \circ \phi(\mathbf{x}) = S \circ \phi(\mathbf{y}_0) - \mathbf{y}_0 + \mathbf{x} + T(\mathbf{x}, \mathbf{y}_0) =: \mathbf{z} + \mathbf{x} + T(\mathbf{x}, \mathbf{y}_0)$$

for $\mathbf{x} \in Q$. This means that $S \circ \phi(Q)$ is very nearly a translation, $\mathbf{z} + Q$, of Q. How big can the volume of $S \circ \phi(Q)$ be? To answer this question, we must see how the term $T(\mathbf{x}, \mathbf{y}_0)$ affects $S \circ \phi(Q)$.

Let Q be the product of intervals $[a_j, b_j]$, where $b_j - a_j = s$ for all j. If $\mathbf{x}, \mathbf{y} \in Q$, then (*) implies $\|T(\mathbf{x}, \mathbf{y}_0)\| \leq M\epsilon \cdot \|\mathbf{x} - \mathbf{y}\| \leq M\epsilon\sqrt{n}|x_j - y_j|$. It follows that each component of $S \circ \phi$ satisfies

$$|(S \circ \phi)_j(\mathbf{x}) - (S \circ \phi)_j(\mathbf{y})| \leq |x_j - y_j|(1 + \sqrt{n}M\varepsilon).$$

In particular, $S \circ \phi(Q)$ is a subset of a cube with sides $s(1 + \sqrt{n}M\epsilon)$. Consequently, $\text{Vol}\,((S \circ \phi)(Q)) \leq s^n(1 + M\epsilon)^n =: C_\epsilon s^n = C_\epsilon |Q|$.

d) Let $\eta > 0$ and let C_ϵ be the constants in part c). Choose ϵ so small that $C_\epsilon < 1 + \eta/|\Delta_\phi(\mathbf{x})|$. Let Q_j be cubes such that $\mathbf{x} \in Q_j^o$ and $\text{Vol}\,(Q_j) \to 0$ as $j \to \infty$. Then by part c) and Exercise 12.4.9,

$$|\det(S)| \cdot \text{Vol}\,(\phi(Q)) = \text{Vol}\,(S \circ \phi(Q_j)) \leq C_\epsilon |Q_j|$$

for j large. Since $S = (D\phi(\mathbf{x}))^{-1}$, we have $\det(S) = |\Delta_\phi(\mathbf{x})|^{-1}$. Hence,

$$\frac{\text{Vol}\,(\phi(Q_j))}{|Q_j|} \leq C_\epsilon |\Delta_\phi(\mathbf{x})| < |\Delta_\phi(\mathbf{x})| + \eta$$

for j large.

Let ϵ be so small that $s - \epsilon\sqrt{n}M > 0$. By repeating the argument in part c), but looking for lower estimates this time, we can show that $S \circ \phi(Q)$ contains a cube with sides $s(1 - \epsilon\sqrt{n}M)$, so $\text{Vol}\,(S \circ \phi(Q_j)) \geq \widetilde{C}_\epsilon |Q_j|$ for some constants $\widetilde{C}_\epsilon \to 1$ as $\epsilon \to 0$. Hence, for ϵ sufficiently small and j large,

$$\frac{\text{Vol}\,(\phi(Q_j))}{|Q_j|} \geq \widetilde{C}_\epsilon |\Delta_\phi(\mathbf{x})| > |\Delta_\phi(\mathbf{x})| - \eta.$$

Combining this estimate with the estimate in the previous paragraph, we conclude that

$$\left| \frac{\text{Vol}\,(\phi(Q_j))}{|Q_j|} - |\Delta_\phi(\mathbf{x})| \right| < \eta$$

for j large, i.e., $\text{Vol}\,(\phi(Q_j))/|Q_j| \to |\Delta_\phi(\mathbf{x})$— as $j \to \infty$.

12.5 Partitions of Unity.

12.5.1. If $(fg)(\mathbf{x}) \neq 0$ then $f(\mathbf{x}) \neq 0$ and $g(\mathbf{x}) \neq 0$. It follows that

$$\{\mathbf{x} : (fg)(\mathbf{x}) \neq 0\} \subseteq \{\mathbf{x} : f(\mathbf{x}) \neq 0\} \cap \{\mathbf{x} : g(\mathbf{x}) \neq 0\}.$$

In particular, $\text{spt}\,(fg) \subseteq \text{spt}\, f \cap \text{spt}\, g$.

12.5.2. Products of $\mathcal{C}^\infty$ functions are $\mathcal{C}^\infty$ functions. Moreover, by Exercise 12.5.1, products of functions with compact support have compact support. Therefore, fg and αf belong to $\mathcal{C}_c^\infty(\mathbf{R}^n)$ when f and g do.

12.5.3. If $f \in \mathcal{C}_c^\infty(\mathbf{R})$ then $f = 0$ on some interval (a, b). If f is also analytic, then by analytic continuation (Theorem 7.56), $f(x) = 0$ for all $x \in (-\infty, \infty)$. In particular, $f(x_0) = 0$, a contradiction.

12.5.4. Fix $j \in \mathbf{N}$. By the Inverse Function Theorem, ϕ^{-1} (hence also $\phi_j \circ \phi^{-1}$) is $\mathcal{C}^1$ on $\phi(V)$. In particular, both ϕ and ϕ^{-1} are continuous and must take open sets (respectively, compact sets) to open sets (respectively, compact sets). Since $K := \phi(\operatorname{spt}\phi_j)$ is a compact subset of $\phi(W_j)$, and $\phi_j \circ \phi^{-1}$ vanishes off K, it is clear that $\phi_j \circ \phi^{-1} \in \mathcal{C}_c^\infty(\mathbf{R}^n)$, $\phi_j \circ \phi^{-1} \geq 0$, and $\operatorname{spt}(\phi_j \circ \phi^{-1}) \subset \phi(W_j)$. Since $\sum \phi_j = 1$ on V, it is also clear that $\sum \phi_j \circ \phi^{-1} = 1$ on $\phi(V)$. Finally, if H is a compact subset of $\phi(V)$ then $\phi^{-1}(H)$ is a compact subset of V. Since $\{\phi_j\}$ is a partition of unity, there is an open set W of V containing H and an $N \in \mathbf{N}$ such that $\phi_j = 0$ on W for all $j \geq N$. Hence $\phi(W)$ is an open set containing H and $\phi_j \circ \phi^{-1} = 0$ on $\phi(W)$ for all $j \geq N$. Thus $\{\phi_j \circ \phi^{-1}\}$ is a partition of unity on $\phi(V)$ subordinate to the covering $\{\phi(W_j)\}$.

12.5.5. Clearly, $\phi_j\psi_k$ belongs to $\mathcal{C}_c^p(\mathbf{R}^n)$, $\phi_j\psi_k \geq 0$, and $\operatorname{spt}(\phi_j\psi_k) \subset V_j \cap W_k$. Also,

$$\sum_{j=1}^\infty \sum_{k=1}^\infty \phi_j\psi_k = \left(\sum_{j=1}^\infty \phi_j\right)\left(\sum_{k=1}^\infty \psi_k\right) = 1 \cdot 1 = 1.$$

Given a compact subset H of V, choose open sets W_1, W_2 containing H and integers N_1, N_2, such that $\phi_j = 0$ on W_1 for $j \geq N_1$ and $\psi_k = 0$ on W_2 for $k \geq N_2$. Then $W := W_1 \cap W_2$ is open, contains H, and $\phi_j\psi_k = 0$ on W for $j, k \geq N := \max\{N_1, N_2\}$. Thus $\{\phi_j\psi_k\}$ is a $\mathcal{C}^p$ partition of unity on V subordinate to the covering $\{V_j \cap W_k\}$.

12.5.6. Since H is a Jordan region, choose a grid $\{Q_\ell\}_{\ell=1}^N$ of a rectangle $R \supset H$ such that

$$\sum_{Q_\ell \cap \partial H \neq \emptyset} |Q_\ell| < \frac{1}{j}.$$

Fix $j \in \mathbf{N}$ and let $\{R_\ell\}$ be a collection of rectangles which satisfies

$$Q_\ell \subset R_\ell^o \quad \text{and} \quad \sum_{\ell=1}^N |R_\ell| < \sum_{\ell=1}^N |Q_\ell| + \frac{1}{j}$$

Then

$$V_j := \bigcup_{R_\ell \cap H \neq \emptyset} R_\ell^o$$

is open, contains H, and $\operatorname{Vol}(V_j \setminus H) < 2/j$.

By Theorem 12.59, choose a $\phi_j \in \mathcal{C}_c^\infty(\mathbf{R}^n)$ such that $\phi_j \equiv 1$ on H, $\operatorname{spt}\phi_j \subset V_j$, and $0 \leq \phi_j \leq 1$. Then

$$\begin{aligned}\int_{\mathbf{R}^n} \phi_j\,dV - \operatorname{Vol}(H) &= \int_{V_j} \phi_j\,dV - \int_H \phi_j\,dV \\ &= \int_{V_j \setminus H} \phi_j\,dV \\ &\leq \operatorname{Vol}(V_j \setminus H) < \frac{2}{j} \to 0\end{aligned}$$

as $j \to \infty$.

12.6 The Gamma Function and Volume.

12.6.1. Using the substitution $u = t^2$, $du = 2t\,dt$, we obtain

$$\int_0^\infty t^2 e^{-t^2}\,dt = \frac{1}{2}\int_0^\infty \sqrt{u}e^{-u}\,du = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{4}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}.$$

12.6.2. Using the substitution $u = -\log x$, $du = -dx/x$ (hence $dx = -e^{-u}\,du$), we obtain

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_0^\infty u^{-1/2}e^{-u}\,du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

12.6.3. Using the substitution $u = e^t$, $du = e^t\,dt$ (hence $dt = du/u$), we obtain

$$\int_{-\infty}^{\infty} e^{\pi t - e^t}\,dt = \int_0^{\infty} u^{\pi-1}e^{-u}\,du = \Gamma(\pi).$$

12.6.4. If $n = 4$ then $\text{Vol}\,(B_r(\mathbf{a})) = 2r^4\pi^2/(4\Gamma(2)) = r^4\pi^2/2$. If $n = 5$ then since $\Gamma(5/2) = (3/2)(1/2)\Gamma(1/2) = 3\sqrt{\pi}/4$, we have $\text{Vol}\,(B_r(\mathbf{a})) = 2r^5\pi^{5/2}/(5\Gamma(5/2)) = 8r^5\pi^2/15$.

12.6.5. Let ψ_n represent the spherical change of variables in $\mathbf{R}^n$. By definition, the first row of D_{ψ_n} looks like $\cos\varphi_1\ -\rho\sin\varphi_1\ 0\ 0\ \ldots\ 0$; the second row looks like $\sin\varphi_1\cos\varphi_2$, $\rho\cos\varphi_1\cos\varphi_2$, $-\rho\sin\varphi_1\sin\varphi_2$, $0, \ldots, 0$; the second to last row looks like

$$\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\theta,\ \rho\cos\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\theta,\ \rho\sin\varphi_1\cos\varphi_2\ldots\sin\varphi_{n-2}\cos\theta,$$

$$\rho\sin\varphi_1\sin\varphi_2\cos\varphi_3\ldots\sin\varphi_{n-2}\cos\theta,\ \ldots,\ -\rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\theta;$$

and the last row looks like

$$\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\theta,\ \rho\cos\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\theta,\ \rho\sin\varphi_1\cos\varphi_2\ldots\sin\varphi_{n-2}\sin\theta,$$

$$\rho\sin\varphi_1\sin\varphi_2\cos\varphi_3\ldots\sin\varphi_{n-2}\sin\theta,\ \ldots,\ \rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\theta.$$

Suppose $\Delta_{\psi_{n-1}} = \rho^{n-2}\sin^{n-3}\varphi_1\ldots\sin\varphi_{n-3}$. Notice that the cofactor $|A_1|$ of $-\rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\theta$ is identical to $\Delta_{\psi_{n-1}}$ if in $D\varphi_{n-1}$, θ is replaced by φ_{n-2} and each entry in the last row of $D\psi_{n-1}$ is multiplied by $\sin\theta$. Similarly, the cofactor $|A_2|$ of $\rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\theta$ is identical to $\Delta_{\psi_{n-1}}$ if in $D\psi_{n-1}$, θ is replaced by φ_{n-2} and each entry in the last row of $D\psi_{n-1}$ is multiplied by $\cos\theta$. Hence it follows from the inductive hypothesis that $|A_1| = \rho^{n-2}\sin^{n-3}\varphi_1\ldots\sin\varphi_{n-3}\sin\theta$ and $|A_2| = \rho^{n-2}\sin^{n-3}\varphi_1\ldots\sin\varphi_{n-3}\cos\theta$. Expanding Δ_{ψ_n} along the last column, we conclude that

$$\begin{aligned}|\Delta_{\psi_n}| &= \rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\theta|A_1| + \rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\theta|A_2|\\ &= \rho^{n-1}\sin^{n-2}\varphi_1\ldots\sin\varphi_{n-2}(\sin^2\theta+\cos^2\theta) = \rho^{n-1}\sin^{n-2}\varphi_1\ldots\sin\varphi_{n-2}.\end{aligned}$$

12.6.6. Use the change of variables $x_1 = a_1\rho\cos\varphi_1, \ldots, x_n = a_n\rho\sin\varphi_1\ldots\sin\varphi_{n-1}\sin\theta$ and repeat the argument in Exercise 12.6.5 to verify that this change has Jacobian

$$a_1\ldots a_n\rho^{n-1}\sin^{n-2}\varphi_1\ldots\sin\varphi_{n-2}.$$

Therefore,

$$\text{Vol}\,(E) = a_1\ldots a_n\text{Vol}\,(B_1(0)) = \frac{2a_1\ldots a_n\pi^{n/2}}{n\Gamma(n/2)}.$$

12.6.7. One way is to use cylindrical coordinates: $x_1 = x_1$, $x_2 = \rho\cos\varphi_1$, $x_3 = \rho\sin\varphi_1\cos\varphi_2, \ldots, x_{n-1} = \rho\sin\varphi_1\ldots\sin\varphi_{n-3}\cos\theta$, $x_n = \rho\sin\varphi_1\ldots\sin\varphi_{n-3}\sin\theta$. By Exercise 12.6.5, the Jacobian of this change of variables is $\rho^{n-2}\sin^{n-3}\varphi_1\ldots\sin\varphi_{n-3}$. Since $x_2^2+\cdots+x_n^2 = \rho^2$, the cone can be described in these new coordinates as $h\rho/r \le x_1 \le h$. Since $\rho \le (r/h)x_1 \le (r/h)h = r$, it follows that if $A_1(0)$ represents the $n-1$- dimensional unit ball, then

$$\begin{aligned}\text{Vol}\,(C) &= \int_C 1\,dV\\ &= \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^r\int_{(h/r)\rho}^h \rho^{n-2}\sin^{n-3}\varphi_1\ldots\sin\varphi_{n-3}\,dx_1\,d\rho\,d\varphi_1\ldots\,d\varphi_{n-3}\,d\theta\\ &= (n-1)\text{Vol}\,(A_1(0))\int_0^r\left(h-\frac{h}{r}\rho\right)\rho^{n-2}\,d\rho\\ &= \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\left(\frac{hr^{n-1}}{n-1}-\frac{h}{r}\frac{r^n}{n}\right)\\ &= \frac{2h\,r^{n-1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\left(\frac{1}{n-1}-\frac{1}{n}\right) = \frac{2h\,r^{n-1}\pi^{(n-1)/2}}{n(n-1)\Gamma((n-1)/2)}.\end{aligned}$$

An even easier way would be to integrate dx_1 last. Fix x_1. Since

$$x_2^2 + \cdots + x_n^2 \le \left(\frac{x_1 r}{h}\right)^2$$

has volume

$$\frac{2(x_1 r/h)^{n-1}\pi^{(n-1)/2}}{(n-1)\Gamma((n-1)/2)},$$

we have

$$\begin{aligned}
\mathrm{Vol}\,(C) &= \int_C 1\,dV \\
&= \int_0^h \int_{A_{rx_1/h}(0)} dV_{n-1}\,dx_1 \\
&= \frac{2\pi^{(n-1)/2}}{(n-1)\Gamma((n-1)/2)}\left(\frac{r^{n-1}}{h^{n-1}}\right)\int_0^h x_1^{n-1}\,dx_1 \\
&= \frac{2h\,r^{n-1}\pi^{(n-1)/2}}{n(n-1)\Gamma((n-1)/2)}.
\end{aligned}$$

12.6.8. By symmetry, we may suppose $k = n$. If ϕ represents the change of variables from rectangular coordinates to spherical coordinates, then

$$x_n^2|\Delta_\phi| = \rho^2 \sin^2\varphi_1 \dots \sin^2\varphi_{n-2}\sin^2\theta\rho^{n-1}\sin^{n-2}\varphi_1\dots\sin\varphi_{n-2}.$$

Therefore,

$$\begin{aligned}
\int_{B_r(0)} x_k^2\,d\mathbf{x} &= \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{r} \rho^{n+1}\sin^n\varphi_1\dots\sin^3\varphi_{n-2}\sin^2\theta\,d\rho\,d\varphi_1\dots\,d\varphi_{n-2}\,d\theta \\
&= \frac{r^{n+2}}{n+2}\cdot\pi\int_0^{\pi}\sin^n\varphi\,d\varphi\cdots\int_0^{\pi}\sin^3\varphi\,d\varphi \\
&= \frac{r^{n+2}}{n+2}\cdot\pi\left(\frac{\Gamma((n+1)/2)\Gamma(1/2)}{\Gamma((n+2)/2)}\right)\cdots\left(\frac{\Gamma(4/2)\Gamma(1/2)}{\Gamma(5/2)}\right) \\
&= \frac{r^{n+2}}{n+2}\cdot\pi\cdot\frac{\pi^{(n-2)/2}}{\Gamma((n+2)/2)} \\
&= \frac{r^{n+2}}{n+2}\cdot\frac{2}{n}\cdot\frac{\pi^{n/2}}{\Gamma(n/2)} = \frac{r^2}{n+2}\mathrm{Vol}\,(B_r(0)).
\end{aligned}$$

12.6.9. By the Mean Value Theorem,

$$|f(\mathbf{x})| = |f(\mathbf{x}) - f(0)| \le \|\nabla f(\mathbf{c})\|\,\|\mathbf{x}\| \le \|\mathbf{x}\|$$

for all $\mathbf{x} \in B_1(0)$. Hence by spherical coordinates,

$$\begin{aligned}
\int_{B_1(0)} |f(\mathbf{x})|^k\,d\mathbf{x} &\le \int_{B_1(0)} \|\mathbf{x}\|^k\,d\mathbf{x} \\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^1 \rho^{n+k-1}\,d\rho \\
&= \frac{2\pi^{n/2}}{(n+k)\Gamma(n/2)} \to 0
\end{aligned}$$

as $k \to \infty$.

12.6.10. a) Let

$$F(x) = \int_0^1 e^{-t}t^{x-1}\log t\,dt \quad \text{and} \quad G(x) = \int_1^\infty e^{-t}t^{x-1}\log t\,dt.$$

By Exercise 4.4.6, $|\log t| \le Ct$ for $t > 1$, so the integrand of G is dominated by $Ce^{-t}t^x \le Ce^{-t/2}$. It follows from the Weierstrass–M–test that G converges uniformly on $\mathbf{R}$.

If $x \ge 1$, then $|e^{-t}t^{x-1}\log t| \le |\log t|$. Since

$$\int_0^1 |\log t|\,dt = (t - t\log t\,\big|_0^1 = 1,$$

it follows that F converges uniformly on $[1, \infty)$. But integrating by parts, we have

$$F(x) = \frac{1}{x}\int_0^1 (e^{-t}t^x \log t - e^{-t}t^{x-1})\,dt = \frac{F(x+1)}{x} + \frac{\Gamma(x)}{x}.$$

Thus F converges uniformly on $(0, 1)$.

We have proved that

$$\int_0^\infty e^{-t}t^{x-1}\log t\,dt$$

converges uniformly on $(0, \infty)$. Differentiating Γ under the integral sign, we conclude that

$$\Gamma'(x) = \int_0^\infty e^{-t}t^{x-1}\log t\,dt$$

exists for all $x > 0$.

b) By repeated differentiation under the integral sign, we obtain

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t}t^{x-1}(\log t)^n\,dt$$

for all $x > 0$. Hence $\Gamma'' \ge 0$, and we conclude by Theorem 5.61 that Γ is convex on $(0, \infty)$.

13.1 Curves.

13.1.1. (ψ, I) runs clockwise. Its speed is $\|\psi'(t)\| = \|(a\cos t, -a\sin t)\| = a$. (σ, J) runs counterclockwise. Its speed is $\|(-2a\sin 2t, 2a\cos 2t)\| = 2a$.

13.1.2. Since $\phi'(t) = \mathbf{a} \neq 0$, C is smooth. Since $t\mathbf{a} + \mathbf{x}_0 = u\mathbf{a} + \mathbf{x}_0$ implies $t = u$, C is simple. Since $\phi(0) = \mathbf{x}_0$ and $\phi(1) = \mathbf{x}_0 + \mathbf{a}$, C contains $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{a}$. Finally, if θ is the angle between $\phi(t_1) - \phi(0)$ and $\phi(t_2) - \phi(0)$, then by (3) in 8.1 we have

$$\cos\theta = \frac{t_1\mathbf{a}\cdot(t_2\mathbf{a})}{|t_1|\,|t_2|\,\|\mathbf{a}\|^2} = \frac{t_1t_2}{|t_1|\,|t_2|} = \pm 1.$$

Hence $\theta = 0$ or π.

13.1.3. The trace of $\phi(\theta) := (f(\theta)\cos\theta, f(\theta)\sin\theta)$ on $I := [0, 2\pi]$ coincides with the graph $r = f(\theta)$. Since

$$\phi'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta)$$

we have $\|\phi'(\theta)\|^2 = |f'(\theta)|^2 + |f(\theta)|^2 \neq 0$. Therefore, (ϕ, I) is a smooth curve.

13.1.4. Let $C = (\phi, (0,1])$ where $\phi(t) = (t, \sin(1/t))$ and set $t_k = 2/((2k+1)\pi)$ for $k \in \mathbf{N}$. Since $\sin(1/t_k) = (-1)^k$, it is clear that $\|\phi(t_k) - \phi(t_{k+1})\| \geq 2$ for each $k \in \mathbf{N}$. Hence by definition,

$$\|C\| \geq \sum_{j=1}^{k} \|\phi(t_j) - \phi(t_{j+1})\| \geq 2k$$

for each $k \in \mathbf{N}$, i.e., $\|C\| = \infty$.

13.1.5. a) This curve evidently lies on the cone $x^2 + y^2 = z^2$. It spirals around this cone from $\phi(0) = (0,1,1)$ to $\phi(2\pi) = (0, e^{2\pi}, e^{2\pi})$. Since $\phi'(t) = (e^t(\sin t + \cos t), e^t(\cos t - \sin t), e^t)$, the arc length is given by

$$L(C) = \int_0^{2\pi} \|\phi'(t)\|\,dt = \sqrt{3}\int_0^{2\pi} e^t\,dt = \sqrt{3}(e^{2\pi} - 1).$$

b) This curve forms a "script vee" from $(-1,1)$ through $(0,0)$ to $(1,1)$. Since $x = y^{3/2}$ implies $x' = 3\sqrt{y}/2$, we have by the explicit form (see the formula which follows (3)) that

$$L(C) = 2\int_0^1 \sqrt{1 + 9y/4}\,dy = \int_0^1 \sqrt{4 + 9y}\,dy = \frac{2(\sqrt{13^3} - 1)}{27}.$$

c) Since $\phi(t) = t^2(1,1,1)$, this curve is the straight line from $(0,0,0)$ to $(4,4,4)$. Hence its arc length is $\sqrt{4^2 + 4^2 + 4^2} = 4\sqrt{3}$.

d) By definition, $\phi'(t) = (-3\cos^2 t\sin t, 3\sin^2 t\cos t)$, hence

$$\|\phi'(t)\| = 3\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t} = 3|\sin t\cos t|.$$

Therefore,

$$L(C) = 4\int_0^{\pi/2} 3|\sin t\cos t|\,dt = 6\sin^2 t\,\Big|_0^{\pi/2} = 6.$$

13.1.6. a) If $\phi(t) = (3\cos t, 3\sin t)$ and $I = [0, \pi/2]$, then $\|\phi'(t)\| = \|(-3\sin t, 3\cos t)\| = 3$, hence

$$\int_C xy^2\,ds = \int_0^{\pi/2} 3\cos t\cdot 9\sin^2 t\cdot 3\,dt = 27.$$

b) If $\phi(t) = (a\cos t, b\sin t)$ and $I = [0, \pi/2]$, then

$$\|\phi'(t)\| = \|(-a\sin t, b\cos t)\| = \sqrt{a^2 + (b^2 - a^2)\cos^2 t},$$

hence

$$\int_C xy\,ds = \int_0^{\pi/2} ab\cos t\sin t\sqrt{a^2+(b^2-a^2)\cos^2 t}\,dt.$$

Making the substitution $u = a^2+(b^2-a^2)\cos^2 t$, $du = -2(b^2-a^2)\cos t\sin t$, we have

$$\int_C xy\,ds = \frac{-ab}{2(b^2-a^2)}\int_{b^2}^{a^2}\sqrt{u}\,du = \frac{ab(a^2+ab+b^2)}{3(a+b)}.$$

c) If $\phi(t) = (2\sin t, 4\sin^2 t, 2\cos t)$ and $I = [0, 2\pi]$, then

$$\|\phi'(t)\| = \|(2\cos t, 8\sin t\cos t, -2\sin t)\| = 2\sqrt{1+16\sin^2 t\cos^2 t},$$

hence

$$\int_C \sqrt{1+yz^2}\,ds = 2\int_0^{2\pi}(1+16\sin^2 t\cos^2 t)\,dt = 12\pi,$$

using the double angle formulas $\sin^2 t = (1-\cos 2t)/2$ and $\cos^2 t = (1+\cos 2t)/2$.

d) Let $\phi_1(t) = (t,0,0)$, $I_1 = [0,1]$; $\phi_2(t) = (t, 2-2t, 0)$, $I_2 = [0,1]$; and $\phi_3(t) = (0,t,0)$, $I_3 = [0,2]$. Then

$$\begin{aligned}\int_C (x+y+z^3)\,ds &= \int_{C_1}(x+y+z^3)\,ds + \int_{C_2}(x+y+z^3)\,ds + \int_{C_3}(x+y+z^3)\,ds\\ &= \int_0^1 t\,dt + \sqrt{5}\int_0^1(2-t)\,dt + \int_0^2 t\,dt = \frac{1}{2} + \frac{3\sqrt{5}}{2} + 2 = \frac{5+3\sqrt{5}}{2}.\end{aligned}$$

13.1.7. a) If $g_k \to g$ uniformly on $\phi(I)$, then $g_k(\phi(t))\|\phi'(t)\| \to g(\phi(t))\|\phi'(t)\|$ uniformly on I. Thus use Theorem 7.10.

b) If $g_k \to g$ monotonically on $\phi(I)$, then $g_k(\phi(t))\|\phi'(t)\| \to g(\phi(t))\|\phi'(t)\|$ monotonically on I. The limit is continuous because ϕ' is continuous. Thus use Dini's Theorem.

13.1.8. By hypothesis, there exist closed, nonoverlapping intervals $J_1, \dots, J_N$ such that $\tau' \neq 0$ on each J_k^0 and $J = \cup_{k=1}^N J_k$. Hence by the Chain Rule and a one-dimensional change of variables, we have

$$\begin{aligned}\int_J g(\psi(u))\|\psi'(u)\|\,du &= \sum_{k=1}^N \int_{J_k^0} g\circ\phi\circ\tau(u)\,|\tau'(u)|\,\|\phi'\circ\tau(u)\|\,du\\ &= \sum_{k=1}^N \int_{\tau(J_k^0)} g\circ\phi(t)\,\|\phi'(t)\|\,dt = \int_{\cup_{k=1}^N \tau(J_k^0)} g\circ\phi(t)\,\|\phi'(t)\|\,dt.\end{aligned}$$

Since τ is 1-1, $\tau(J_k)\setminus\tau(J_k^0)$ consists of two points, so this last integral is unchanged if J_k^0 is replaced by J_k. Since $\cup_{k=1}^N \tau(J_k) = \tau(\cup_{k=1}^N J_k) = \tau(J) = I$, we conclude that

$$\int_J g(\psi(u))\|\psi'(u)\|\,du = \int_I g(\phi(t))\|\phi'(t)\|\,dt.$$

13.1.9. It is clear that $(x,y) = \phi(t)$ implies $x^3+y^3 = 3xy$. We examine the trace of $\phi(t)$ as $t \to -\infty$, $t \to -1-$, $t \to -1+$, and $t \to \infty$. Notice once and for all that

$$\phi'(t) = \left(\frac{3(1-2t^3)}{(1+t^3)^2}, \frac{3(2t-t^4)}{(1+t^3)^2}\right).$$

As $t \to -\infty$, $(x,y) \to (0,0)$ and $dy/dx = (2t-t^4)/(1-2t^3) \to -\infty$. Thus the trace of $\phi(t)$ approaches $(0,0)$ and is asymptotic to the negative y axis as $t \to -\infty$.

As $t \to -1-$, $x \to \infty$, $y \to -\infty$, $y/x = t \to -1$, and $dy/dx \to -1$. Thus the trace of $\phi(t)$ lies in the fourth quadrant and is asymptotic to the line $y = -x$ as $t \to -1-$.

As $t \to -1+$, $x \to -\infty$, $y \to \infty$, $y/x = t \to -1$, and $dy/dx \to -1$. Thus the trace of $\phi(t)$ lies in the second quadrant and is asymptotic to the line $y = -x$ as $t \to -1+$.

As $t\to\infty$, $(x,y)\to(0,0)$ and $dy/dx=(2t-t^4)/(1-2t^3)\to\infty$. Thus the trace of $\phi(t)$ approaches $(0,0)$ and is asymptotic to the positive y axis as $t\to\infty$.

Finally, $\phi(0)=(0,0)$ and $dy/dx\to 0$ as $t\to 0$. Thus the trace of $\phi(t)$ is asymptotic to the x axis as $t\to 0$.

13.1.10. a) If $\psi(t)=t\mathbf{a}+\mathbf{b}$, then $\psi'(t)=\mathbf{a}$ and $\ell(t)=\|\mathbf{a}\|(t-t_0)$ for all t. It follows that $\theta(t)/\ell(t)=0$ for all t, i.e., $\kappa(\mathbf{x}_0)=0$ for all $\mathbf{x}_0$ on the given line.

b) If $\psi(t)=(a+r\cos t, b+r\sin t)$, then $\psi'(t)=(-r\sin t, r\cos t)$. Suppose $t>t_0$ is near t_0 and $\mathbf{x}_0=\psi(t_0)$. Then $\ell(t)=r(t-t_0)$, and $\theta(t)=t-t_0$. Thus $\theta(t)/\ell(t)=1/r$ for all t. A similar argument works for the case $t<t_0$. Therefore, $\kappa(\mathbf{x}_0)=1/r$ for all $\mathbf{x}_0$ on the circle C.

13.1.11. a) Let $s=\ell(t)$. By Definition 13.6 and the Fundamental Theorem of Calculus, $ds/dt=\|\phi'(t)\|$, hence by the Inverse Function Theorem, $dt/ds=(\ell^{-1})'(s)=1/\|\phi'(t)\|$, where $t=\ell^{-1}(s)$. Since $\nu(s)=\phi(\ell^{-1}(s))$, it follows from the Chain Rule that

$$\nu'(s):=\frac{d\nu}{ds}(s)=\frac{d\phi}{dt}(t)\frac{dt}{ds}=\frac{1}{\|\phi'(t)\|}\cdot\phi'(t)=\frac{\phi'(t)}{\|\phi'(t)\|}.$$

In particular, $\|\nu'(s)\|=1$ for all $s\in[0,L]$.

b) By part a) and the Dot Product Rule,

$$0=(\nu'(s)\cdot\nu'(s))'=2\nu'(s)\cdot\nu''(s).$$

Therefore, $\nu''(s)$ is orthogonal to $\nu'(s)$ for all $s\in[0,L]$.

c) Let θ_s represent the angle between $\nu'(s)$ and $\nu'(s_0)$ and suppose for simplicity that $s>s_0$. By part a) and the law of cosines,

$$\|\nu'(s)-\nu'(s_0)\|^2=2-2\cos\theta_s=4\sin^2\frac{\theta_s}{2}.$$

Hence

$$\kappa(\mathbf{x}_0)=\lim_{s\to s_0}\left|\frac{2\sin(\theta_s/2)}{s-s_0}\right|\left|\frac{\theta_s}{2\sin(\theta_s/2)}\right|=\lim_{s\to s_0}\frac{\|\nu'(s)-\nu'(s_0)\|}{|s-s_0|}=\|\nu''(s_0)\|.$$

d) Since $\phi(t)=\nu(\ell(t))$, we have by the Chain Rule that $\phi'(t)=\nu'(\ell(t))\cdot\ell'(t)=\nu'(\ell(t))\cdot\|\phi'(t)\|$ and $\phi''(t)=\nu''(\ell(t))\|\phi'(t)\|^2+\nu'(\ell(t))\|\phi'(t)\|'$. Therefore,

$$\begin{aligned}\phi'(t_0)\times\phi''(t_0)&=(\nu'(\ell(t_0))\cdot\|\phi'(t_0)\|)\times(\nu''(\ell(t_0))\|\phi'(t_0)\|^2+\nu'(\ell(t_0))\|\phi'(t_0)\|')\\&=\|\phi'(t_0)\|^3(\nu'(s_0)\times\nu''(s_0)).\end{aligned}$$

In particular, $(\phi'(t_0)\times\phi''(t_0))/\|\phi'(t_0)\|^3=\nu'(s_0)\times\nu''(s_0)$. But the angle between ν' and ν'' is $\pi/2$ and $\sin(\pi/2)=1$. Therefore, $\|\nu'(s_0)\times\nu''(s_0)\|=\|\nu''(s_0)\|=\kappa(\mathbf{x}_0)$.

e) Use the trivial parameterization $\phi(t)=(t,f(t),0)$ and apply part d). We obtain

$$\kappa(\phi(t))=\frac{\|(1,f'(t),0)\times(0,f''(t),0)\|}{\|(1,f'(t),0)\|^3}=\frac{|f''(t)|}{(1+(f'(t))^2)^{3/2}}.$$

13.2 Oriented Curves.

13.2.1. a) Let $(x,y,z)=\phi(t)$. Then $y^2+9z^2=9\sin^2 t+9\cos^2 t=9$. The trace spirals around the elliptical cylinder $y^2+9z^2=9$.

b) Let $(x,y,z)=\phi(t)$. Then $z=x$ and $y^2=x^3$. The trace looks like a gull in flight (called a cubical parabola) traced in the $z=x$ plane.

c) Let $(x,y,z)=\phi(t)$. Then $y=x^2$ and $z=\sin x$. Thus the trace looks like a sine wave traced on the parabolic cylinder $y=x^2$.

d) Let $(x,y,z)=\phi(t)$. Then $y^2+z^2=\sin^2 t+\cos^2 t=1$ and $x=z$. Thus the trace looks like the ellipse sliced by the plane $x=z$ out of the cylinder $y^2+z^2=1$.

e) Let $(x,y,z)=\phi(t)$. Then $y=x$ and $x=\sin z$. The trace looks like a sine wave traced vertically on the plane $y=x$.

13.2.2. a) Let $\phi(t)=(t,t^2)$ and $I=[1,3]$. Then $F\cdot\phi'=(t^3,t^2-t)\cdot(1,2t)=3t^3-2t^2$. Hence

$$\int_C F\cdot T\,ds=\int_1^3(3t^3-2t^2)\,dt=\frac{128}{3}.$$

b) Let $\phi(t) = (-1, \cos t, \sin t/\sqrt{2})$ and $I = [0, 2\pi]$. Then

$$F \cdot \phi' = (\sqrt{-1 + \cos^3 t + 5}, \sin t/\sqrt{2}, 1) \cdot (0, -\sin t, \cos t/\sqrt{2}) = (\cos t - \sin^2 t)/\sqrt{2}.$$

Hence

$$\int_C F \cdot T\, ds = \frac{1}{\sqrt{2}} \int_0^{2\pi} (\cos t - \sin^2 t)\, dt = \frac{-\pi\sqrt{2}}{2}.$$

c) This curve has two pieces: C_1 where $x \geq 0$, and C_2 where $x \leq 0$. To parameterize C_1, set $\phi(t) = (\cos t, \cos t, \sin t/\sqrt{3})$ and $I = [-\pi/2, \pi/2]$. (Notice when viewed from far out the positive y axis that this parameterization is oriented clockwise because the x axis lies on the *left* side of the yz plane.) Then

$$F \cdot \phi' = (\sin t/\sqrt{3}, -\sin t/\sqrt{3}, 2\cos t) \cdot (-\sin t, -\sin t, \cos t/\sqrt{3}) = 2\cos^2 t/\sqrt{3}.$$

Therefore,

$$\int_{C_1} F \cdot T\, ds = \frac{2}{\sqrt{3}} \int_{-\pi/2}^{\pi/2} \cos^2 t\, dt = \frac{\pi}{\sqrt{3}}.$$

To parameterize C_2, set $\psi(t) = (\cos t, -\cos t, \sin t/\sqrt{3})$ and $J = [\pi/2, 3\pi/2]$. Then

$$F \cdot \psi' = (\sin t/\sqrt{3}, -\sin t/\sqrt{3}, 0) \cdot (-\sin t, \sin t, \cos t/\sqrt{3}) = -2\sin^2 t/\sqrt{3}.$$

Therefore,

$$\int_{C_2} F \cdot T\, ds = \frac{-2}{\sqrt{3}} \int_{\pi/2}^{3\pi/2} \sin^2 t\, dt = \frac{-\pi}{\sqrt{3}}.$$

In particular, $\int_C F \cdot T\, ds = \pi/\sqrt{3} - \pi/\sqrt{3} = 0$.

13.2.3. a) Let C_1 represent the horizontal piece and C_2 represent the vertical piece. On C_1, $y = 1$ hence $dy = 0$, and

$$\int_{C_1} y\, dx + x\, dy = \int_0^1 1\, dx = 1.$$

On C_2, $x = 2$ hence $dx = 0$, and

$$\int_{C_2} y\, dx + x\, dy = \int_1^3 2\, dy = 4.$$

Therefore, the integral over both pieces is $4 + 1 = 5$.

b) If these surfaces intersect, then $z = 1 - z^2$, i.e., $z = (-1 \pm \sqrt{5})/2$. Since $z \geq 0$, the solution with both minus signs is extraneous. Hence these surfaces intersect to form a circle in the plane $z = z_0 := (-1 + \sqrt{5})/2$ of radius $\sqrt{z_0}$. We can parameterize this intersection by $\phi(t) = (\sqrt{z_0}\cos t, \sqrt{z_0}\sin t, z_0)$ and $I = [0, 2\pi]$. Since $\phi'(t) = (-\sqrt{z_0}\sin t, \sqrt{z_0}\cos t, 0)$, we have

$$\omega = dx + (x + y)\, dy + (x^2 + xy + y^2)\, dz = (-\sqrt{z_0}\sin t + (\sqrt{z_0}\cos t + \sqrt{z_0}\sin t)\sqrt{z_0}\cos t)\, dt.$$

Therefore,

$$\int_C \omega = \int_0^{2\pi} (-\sqrt{z_0}\sin t + z_0\cos^2 t + z_0 \sin t \cos t)\, dt = \pi z_0 = \frac{\pi(-1 + \sqrt{5})}{2}.$$

c) Let C_1 represent the piece in $y = c$, C_2 represent the piece in $x = b$, C_3 represent the piece in $y = d$, and C_4 represent the piece in $x = a$. Then

$$\int_{C_1} xy\, dx + (x + y)\, dy = \int_a^b cx\, dx = \frac{c(b^2 - a^2)}{2},$$

$$\int_{C_2} xy\, dx + (x + y)\, dy = \int_c^d (b + y)\, dy = b(d - c) + \frac{d^2 - c^2}{2},$$

$$\int_{C_3} xy\, dx + (x + y)\, dy = \int_b^a d \cdot x\, dx = \frac{-d(b^2 - a^2)}{2},$$

and

$$\int_{C_4} xy\,dx + (x+y)\,dy = \int_d^c (a+y)\,dy = -a(d-c) - \frac{(d^2-c^2)}{2}.$$

In particular,

$$\int_C \omega = (b-a)(d-c) - \frac{(d-c)(b^2-a^2)}{2} = \left(\frac{2-a-b}{2}\right)|R|.$$

d) Let $x=t^2$, $y=t^2$, $z=t$, and let t run from 1 to 0. Then $dx=dy=2t\,dt$, $dz=dt$, and

$$\int_C \sqrt{x}\,dx + \cos y\,dy - dz = \int_1^0 (2t^2 + 2t\cos(t^2) - 1)\,dt = \frac{1}{3} - \sin(1).$$

13.2.4. a) Since $\tau'(u) = \delta > 0$, (ψ, J) and (ϕ, I) are orientation equivalent by Definition 13.18.

b) Let $I=[a,b]$ and set $\tau(u) = (u-a)/(b-a)$. Then $\tau'(u) = 1/(b-a) > 0$ and $\tau(I) = [0,1]$. Observe that $t = \tau(u)$ implies $u = a + (b-a)t$ and set $\psi(t) = \phi(a+(b-a)t)$ for $t \in [0,1]$. Then $\phi = \psi \circ \tau$, hence $(\psi, [0,1])$ and (ϕ, I) are orientation equivalent.

c) If $C = \{(\phi_j, I_j) : j = 1, \dots, N\}$, then by repeating the proof of part b), we can choose smooth curves $\{(\psi_j, [(j-1)/N, j/N]) : j = 1, \dots, N\}$ such that $(\psi_j, [(j-1)/N, j/N])$ and (ϕ_j, I_j) are orientation equivalent with transition τ_j. Define ψ (respectively τ) on $[0,1]$ by $\psi(t) = \psi_j(t)$ (respectively $\tau(t) = \tau_j(t)$) when $t \in ((j-1)/N, j/N)$, and $\psi(t) = 0$ (respectively $\tau(t) = 0$) otherwise. Then on $(0,1) \setminus \{j/N : j = 1, \dots, N\}$, ψ and τ are $\mathcal{C}^1$ and $\tau' > 0$. Moreover, $\psi = \phi_j \circ \tau$ on $((j-1)/N, j/N)$.

13.2.5. The easy way is to apply Theorem 12.65 directly.

If you want a proof which avoids this "enrichment" result, notice by hypothesis, there exist closed, nonoverlapping intervals $J_1, \dots, J_N$ such that $\tau' > 0$ on each J_k^0 and $J = \cup_{k=1}^N J_k$. This means that τ is increasing on each J_k, so by the one-dimensional change of variables formula (Theorem 5.34), we have

$$\begin{aligned}
\int_J F(\psi(u)) \cdot \psi'(u)\,du &= \sum_{k=1}^N \int_{J_k} F \circ \phi \circ \tau(u) \cdot \phi' \circ \tau(u)\ |\tau'(u)|\,du \\
&= \sum_{k=1}^N \int_{\tau(J_k)} F \circ \phi(t) \cdot \phi'(t)\,dt \\
&= \int_{\cup_{k=1}^N \tau(J_k)} F(\phi(t)) \cdot \phi'(t)\,dt = \int_I F(\phi(t)) \cdot \phi'(t)\,dt.
\end{aligned}$$

13.2.6. Since f is continuously differentiable and nonzero on $[a,b]$, we have by the Intermediate Value Theorem that either $f' > 0$ on $[a,b]$ or $f' < 0$ on $[a,b]$.

Suppose first that $f' > 0$ on $[a,b]$. Then $(f^{-1})'(u) = 1/f'(t) > 0$ for $t = f^{-1}(u)$, i.e., $\tau(u) := f^{-1}(u)$ is an orientation equivalent change of variables. Let $\phi(t) = (t, f(t))$ and $\psi(u) = (f^{-1}(u), u)$. Then ϕ is the trivial parameterization of $y = f(x)$, ψ is the trivial parameterization of $x = f^{-1}(y)$, and

$$\phi \circ \tau(u) = \phi(f^{-1}(u)) = (f^{-1}(u), f(f^{-1}(u))) = (f^{-1}(u), u) = \psi(u).$$

Hence $(\phi, [a,b])$ and $(\psi, [f(a), f(b)])$ are orientation equivalent.

If $f' < 0$ on $[a,b]$, then set $\psi(u) = (f^{-1}(-u), -u)$ for $u \in [-f(a), -f(b)]$ and $\tau(u) = f^{-1}(-u)$. Then $\tau'(u) > 0$ and

$$\phi \circ \tau(u) = \phi(f^{-1}(u)) = (f^{-1}(-u), -u) = \psi(u).$$

Hence, $(\phi, [a,b])$ and $(\psi, [-f(a), -f(b)])$ are orientation equivalent. In particular, the explicit curve $y = f(x)$, as x runs from a to b, is orientation equivalent to the explicit curve $x = f^{-1}(y)$, as y runs from $f(a)$ to $f(b)$.

13.2.7. a) Suppose $C(x) := L((x_1, y); (x, y)) \subset V$. Then

$$\int_{C(x)} F \cdot T\,ds = \int_{C(x)} P\,dx + Q\,dy = \int_{x_1}^x P(u,y)\,du + 0.$$

Therefore, $(\partial/\partial x)\int_{C(x)} F\cdot T\,ds = P(x,y)$. Similarly, $(\partial/\partial y)\int_{C(y)} F\cdot T\,ds = Q(x,y)$ for any vertical line segment $C(y)\subset V$ terminating at (x,y).

b) If $C(x,y)$ and $D(x,y)$ are piecewise smooth curves from (x_0,y_0) to (x,y) and let C represent the curve $C(x,y)$ followed by $-D(x,y)$, i.e., $D(x,y)$ in reverse orientation. Then C is piecewise smooth and closed and it follows from hypothesis that

$$0=\int_C F\cdot T\,ds=\int_{C(x,y)} F\cdot T\,ds-\int_{D(x,y)} F\cdot T\,ds,\ \text{ i.e., }\int_{C(x,y)} F\cdot T\,ds=\int_{D(x,y)} F\cdot T\,ds.$$

The converse is proved similarly.

c) Suppose F is conservative, i.e., $F=(f_x,f_y)$ for some $\mathcal{C}^1$ function f on V. Let $C=(\phi,[a,b])$ be a closed smooth curve and $\phi=(\psi,\sigma)$. Then by definition and the Chain Rule,

$$\int_C F\cdot T\,ds=\int_a^b (f_x(\phi(t))\psi'(t)+f_y(\phi(t))\sigma'(t))\,dt=\int_a^b (f\circ\phi)'(t)\,dt=f\circ\phi(b)-f\circ\phi(a).$$

Since C is closed, this last difference is zero. Thus the integral over C is zero. If $C=\{\phi_j,[a_j,b_j]\}$ is piecewise smooth, then the integral over C breaks into a finite sum of smooth pieces. Telescoping, we obtain

$$\int_C F\cdot T\,ds=\sum_{j=1}^N f\circ\phi_j(b_j)-f\circ\phi_j(a_j)=f\circ\phi_N(b_N)-f\circ\phi_1(a_1)=0$$

since C is closed.

Conversely, by part b) the function $f(x,y):=\int_{C(x,y)} F\cdot T\,ds$ is well-defined. If $C(x,y)$ ends in a horizontal line segment, then by part a), $f_x=P$. If $C(x,y)$ ends in a vertical line segment, then by part a), $f_y=Q$. Thus $F=(P,Q)=(f_x,f_y)$ is conservative by definition.

d) By part c), $F=(f_x,f_y)$ for some f defined on V. Since F is $\mathcal{C}^1$, it follows that f is $\mathcal{C}^2$ on V. In particular, $P_y=f_{xy}=f_{yx}=Q_x$ by Theorem 11.2.

13.2.8. Clearly, $a=1$, $b=f(1)-f(0)$, and $c=\sqrt{1+(f(1)-f(0))^2}$. Since $\phi(x):=\sqrt{1+x^2}$ is convex on $[0,1]$, it follows from the Fundamental Theorem of Calculus, Jensen's Inequality, Definition 13.6, and the trivial inequality $\sqrt{1+A^2}\le 1+|A|$ that

$$\begin{aligned}c&=\phi\left(\int_0^1 f'(x)\,dx\right)\le\int_0^1 \phi(f'(x))\,dx\\&=L\le\int_0^1(1+|f'(x)|\,dx)=1+\int_0^1 f'(x)\,dx=a+b.\end{aligned}$$

13.3 Surfaces.

13.3.1. a) If (ϕ,E) is the parameterization given in Example 13.33, then

$$\|\phi_u\times\phi_v\|=\|(v\cos u, v\sin u,-v)\|=\sqrt{2}v.$$

Thus

$$A(S)=\int_a^b\int_0^{2\pi}\sqrt{2}v\,du\,dv=\sqrt{2}\pi(b^2-a^2).$$

b) If (ϕ,E) is the parameterization given in Example 13.31, then

$$\|\phi_u\times\phi_v\|=\|(a^2\cos u\cos^2 v, a^2\sin u\cos^2 v, a^2\sin v\cos v)\|=a^2|\cos v|.$$

Thus

$$A(S)=\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} a^2|\cos v|\,dv\,du=4\pi a^2.$$

c) If (ϕ,E) is the parameterization given in Example 13.31, then

$$\begin{aligned}\|\phi_u\times\phi_v\|&=\|(b(a+b\cos v)\cos u\cos v, b(a+b\cos v)\sin u\cos v, b(a+b\cos v)\sin v)\|\\&=b|a+b\cos v|.\end{aligned}$$

Since $a > b$, it follows that

$$A(S) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} b(a + b\cos v)\,du\,dv = 4\pi^2 ab.$$

13.3.2. a) The trivial parameterization is (ϕ, E), where $\phi(u,v) = (u, v, u^2 - v^2)$ and $E = \{(u,v) : -1 \le u \le 1,\ -|u| \le v \le |u|\}$. The boundary is $y = \pm x$, $z = 0$, and $z = 1 - y^2$, $x = \pm 1$. Since

$$\|N_\phi\| = \|(-2u, 2v, 1)\| = \sqrt{1 + 4u^2 + 4v^2},$$

we have

$$\iint_S g\,d\sigma = 4\int_0^1 \int_0^u (1 + 4u^2 + 4v^2)\,dv\,du = \frac{22}{3}.$$

b) Let $\phi(u,v) = (u, u^3, v)$ and $E = [0,2] \times [0,4]$. The boundary is $y = x^3$, $z = 0, 4$, and $(0,0,z)$, $(2,8,z)$, for $0 \le z \le 4$. Since $\|N_\phi\| = \|(3u^2, -1, 0)\| = \sqrt{1 + 9u^4}$, we have

$$\iint_S g\,d\sigma = \int_0^4 \int_0^2 u^3 v\sqrt{1 + 9u^4}\,dv\,du = 8\int_0^2 u^3\sqrt{1 + 9u^4}\,du = \frac{4}{27}(145^{3/2} - 1).$$

c) Because these surfaces intersect at $z = 3/\sqrt{2}$, a parameterization of this "spanish olive half" is given by (ψ, B) where

$$\psi(u,v) = (3\cos u\cos v, 3\sin u\cos v, 3\sin v)$$

and $B = [0, 2\pi] \times [\pi/4, \pi/2]$. The boundary is $x^2 + y^2 = 9$, $z = 0$, and $2x^2 + 2y^2 = 9$, $z = 3/\sqrt{2}$. As in Exercise 13.3.1b, $\|N_\psi\| = 9|\cos v|$, so

$$\begin{aligned}\iint_S g\,d\sigma &= 9\int_{\pi/4}^{\pi/2}\int_0^{2\pi} (3\cos u\cos v + 3\sin u\cos v + 3\sin v)\cos v\,dv\,du \\ &= 54\pi\int_{\pi/4}^{\pi/2} \sin v\cos v\,dv = \frac{27\pi}{2}.\end{aligned}$$

13.3.3. Parameterize this ellipsoid using $\phi(u,v) = (a\cos u\cos v, b\sin u\cos v, c\sin v)$ and $E = [0,2\pi]\times[-\pi/2, \pi/2]$. Since

$$\|N_\phi\| = |\cos v|\sqrt{a^2b^2\sin^2 v + a^2c^2\sin^2 u\cos^2 v + b^2c^2\cos^2 u\cos^2 v}$$

is nonzero when $v \ne 0$ and $v \ne 2\pi$, this gives a smooth $\mathcal{C}^\infty$ parameterization of the ellipse except at the north and south poles, i.e., the points $(0,0,c)$ and $(0,0,-c)$. A smooth $\mathcal{C}^\infty$ parameterization at these poles can be given using the trivial parameterization, i.e., $\psi(u,v) = (u, v, \pm c\sqrt{1 - u^2/a^2 - v^2/b^2})$ and $B = \{(u,v) : u^2/a^2 + v^2/b^2 \le 1/2\}$. Thus the ellipse is a piecewise smooth $\mathcal{C}^\infty$ surface.

13.3.4. a) Parameterize S by (ϕ, E), where $\phi(u,v) = (u, v, 0)$. Since $\|N_\phi\| = \|(0,0,1)\| = 1$, we have

$$\iint_S g\,d\sigma = \int_E 1g(u,v,0)\,d(u,v).$$

Thus $\int_S d\sigma = \text{Area}\,(S)$ by Theorem 12.22.

b) Parameterize the surface S using $\phi(u,v) = (u, v, f(u))$, $E = [a,b] \times [c,d]$. Since

$$\|N_\phi\| = \|(f'(u), 0, 1)\| = \sqrt{1 + (f'(u))^2},$$

we have

$$A(S) = \int_c^d \int_a^b \sqrt{1 + (f'(u))^2}\,du\,dv = (d - c)L(C)$$

by Definition 13.6.

c) Parameterize the surface by $\phi(u,v) = (u, f(u)\cos v, f(u)\sin v)$, $E = [a,b] \times [0, 2\pi]$. Since

$$\|N_\phi\| = \|(f(u)f'(u), -f(u)\cos v, -f(u)\sin v,)\| = |f(u)|\sqrt{1 + (f'(u))^2},$$

we have

$$A(S) = \int_0^{2\pi} \int_a^b |f(u)|\sqrt{1+(f'(u))^2}\,du\,dv = 2\pi \int_a^b |f(x)|\sqrt{1+(f'(x))^2}\,dx.$$

13.3.5. By Theorem 13.36, $N_\psi = \Delta_\tau N_\phi \circ \tau$. Hence by Theorem 12.65,

$$\begin{aligned}\iint_B g(\psi(s,t)\|N_\psi(s,t)\| &= \iint_B g(\phi(\tau(s,t)))|\Delta_\tau(s,t)|\,\|N_\phi(\tau(s,t))\| \\ &= \iint_E g(\phi(u,v))\|N_\phi(u,v)\|.\end{aligned}$$

13.3.6. Parameterize S by $\phi(x,y) = (x,y,(x^2+y^2)/2)$ and $E = B_{\sqrt{8}}(0,0)$. Notice that $\|N_\phi\| = \|(-x,-y,1)\| = \sqrt{1+x^2+y^2}$. By the Mean Value Theorem and hypothesis, given $(x,y) \in B_3(0,0)$,

$$|f(x,y) - f(0,0)| \le |\nabla f(c,d)\cdot(x,y)| \le \|(x,y)\|$$

for some $(c,d) \in L((x,y);(0,0)) \subset B_3(0,0)$. Since $\|(x,y)\| \le \sqrt{1+x^2+y^2} = \|N_\phi\|$, it follows that

$$\iint_S |f(x,y) - f(0,0)|\,d\sigma \le \int_E (1+x^2+y^2)\,d(x,y) = \int_0^{2\pi} \int_0^{\sqrt{8}} (1+r^2)r\,dr\,d\theta = 40\pi.$$

13.3.7. Suppose that (ϕ, E) is a $\mathcal{C}^p$ parameterization of S which satisfies $(x_0,y_0,z_0) = \phi(u_0,v_0)$ and $N_\phi(u_0,v_0) \ne 0$. Then one the components of N_ϕ is nonzero, say $\partial(\phi_1,\phi_2)/\partial(u,v) \ne 0$. Consider the function $F(u,v,x,y) = (\phi_1(u,v)-x, \phi_2(u,v)-y)$. Since $F(u_0,v_0) = (0,0)$ and

$$\frac{\partial(F_1,F_2)}{\partial(u,v)} = \frac{\partial(\phi_1,\phi_2)}{\partial(u,v)} \ne 0,$$

it follows from the Implicit Function Theorem that there is an open set V containing (x_0,y_0) and a continuously differentiable function $g : V \to \mathbf{R}^2$ such that

$$\phi_1(g(x,y)) = x \quad \text{and} \quad \phi_2(g(x,y)) = y.$$

Set $f(x,y) = \phi_3(g(x,y))$. Then f is $\mathcal{C}^p$ and $\phi \circ g(V)$ coincides with the graph of $z = f(x,y)$, $(x,y) \in V$. Since $z = f(x,y)$ has a tangent plane at (x_0,y_0,z_0) by Theorem 11.22, it follows that S has a tangent plane at (x_0,y_0,z_0).

13.3.8. Let $(x,y,z) = \psi(u,v)$. Then

$$\|\psi_u\|^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \quad \|\psi_v\|^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$

and

$$\psi_u \cdot \psi_v = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}.$$

Therefore,

$$\begin{aligned}E^2G^2 - F^2 = {} & \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial y}{\partial v}\right)^2 - 2\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right)\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) + \left(\frac{\partial x}{\partial v}\right)^2\left(\frac{\partial y}{\partial u}\right)^2 \\ & + \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial z}{\partial v}\right)^2 - 2\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right) + \left(\frac{\partial x}{\partial v}\right)^2\left(\frac{\partial z}{\partial u}\right)^2 \\ & + \left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial z}{\partial v}\right)^2 - 2\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\right)\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + \left(\frac{\partial y}{\partial v}\right)^2\left(\frac{\partial z}{\partial u}\right)^2 \\ = {} & \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right)^2 \\ & + \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 = \|N_\psi\|^2.\end{aligned}$$

13.3.9. Let $(x, y, z) = \phi(u, v)$ and $(u, v) = \psi(t)$. Notice for any $a, b, c \in \mathbf{R}$ that

$$a\frac{\partial(y,z)}{\partial(u,v)} + b\frac{\partial(z,x)}{\partial(u,v)} + c\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} a & b & c \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix}$$

and by the Chain Rule that

$$(\phi \circ \psi)' = (x_u u_t + x_v v_t, y_u u_t + y_v v_t, z_u u_t + z_v v_t).$$

It follows that

$$\begin{aligned}(\phi \circ \psi)' \cdot (\phi_u \times \phi_v) &= (x_u u_t + x_v v_t)\frac{\partial(y,z)}{\partial(u,v)} + (y_u u_t + y_v v_t)\frac{\partial(z,x)}{\partial(u,v)} \\ &\quad + (z_u u_t + z_v v_t)\frac{\partial(x,y)}{\partial(u,v)} \\ &= u_t \det\begin{bmatrix} x_u & y_u & z_u \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} + v_t \det\begin{bmatrix} x_v & y_v & z_v \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} = 0.\end{aligned}$$

13.4 Oriented Surfaces.

13.4.1. a) The boundary is $9 = x^2 + z^2$, $y = 0$, with counterclockwise orientation when viewed from far out the positive y axis. Since the x axis lies to the left of the yz plane, we can parameterize this curve by $\phi(t) = (3\sin t, 0, 3\cos t)$, $I = [0, 2\pi]$. Thus

$$\int_{\partial S} F \cdot T\, ds = \int_0^{2\pi} (3\sin t + 3\cos t)(-3\sin t)\, dt = -9\pi.$$

b) This boundary is a triangle (oriented in the clockwise direction when viewed from the origin) consisting of three line segments, C_1 (which lies in the yz plane), C_2 (which lies in the xz plane), C_3 (which lies in the xy plane). Parameterize C_1 by $\phi_1(t) = (0, -t, 1 + 2t)$, $I_1 = [-1/2, 0]$; C_2 by $\phi_2(t) = (t, 0, 1 - t)$, $I_2 = [0, 1]$; and C_3 by $\phi_3(t) = (-t, (1 + t)/2, 0)$, $I_3 = [-1, 0]$. Then,

$$\begin{aligned}\int_{\partial S} F \cdot T\, ds &= \int_{C_1} F \cdot T\, ds + \int_{C_2} F \cdot T\, ds + \int_{C_3} F \cdot T\, ds \\ &= \int_{-1/2}^{0} (t, -t, 0) \cdot (0, -1, 2)\, dt + \int_0^1 (t, -t, t - 2t^2 + t^3) \cdot (1, 0, -1)\, dt \\ &\quad + \int_{-1}^{0} ((-3t - 1)/2, (3t + 1)/2, 0) \cdot (-1, 1/2, 0)\, dt = -\frac{1}{12}.\end{aligned}$$

c) This boundary has two pieces, the circle C_1 described by $x^2 + y^2 = 4$, $z = 4$, oriented in the clockwise direction when viewed from high up the positive z axis, and the circle C_2 described by $x^2 + y^2 = 1$, $z = 1$, oriented in the counterclockwise direction when viewed from high up the positive z axis. Using the parameterizations $\phi_1(t) = (2\sin t, 2\cos t, 4)$, $I_1 = [0, 2\pi]$, and $\phi_2(t) = (\cos t, \sin t, 1)$, $I_2 = [0, 2\pi]$, we have

$$\begin{aligned}\int_{\partial S} F \cdot T\, ds &= \int_{C_1} F \cdot T\, ds + \int_{C_2} F \cdot T\, ds \\ &= \int_0^{2\pi} (10\cos t + \cos 4, 8\sin t - \sin 4, 6\cos 4\sin t + 8\sin 4\cos t) \cdot (2\cos t, -2\sin t, 0)\, dt \\ &\quad + \int_0^{2\pi} (5\sin t + \cos 1, 4\cos t - \sin 1, 3\cos 1\cos t + 2\sin 1\sin t) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_0^{2\pi} (24\cos^2 t - 21\sin^2 t + (2\cos 4 - \sin 1)\cos t + (2\sin 4 - \cos 1)\sin t)\, dt \\ &= 3\pi.\end{aligned}$$

13.4.2. a) Use the trivial parameterization $\phi(u,v) = (u, v, u^2 + v^2)$, $E = B_1(0,0)$. Then $N_\phi = (-2u, -2v, 1)$ points upward, i.e., the wrong way. Thus

$$\iint_S F \cdot \mathbf{n}\, d\sigma = -\int_{B_1(0,0)} (u, v, u^2 + v^2) \cdot (-2u, -2v, 1)\, d(u,v) = \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{\pi}{2}.$$

b) Let $\phi(u,v) = (u, 2\cos v, 2\sin v)$, $E = [0,1] \times [0,\pi]$. Then $N_\phi = (0, -2\cos v, -2\sin v)$ points inward, the wrong way. Hence,

$$\begin{aligned}\iint_S F \cdot \mathbf{n}\, d\sigma &= -\int_0^{\pi} \int_0^1 (u^2 + 4\cos^2 v, 4\sin v \cos v, 4\sin^2 v) \cdot (0, -2\cos v, -2\sin v)\, du\, dv \\ &= \int_0^{\pi} 8 \sin v\, dv = 16.\end{aligned}$$

c) If (ϕ, E) is the parameterization given in Example 13.32, then

$$N_\phi = (b(a + b\cos v)\cos u \cos v, b(a + b\cos v)\sin u \cos v, b(a + b\cos v)\sin v),$$

so

$$\begin{aligned}F \cdot N_\phi &= ((a + b\cos v)\sin u, -(a + b\cos v)\cos u, b\sin v)\cdot \\ &\quad \cdot (b(a + b\cos v)\cos u \cos v, b(a + b\cos v)\sin u \cos v, b(a + b\cos v)\sin v) \\ &= b(a + b\cos v)^2 \sin u \cos u \cos v - b(a + b\cos v)^2 \sin u \cos u \cos v \\ &\quad + b^2(a + b\cos v)\sin^2 v \\ &= ab^2 \sin^2 v + b^3 \sin^2 v \cos v.\end{aligned}$$

Hence

$$\iint_S F \cdot \mathbf{n}\, d\sigma = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (ab^2 \sin^2 v + b^3 \sin^2 v \cos v)\, dv\, du = 2\pi^2 ab^2.$$

d) Use the trivial parameterization $\phi(u,v) = (u, v, u^2)$, $E = B_1(0,0)$. Then $N_\phi = (-2u, 0, 1)$ points upward and

$$\begin{aligned}\iint_S F \cdot \mathbf{n}\, d\sigma &= \int_{B_1(0,0)} (u^4 - 2u^3 v^2) d(u,v) \\ &= \int_0^{2\pi} \int_0^1 (r^4 \cos^4 \theta - 2r^5 \cos^3 \theta \sin^2 \theta) r\, dr\, d\theta \\ &= \frac{1}{6} \int_0^{2\pi} \cos^4 \theta\, d\theta - \frac{2}{7} \int_0^{2\pi} \cos^3 \theta \sin^2 \theta\, d\theta = \frac{\pi}{8}.\end{aligned}$$

13.4.3. a) Using the trivial parameterization $z = x^4 + y^2$, we see that $N_\phi = (4x^3, -2y, 1)$ points upward. Thus

$$\iint_S \omega = \int_0^1 \int_0^1 (x, y, x^4 + y^2) \cdot (4x^3, -2y, 1)\, dx\, dy = -\frac{14}{15}.$$

b) By the calculation which follows Definition 13.28,

$$N_\phi = (a^2 \cos u \cos^2 v, a^2 \sin u \cos^2 v, a^2 \sin v \cos v)$$

points outward. Thus

$$\begin{aligned}\iint_S \omega &= \int_0^{2\pi} \int_0^{\pi/2} (a\cos u \cos v, a \sin u \cos v, 0)\cdot \\ &\quad \cdot (a^2 \cos u \cos^2 v, a^2 \sin u \cos^2 v, a^2 \sin v \cos v)\, dv\, du \\ &= \int_0^{2\pi} \int_0^{\pi/2} a^3 \cos^3 v\, dv\, du \\ &= 2\pi a^3 \int_0^{\pi/2} \cos v (1 - \sin^2 v)\, dv = \frac{4\pi a^3}{3}.\end{aligned}$$

c) Using the trivial parameterization $z = \sqrt{a^2 - x^2 - y^2}$, we see that $N_\phi = (x/z, y/z, 1)$ points upward. Thus

$$\begin{aligned}\iint_S \omega &= \int_{B_b(0,0)} (xz, 1, z) \cdot (x/z, y/z, 1)\, d(x,y) \\ &= \int_0^{2\pi} \int_0^b (r^2 \cos^2\theta + \frac{r\sin\theta}{\sqrt{a^2-r^2}} + \sqrt{a^2-r^2}) r\, dr\, d\theta \\ &= \pi \int_0^b r^3\, dr + 2\pi \int_0^b \sqrt{a^2-r^2}\, r\, dr = (3b^4 + 8a^3 - 8(a^2-b^2)^{3/2})\frac{\pi}{12}.\end{aligned}$$

d) If $\phi(u,v) = ((v/2)\cos u, (v/2)\sin u, v)$ then $N_\phi = ((v/2)\cos u, (v/2)\sin u, -v/4)$ points away from the z axis. Thus

$$\begin{aligned}\iint_S \omega &= \int_0^{2\pi} \int_0^2 ((v/2)\cos u, (v/2)\sin u, v^2) \cdot ((v/2)\cos u, (v/2)\sin u, -v/4)\, dv\, du \\ &= \frac{\pi}{2} \int_0^2 (v^2 - v^3)\, dv = -\frac{2\pi}{3}.\end{aligned}$$

13.4.4. By Theorem 13.36, $N_\psi = \Delta_\tau N_\phi \circ \tau$. Since $|\Delta_\tau| = \Delta_\tau$, it follows from Theorem 12.65 that

$$\begin{aligned}\int_B F(\psi(s,t) \cdot N_\psi(s,t)\, d(s,t) &= \int_B |\Delta_\tau(s,t)| F(\phi(\tau(s,t))) \cdot N_\phi(\tau(s,t))\, d(s,t) \\ &= \int_E F(\phi(u,v)) \cdot N_\phi(u,v)\, d(u,v).\end{aligned}$$

13.4.5. By definition,

$$\begin{aligned}\iiint_E P_x\, dV &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} P_x\, dx\, dy\, dz \\ &= \int_0^1 \int_0^{1-z} (P(1-y-z, y, z) - P(0,y,z))\, dy\, dz.\end{aligned}$$

Similarly,

$$\iiint_E Q_y\, dV = \int_0^1 \int_0^{1-z} (Q(x, 1-x-z, z) - Q(x,0,z))\, dx\, dz$$

and

$$\iiint_E R_z\, dV = \int_0^1 \int_0^{1-y} (R(x,y,1-x-y) - R(x,y,0))\, dx\, dy.$$

Let $\omega = P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy$. The tetrahedron ∂E has four faces, S_1 in $z=0$, S_2 in $x=0$, S_3 in $y=0$, and the slanted face S_4. By definition,

$$\iint_{S_1} \omega = -\int_0^1 \int_0^{1-y} R(x,y,0)\, dx\, dy,$$

$$\iint_{S_2} \omega = -\int_0^1 \int_0^{1-z} P(0,y,z)\, dy\, dz, \quad \text{and} \quad \iint_{S_3} \omega = -\int_0^1 \int_0^{1-z} Q(x,0,z)\, dx\, dz.$$

On the other hand, using trivial parameterizations, we have

$$\begin{aligned}\iint_{S_4} \omega = \int_0^1 \int_0^{1-u} P(1-u-v, u, v)\, dv\, du &+ \int_0^1 \int_0^{1-u} Q(u, 1-u-v, v)\, dv\, du \\ &+ \int_0^1 \int_0^{1-v} R(u, v, 1-u-v)\, du\, dv.\end{aligned}$$

Therefore,

$$\iint_{\partial E} \omega = \iiint_E (P_x + Q_y + R_z)\, dV.$$

13.4.6. The tetrahedron T has three faces, T_1 in $x=0$, T_2 in $y=0$, and T_3 in $z=0$. To evaluate the integral over T_1, let $\phi(y,z) = (0,y,z)$ and E be the triangle with vertices $(0,0,0)$, $(0,1,0)$, and $(0,0,1)$. Since $N_\phi = (1,0,0)$ points toward the positive x axis, we have

$$\begin{aligned}
\iint_{T_1} \omega &= -\iint_E (R_y - Q_z)\, dA \\
&= -\int_0^1 \int_0^{1-z} R_y\, dy\, dz + \int_0^1 \int_0^{1-y} Q_z\, dz\, dy \\
&= -\int_0^1 (R(0,1-z,z) - R(0,0,z))\, dz + \int_0^1 (Q(0,y,1-y) - Q(0,y,0))\, dy.
\end{aligned}$$

Similarly,

$$\iint_{T_2} \omega = -\int_0^1 (P(x,0,1-x) - P(x,0,0))\, dx + \int_0^1 (R(1-z,0,z) - R(0,0,z))\, dz$$

and

$$\iint_{T_3} \omega = -\int_0^1 (Q(1-y,y,0) - Q(0,y,0))\, dy + \int_0^1 (P(x,1-x,0) - P(x,0,0))\, dx.$$

Therefore,

$$\begin{aligned}
\iint_T \omega = &-\int_0^1 R(0,1-z,z)\, dz + \int_0^1 Q(0,y,1-y)\, dy - \int_0^1 P(x,0,1-x)\, dx \\
&+ \int_0^1 R(1-z,0,z)\, dz - \int_0^1 Q(1-y,y,0)\, dy + \int_0^1 P(x,1-x,0)\, dx.
\end{aligned}$$

On the other hand, ∂T has three pieces: C_1 which runs from $(1,0,0)$ to $(0,0,1)$, C_2 which runs from $(0,0,1)$ to $(0,1,0)$, and C_3 which runs from $(0,1,0)$ to $(1,0,0)$. To parameterize C_1, let $\phi(t) = (1-t,0,t)$ and $I = [0,1]$. Then

$$\begin{aligned}
\int_{C_1} P\, dx + Q\, dy + R\, dz &= \int_0^1 (P,Q,R) \cdot (-1,0,1)\, dt \\
&= -\int_0^1 P(1-t,0,t)\, dt + \int_0^1 R(1-t,0,t)\, dt \\
&= -\int_0^1 P(x,0,1-x)\, dx + \int_0^1 R(1-z,0,z)\, dz.
\end{aligned}$$

Similarly,

$$\int_{C_2} P\, dx + Q\, dy + R\, dz = -\int_0^1 R(0,1-z,z)\, dz + \int_0^1 Q(0,y,1-y)\, dy$$

and

$$\int_{C_2} P\, dx + Q\, dy + R\, dz = -\int_0^1 Q(1-y,y,0)\, dy + \int_0^1 P(x,1-x,0)\, dx.$$

Therefore,

$$\int_{\partial T} P\, dx + Q\, dy + R\, dz = \iint_T (R_y - Q_z)\, dy\, dz + (P_z - R_x)\, dz\, dx + (Q_x - P_y)\, dx\, dy.$$

13.4.7. a) By definition, given $\mathbf{x} \in S$ there is a parametrization (ϕ_x, E_x) which is smooth at $\mathbf{x}$, i.e., such that $N_{\phi_x}(u_0,v_0) \neq 0$ for $\mathbf{x} = \phi(u_0,v_0)$. Since N_{ϕ_x} is continuous, it follows from the sign preserving property that there is an $r(\mathbf{x}) > 0$ such that N_{ϕ_x} is nonzero on a relative closed ball $E_x := E \cap \overline{B_{r(\mathbf{x})}(\mathbf{x})}$. Hence (ϕ_x, E_x) is a smooth parametrization of the surface $\phi_x(E_x)$. Moreover, by the Borel Covering Lemma, there exist relative closed balls E_{x_j} such that $E = \cup E_{x_j}$. In particular, the smooth parametrizations (ϕ_{x_j}, E_{x_j}) satisfy $S = \cup_{j=1}^N \phi_{x_j}(E_{x_j})$.

b) All we need do is make the relative closed balls E_j in part a) nonoverlapping. Let E_j be ordered from largest radius to the smallest. Set $S_1 = \phi(E_1)$ and $S_2 = \phi(E_2 \setminus E_1)$. In general, set $S_j = \phi(E_j \setminus (\cup_{k=1}^{j-1} E_k)$. Then each S_j is nonoverlapping and has a smooth parametrization.

If S is oriented, we can repeat the entire process making sure that not only are the (ϕ_x, E_x) smooth, but also "orientable."

13.5 Theorems of Green and Gauss.

13.5.1. a) Let E be the portion of the disc $B_2(0,0)$ which lies in the first quadrant. By Green's Theorem,

$$\int_C F \cdot T\, ds = \iint_E (y-0)\, dA = \int_0^{\pi/2} \int_0^2 r^2 \sin\theta\, dr\, d\theta = \frac{8}{3}.$$

b) By Green's Theorem,

$$\int_C F \cdot T\, ds = \int_0^2 \int_0^3 \left(\frac{1}{x+1} - e^y\right) dy\, dx = \int_0^2 \left(\frac{3}{x+1} + 1 - e^3\right) dx = 3\log 3 + 2(1-e^3).$$

c) Let $E = B_2(0,0) \setminus B_1(0,0)$. By Green's Theorem,

$$\begin{aligned} \int_C F \cdot T\, ds &= -\iint_E (y^2 - 2yf'(x^2+y^2))\, dA \\ &= \int_1^2 \int_0^{2\pi} (2r^2 f'(r^2)\sin\theta - r^3 \sin^2\theta)\, d\theta\, dr \\ &= 0 - \pi \int_1^2 r^3\, dr = \frac{-15\pi}{4}. \end{aligned}$$

13.5.2. a) By Green's Theorem,

$$\int_C \omega = \int_c^d \int_a^b (y-1)\, dx\, dy = (b-a)(c-d)(c+d-2)/2.$$

b) By Green's Theorem,

$$\int_C \omega = -\int_0^1 \int_{x^2}^x (2x - f(x))\, dy\, dx = \int_0^1 (2x^3 - 2x^2)\, dx - \int_0^1 (x^2 - x) f(x)\, dx = -1/6.$$

c) Since

$$\frac{\partial}{\partial x}(e^x \sin y) - \frac{\partial}{\partial y}(-e^x \cos y) = e^x \cos y - e^x \cos y = 0,$$

it follows from Green's Theorem that $\int_C \omega = 0$ for all such curves C.

13.5.3. a) By Gauss' Theorem,

$$\iint_S F \cdot \mathbf{n}\, d\sigma = \int_0^3 \int_0^2 \int_0^1 (2 + e^z)\, dx\, dy\, dz = 2(5 + e^3).$$

b) Let E be the cylinder whose boundary is S. By Gauss' Theorem,

$$\begin{aligned} \iint_S F \cdot \mathbf{n}\, d\sigma &= 2\iiint_E (x+y+z)\, dV = 2\int_0^{2\pi} \int_0^1 \int_0^1 (r\cos\theta + r\sin\theta + z) r\, dz\, dr\, d\theta \\ &= 0 + 0 + 4\pi \int_0^1 \int_0^1 rz\, dz\, dr = \pi. \end{aligned}$$

c) By Gauss' Theorem,

$$\iint_S F\cdot\mathbf{n}\,d\sigma = \int_{-1}^{1}\int_{x^2}^{2-x^2}\int_0^z 3\,dx\,dy\,dz = \frac{3}{2}\int_{-1}^{1}((2-x^2)^2-(x^2)^2)\,dx = 8.$$

d) Let E be the ellipsoid $\{(x,y,z) : x^2/a^2+y^2/b^2+z^2/c^2\le 1\}$. By Gauss' Theorem,

$$\begin{aligned}\iint_S F\cdot\mathbf{n}\,d\sigma &= \iiint_E (|x|+|y|+|z|)\,dV\\ &= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (a\rho\cos\theta\sin\varphi + b\rho\sin\theta\sin\varphi + c\rho\cos\varphi)abc\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= 2abc\int_0^{\pi/2}\int_0^{\pi/2}(a\cos\theta\sin\varphi + b\sin\theta\sin\varphi + c\cos\varphi)\sin\varphi\,d\varphi\,d\theta\\ &= 2abc\int_0^{\pi/2}\left((a+b)\sin^2\varphi + \frac{c\pi}{2}\cos\varphi\sin\varphi\right)\,d\varphi = \pi abc(a+b+c)/2.\end{aligned}$$

13.5.4. a) By Gauss' Theorem,

$$\iint_S \omega = \int_{-2}^{2}\int_{x^2}^{4}\int_0^1 (yz+2y+1)\,dz\,dy\,dx = 2\int_0^2 (24-x^2-5x^4/4)\,dx = 224/3.$$

b) Let H be the solid hyperboloid whose boundary is S, let A be the upper semidisk $\{(x,z) : z\ge 0, x^2+z^2\le 1\}$, and B be the upper semiannulus $\{(x,z) : z\ge 0, 1\le x^2+z^2\le 2\}$. By Gauss' Theorem,

$$\begin{aligned}\iint_S \omega &= \iiint_H y|z|\,dV = 2\int_A\int_0^1 yz\,dy\,d(x,z) + 2\int_B\int_{\sqrt{x^2+z^2-1}}^{1} yz\,dy\,d(x,z)\\ &= \int_A z\,d(x,z) + \int_B (2-x^2-z^2)z\,d(x,z)\\ &= \int_0^\pi\int_0^1 r^2\sin\theta\,dr\,d\theta + \int_0^\pi\int_1^{\sqrt{2}}(2r^2-r^4)\sin\theta\,dr\,d\theta = 2(8\sqrt{2}-2)/15.\end{aligned}$$

c) Let E represent the three dimensional region whose boundary is S. Since $y=4-x^2-z^2$ and $y=5-4x-2z$ imply $(x-2)^2+(z-1)^2=4$, the projection of E onto the xz plane is the disc D centered at $(2,0,1)$ of radius 2. By Gauss' Theorem,

$$\iint_S \omega = \iiint_E 3\,dV = 3\int_D\int_{5-4x-2z}^{4-x^2-z^2} dy\,d(x,z) = 3\int_D (4x+2z-x^2-z^2-1)\,d(x,z).$$

Using the change of variables $x=2+r\cos\theta$, $z=1+r\sin\theta$, $dx\,dy = r\,dr\,d\theta$, we conclude

$$\iint_S \omega = 3\int_0^{2\pi}\int_0^2 (4-r^2)r\,dr\,d\theta = 24\pi.$$

13.5.5. a) Let $P=-y$ and $Q=x$. Then $Q_x - P_y = 2$ and we have by Green's Theorem that

$$\frac{1}{2}\int_{\partial E} x\,dy - y\,dx = \iint_E dA = \text{Area}\,(E).$$

b) If $x = 3t/(1+t^3)$ and $y=3t^2/(1+t^3)$, then

$$xdy - ydx = \frac{3t}{1+t^3}\frac{3(1-2t^3)\,dt}{(1+t^3)^2} - \frac{3t^2}{1+t^3}\frac{3(2t-t^4)\,dt}{(1+t^3)^2} = \frac{9t^2\,dt}{(1+t^3)^2}.$$

Thus by part a),

$$\text{Area}\,(E) = \frac{1}{2}\int_0^\infty \frac{9t^2\,dt}{(1+t^3)^2} = \frac{3}{2}\int_1^\infty \frac{du}{u^2} = \frac{3}{2}.$$

c) Since $\text{div}\,(x,y,z) = 3$, we have by Gauss' Theorem that

$$\text{Vol}\,(E) = \frac{1}{3}\int_{\partial E} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy.$$

d) If $x = (a + b\cos v)\cos u$, $y = (a + b\cos v)\sin u$, and $z = b\sin v$, then

$$\begin{aligned} &x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \\ &= \big(b(a+b\cos v)^2\cos^2 u\cos v + b(a+b\cos v)^2\sin^2 u\cos v + b^2(a+b\cos v)\sin^2 v\big)\;du\,dv \\ &= \big(b(a+b\cos v)^2\cos v + b^2(a+b\cos v)\sin^2 v\big)\;du\,dv. \end{aligned}$$

Thus by part c),

$$\begin{aligned} \text{Vol}\,(E) &= \frac{1}{3}\int_{\partial E} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy \\ &= \frac{2\pi}{3}\int_{-\pi}^{\pi}\big(b(a+b\cos v)^2\cos v + b^2(a+b\cos v)\sin^2 v\big)\;dv \\ &= \frac{2\pi}{3}\big(0 + 2ab^2\pi + 0 + ab^2\pi + 0\big) = 2\pi^2 ab^2. \end{aligned}$$

13.5.6. a) Parameterize ∂E by $\phi(t) = (\cos t, \sin t)$, $I = [0, 2\pi]$. Then

$$\int_{\partial E} P\,dx + Q\,dy = \int_0^{2\pi}(-\sin^2 t - \cos^2 t)\,dt = -2\pi.$$

On the other hand, $Q_x = P_y = (x^2-y^2)/(x^2+y^2)^2$, so $\iint_E (P_y - Q_x)\,dA = 0$.

b) Let $F = (x(x^2+y^2+z^2)^{-3/2}, y(x^2+y^2+z^2)^{-3/2}, z(x^2+y^2+z^2)^{-3/2})$ and $E = B_1(0,0,0)$. Then $\text{div}\,F = 0$. On the other hand, since $F = (x,y,z)$ on ∂E, we have

$$\iint_{\partial E} F\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}(\cos^2 u\cos^3 v + \sin^2 u\cos^3 v + \sin^2 u\cos v)\,dv\,du = 4\pi \neq 0.$$

13.5.7. By Exercise 12.2.3 and Gauss' Theorem,

$$\begin{aligned} \lim_{r\to 0}\frac{1}{\text{Vol}\,(B_r(\mathbf{x}_0))}\iint_{\partial B_r(\mathbf{x}_0)} F\cdot\mathbf{n}\,d\sigma &= \lim_{r\to 0}\frac{1}{\text{Vol}\,(B_r(\mathbf{x}_0))}\iiint_{B_r(\mathbf{x}_0)}\text{div}\,F\,dV \\ &= \text{div}\,F(\mathbf{x}_0). \end{aligned}$$

13.5.8. The sum rules are obvious. By the product rule for partial derivatives,

$$\begin{aligned} \nabla\times(fF) &= \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ fF_1 & fF_2 & fF_3\end{bmatrix} \\ &= (f_yF_3 + f(F_3)_y - f_zF_2 - f(F_2)_z, f_zF_1 + f(F_1)_z - f_xF_3 - f(F_3)_x, \\ &\qquad f_xF_2 + f(F_2)_x - f_yF_1 - f(F_1)_y) \\ &= (\nabla f\times F) + f(\nabla\times F). \end{aligned}$$

Similarly,

$$\nabla\cdot(F\times G) = \nabla\cdot\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ F_1 & F_2 & F_3\\ G_1 & G_2 & G_3\end{bmatrix} = (\nabla\times F)\cdot G - (\nabla\times G)\cdot F.$$

13.5.9. a) By hypothesis, $F = (f_x, f_y)$. Thus by Green's Theorem and Theorem 11.2,

$$\int_{\partial E} F \cdot T\, ds = \iint_E (f_{yx} - f_{xy})\, dA = 0.$$

b) By definition and Theorem 11.2,

$$\begin{aligned} \text{curl grad}\, f &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{bmatrix} \\ &= (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}) = (0,0,0). \end{aligned}$$

Similarly,

$$\text{div curl}\, F = (F_3)_{yx} - (F_2)_{zx} + (F_1)_{zy} - (F_3)_{xy} + (F_2)_{xz} - (F_1)_{yz} = 0.$$

c) Since $F = \nabla f$, we have $\nabla(fF) = \nabla f \cdot F + f \cdot \nabla F = F \cdot F + f \cdot \nabla \cdot \nabla f$ by Exercise 13.5.8. But $\nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz} = 0$ by hypothesis. Hence it follows from Gauss' Theorem that

$$\iint_{\partial E} (fF) \cdot \mathbf{n}\, d\sigma = \iiint_E F \cdot F\, dV.$$

13.5.10. a) By definition, $\nabla \cdot \nabla u = \nabla \cdot (u_x, u_y, u_z) = u_{xx} + u_{yy} + u_{zz}$.
b) By Gauss' Theorem and Exercise 13.5.8,

$$\iint_{\partial E} u \nabla v \cdot \mathbf{n}\, d\sigma = \iiint_E \nabla \cdot (u \nabla v)\, dV = \iiint_E (\nabla u \cdot \nabla v + u \Delta v)\, dV.$$

c) This follows immediately from Gauss' Theorem since by Exercise 13.5.8 and part a),

$$\nabla \cdot (u\nabla v - v \nabla u) = \nabla u \cdot \nabla v + u \Delta v - \nabla v \cdot \nabla u - v \Delta u = u \Delta v - v \Delta u.$$

d) Apply part b) to $u = v$. Since u is harmonic, we have

$$\iiint_E \|\nabla u\|^2\, dV = 0.$$

Thus $\nabla u = 0$, i.e., u is constant on E. Since u is continuous on $\overline{E}$, it follows that u is constant on $\overline{E}$. Since u is zero on ∂E, we conclude that u is zero on E.

e) By Green's Theorem,

$$\int_{\partial E} (u_x\, dy - u_y\, dx) = \iint_E (u_{xx} - (-u_{yy}))\, dA = \iint_E \Delta u\, dA.$$

If u is harmonic on E, then the integral on the left is zero. Conversely, if the integral on the left is zero, then $\iint_E \Delta u\, dA = 0$ for all such regions $E \subset V$. Fix $\mathbf{x}_0 \in V$ and set $E = B_r(\mathbf{x}_0)$. By Exercise 12.2.3, we have

$$\Delta u(\mathbf{x}_0) = \lim_{r \to 0} \frac{1}{\pi r^2} \iint_{B_r(\mathbf{x}_0)} \Delta u\, dA = 0.$$

Thus u is harmonic at $\mathbf{x}_0$.

13.6 Stokes's Theorem.

13.6.1. a) The trivial parameterization of $z = -x$, $x^2 + y^2 \le 1$, has normal $(1, 0, 1)$, whose induced orientation on C is counterclockwise. Since $\text{curl}\, F = (xz, -yz, -2xy)$, it follows from Stokes's Theorem that

$$\begin{aligned} \int_C F \cdot T\, ds &= \iint_{B_1(0,0)} (-x^2, xy, -2xy) \cdot (1,0,1)\, dA \\ &= -\int_0^{2\pi} \int_0^1 (r^3 \cos^2\theta + 2r^2 \sin\theta \cos\theta)\, dr\, d\theta \\ &= -\pi \int_0^1 r^3\, dr + 0 = -\pi/4. \end{aligned}$$

b) The trivial parameterization of $z = y^3$, $x^2 + y^2 \le 3$, has normal $(0, -3y^2, 1)$, whose induced orientation on C is counterclockwise (the wrong way). Since $\operatorname{curl} F = (ze^y, 1, y)$, it follows from Stokes's Theorem that

$$\begin{aligned}\int_C F \cdot T\, ds &= -\iint_{B_{\sqrt{3}}(0,0)} (y^3 e^y, 1, y) \cdot (0, -3y^2, 1)\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} (3r^3 \sin^2\theta - r^2 \sin\theta)\, dr\, d\theta \\ &= 3\pi \int_0^{\sqrt{3}} r^3\, dr + 0 = 27\pi/4.\end{aligned}$$

13.6.2. a) The boundary of S has two smooth pieces. It might be better to change the surface. Let

$$E = \{(x, y, 0) : x^2 \le y \le 1 \text{ and } -1 \le x \le 1\}.$$

Then E is a surface whose boundary equals ∂S, hence by Stokes's Theorem,

$$\iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma = \iint_E \operatorname{curl} F \cdot \mathbf{n}\, d\sigma.$$

But on E, $\mathbf{n} = (0, 0, 1)$, and the third component of $\nabla \times F$ is

$$\frac{\partial}{\partial x}(y \cos z^3) - \frac{\partial}{\partial y}(x \sin z^3) = 0.$$

Thus

$$\iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma = \iint_E \operatorname{curl} F \cdot \mathbf{n}\, d\sigma = 0.$$

b) The boundary of S is given by $x^2 + y^2 = 3$, $z = 0$, oriented in the counterclockwise direction. Using the parameterization $\phi(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t, 0)$, $t \in [0, 2\pi]$, we have by Stokes's Theorem that

$$\begin{aligned}\iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} (\sqrt{3}\sin t, 0, \sqrt{3}\sin t) \cdot (-\sqrt{3}\sin t, \sqrt{3}\cos t, 0)\, dt \\ &= -3 \int_0^{2\pi} \sin^2 t\, dt = -3\pi.\end{aligned}$$

c) The boundary of S is given by $x^2 + y^2 = 10$, $z = 0$, oriented in the clockwise direction. Using the parameterization $\phi(t) = (\sqrt{10}\sin t, \sqrt{10}\cos t, 0)$, $t \in [0, 2\pi]$, we have by Stokes's Theorem that

$$\begin{aligned}\iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} (\sqrt{10}\sin t, \sqrt{10}\sin t, mess) \cdot (\sqrt{10}\cos t, -\sqrt{10}\sin t, 0)\, dt \\ &= \int_0^{2\pi} (10 \sin t \cos t - 10 \sin^2 t)\, dt = -10\pi.\end{aligned}$$

d) The boundary of S has three smooth pieces, C_1 (given by $y = 0$, $z = 0$, $0 \le x \le 1$, oriented left to right), C_2 (given by $y = (1 - x)/2$, $z = 0$, $0 \le x \le 1$, oriented right to left), and C_3 (given by $x = 0$, $z = 0$, $0 \le y \le 1/2$, oriented top to bottom). Thus

$$\int_{C_1} F \cdot T\, ds = \int_0^1 F_1(x, 0, 0)\, dx = 0, \qquad \int_{C_3} F \cdot T\, ds = \int_0^{1/2} F_2(0, y, 0)\, dy = 0,$$

and

$$\int_{C_2} F \cdot T\, ds = -\int_0^1 F(x, (1 - x)/2, 0) \cdot (1, -1/2, 0)\, dx = \frac{1}{2} \int_0^1 (x^2 - x)\, dx = -1/12.$$

Thus by Stokes's Theorem

$$\iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma = \int_{\partial S} F \cdot T\, ds = -1/12.$$

13.6.3. a) Since $\operatorname{div} F = x^2 + y^2 + z^2$, we have by Gauss' Theorem that

$$\iint_S F \cdot \mathbf{n}\, d\sigma = \iiint_{B_1(0,0,0)} (x^2 + y^2 + z^2)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta = 4\pi/5.$$

b) If $R_y = 0$ and $Q_z = -xy$ then $R = g(x,z)$ and $Q = -xyz + f(x,y)$. Thus $Q_x = -yz + f_x$ and we may set $f = 0$, i.e., $Q = -xyz$. Since $P_y = 0$ implies $P = h(x,z)$, we have $P_z = h_z$ and $R_x = g_x$. Hence we may set $h_z = xz$ and $g = 0$, i.e., $P = xz^2/2$ and $R = 0$. Since $z = y$, the projection of ∂S onto the xy plane is given by $x^2 + 2y^2 = 1$. Thus we can parameterize ∂S by $\phi(t) = (\cos t, \sin t/\sqrt{2}, \sin t/\sqrt{2})$, $t \in [0, 2\pi]$. Hence,

$$\begin{aligned}(P,Q,R) \cdot \phi'(t) &= (\cos t \sin^2 t/4, -\cos t \sin^2 t/2, 0) \cdot (-\sin t, \cos t/\sqrt{2}, \cos t/\sqrt{2}) \\ &= -\sin^3 t \cos t/4 - \sin^2 t \cos^2 t/(2\sqrt{2}).\end{aligned}$$

We conclude by Stokes's Theorem that

$$\iint_S F \cdot \mathbf{n}\, d\sigma = -\int_0^{2\pi} (\sin^3 t \cos t/4 + \sin^2 t \cos^2 t/(2\sqrt{2}))\, dt = -\pi/(8\sqrt{2}).$$

c) If $R_y = x$ and $Q_z = 0$ then $R = xy + f(x,z)$ and $Q = g(x,y)$. Thus $R_x = y + f_x$ and we may set $f = 0$, i.e., $R = xy$. Since $P_z = -y$ implies $P = -yz + h(x,y)$ and $P_y = -z + h_y$, we may set $Q = 0 = h$, i.e., $P = -yz$. Now ∂S has two pieces: C_1 given by $\phi(t) = (\sin t, 2, \cos t)$, $t \in [0, 2\pi]$, and C_2 given by $\psi(t) = (2\cos t, 4, 2\sin t)$, $t \in [0, 2\pi]$. Thus

$$(P,Q,R) \cdot \phi'(t) = (-2\cos t, 0, 2\sin t) \cdot (\cos t, 0, -\sin t) = -2\cos^2 t - 2\sin^2 t = -2,$$

and

$$(P,Q,R) \cdot \psi'(t) = (-8\sin t, 0, 8\cos t) \cdot (-2\sin t, 0, 2\cos t) = 16\cos^2 t + 16\sin^2 t = 16.$$

We conclude by Stokes's Theorem that

$$\iint_S F \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} (16 - 2)\, dt = 28\pi.$$

d) Since $\operatorname{div} F = 2y + 2$, we have by Gauss' Theorem that

$$\begin{aligned}\iint_S F \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} \int_0^2 \int_{r^2-4}^{4-r^2} (2r\sin\theta + 2) r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 (8 - 2r^2)(2r\sin\theta + 2) r\, dz\, dr\, d\theta \\ &= 0 + 4\pi \int_0^2 (8r - 2r^3)\, dr = 32\pi.\end{aligned}$$

e) The boundary of S is given by $x^2 + y^2 = 1$, $z = 6$, and must be oriented in the clockwise direction when viewed from high up the z axis. Thus it can be parameterized by $\phi(t) = (\sin t, \cos t, 6)$, $t \in [0, 2\pi]$.

If $P_z = 2z$ and $R_x = 0$, then $P = z^2 + f(x,y)$ and $R = g(y,z)$. Thus $P_y = f_y$ and we may set $f = 0$, $Q_x = 1$. Thus $Q = x + h(y,z)$ and $Q_z = h_z = 0$, $R_y = g_y = 2y$. Therefore, set $R = y^2$, $Q = x$, and $R = y^2$. In particular, $(P,Q,R) \circ \phi = (36, \sin t, \cos^2 t)$. Since $\phi'(t) = (\cos t, -\sin t, 0)$, it follows from Stokes's Theorem that

$$\iint_S F \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} (36\cos t - \sin^2 t)\, dt = -\pi.$$

13.6.4. a) Let E be the solid cylinder whose boundary is S and $F = (xy, x^2 - z^2, xz)$. Since $\operatorname{div} F = x + y$, it follows from Gauss' Theorem that

$$\iint_S \omega = \iiint_E (x+y)\,dV = \int_0^{2\pi}\int_0^3\int_0^2 (x + r\cos\theta)r\,dx\,dr\,d\theta = 2\pi\int_0^3\int_0^2 xr\,dx\,dr = 18\pi.$$

b) The boundary of S consists of two pieces. C_1: $x^2 + z^2 = 8$, $y = 1$, oriented in the clockwise direction when viewed from far out the y axis, and C_2: $x^2 + z^2 = 8$, $y = 0$, oriented in the counterclockwise direction when viewed from far out the y axis.

Let curl $(P, Q, R) = (x - 2z, -y, 0)$. If $R_y = x$ and $Q_z = 2z$, then $R = xy + f(x, z)$ and $Q = z^2 + g(x, y)$. Thus $R_x = y + f_x$ and we can set $f = 0$, $P = 0$, and $g = 0$. Hence $(P, Q, R) = (0, z^2, xy)$. Parameterize C_1 by $\phi(t) = (\sqrt{8}\cos t, 1, \sqrt{8}\sin t)$, $t \in [0, 2\pi]$. Then

$$\int_{C_1} (P, Q, R)\cdot T\,ds = \int_0^{2\pi} (0, 8\sin^2 t, \sqrt{8}\cos t)\cdot(-\sqrt{8}\sin t, 0, \sqrt{8}\cos t)\,dt = 8\pi.$$

Similarly,

$$\int_{C_2} (P, Q, R)\cdot T\,ds = \int_0^{2\pi} (0, 8\cos^2 t, 0)\cdot(\sqrt{8}\cos t, 0, -\sqrt{8}\sin t)\,dt = 0.$$

Therefore, $\iint_S \omega = 8\pi + 0 = 8\pi$.

c) Since $F = (e^y\cos x, x^2 z, x + y + z)$ implies $\operatorname{div} F = -e^y \sin x + 1$, it follows from Gauss' Theorem that

$$\iint_S \omega = \int_0^3\int_0^1\int_0^{\pi/2} (1 - e^y\sin x)\,dx\,dy\,dz = 3\int_0^1 (\frac{\pi}{2} - e^y)\,dy = 3(1-e) + 3\pi/2.$$

d) The boundary of S is given by $2x^2 + z^2 = 1$, $y = x$, oriented in the counterclockwise direction when viewed from far out the x axis. It can be parameterized by $\phi(t) = (\cos t/\sqrt{2}, \cos t/\sqrt{2}, \sin t)$, $t \in [0, 2\pi]$.

Let curl $(P, Q, R) = (x, -y, \sin y)$. If $R_y = x$ and $Q_z = 0$, then $R = xy + f(x, z)$ and $Q = g(x, y)$. Thus $R_x = y + f_x$ and we can set $f = 0$, $P = 0$, and $g = 0$. Hence $P = \cos y$, $Q = 0$, $R = xy$, and $(P, Q, R)\circ\phi = (\cos(\cos t/\sqrt{2}), 0, \cos^2 t/2)$. Therefore, it follows from Stokes's Theorem that

$$\iint_S \omega = \int_0^{2\pi}\left(-\frac{\sin t}{\sqrt{2}}\cos\left(\frac{\cos t}{\sqrt{2}}\right) + \frac{\cos^2 t}{2}\cos t\right)dt = \int_{1/\sqrt{2}}^{1/\sqrt{2}} \cos u\,du + 0 = 0.$$

13.6.5. Let $F = (P, Q, R)$ and $\phi(u, v) = (u, v, 0)$. Then $N_\phi = (0, 0, 1)$ and $(\nabla\times F)\cdot N_\phi = Q_x - P_y$. Therefore, applying Stokes's Theorem, we obtain

$$\int_{\partial E} P\,dx + Q\,dy = \int_{\partial E} (P, Q, R)\cdot T\,ds = \iint_E (\nabla\times F)\cdot\mathbf{n}\,d\sigma = \iint_E (Q_x - P_y)\,dA.$$

13.6.6. Let $\epsilon > 0$. Since F is $\mathcal{C}^1$ on $B_1(\mathbf{x}_0)$, we can choose r so small that

$$|((\nabla\times F)(\mathbf{x}) - (\nabla\times F)(\mathbf{x}_0))\cdot\mathbf{n}| < \epsilon$$

for all $\mathbf{x} \in B_r(\mathbf{x}_0)$. Hence by Stokes's Theorem,

$$\begin{aligned}&\left|\frac{1}{\sigma(S_r)}\int_{\partial S_r} F\cdot T\,ds - (\nabla\times F)(\mathbf{x}_0)\cdot\mathbf{n}\right|\\ &\quad= \left|\frac{1}{\sigma(S_r)}\iint_{S_r} (\nabla\times F)(\mathbf{x})\cdot\mathbf{n}\,d\sigma(\mathbf{x}) - (\nabla\times F)(\mathbf{x}_0)\cdot\mathbf{n}\right|\\ &\quad\le \frac{1}{\sigma(S_r)}\iint_{S_r} |((\nabla\times F)(\mathbf{x}) - (\nabla\times F)(\mathbf{x}_0))\cdot\mathbf{n}|\,d\sigma < \epsilon.\end{aligned}$$

In particular, $\int_{\partial S_r} F\cdot T\,ds/\sigma(S_r) \to (\nabla\times F)(\mathbf{x}_0))\cdot\mathbf{n}$ as $r\to 0$.

13.6.7. a) Let θ be the angle between F and T. Since $\theta \in [0, \pi/2]$, $\cos\theta \geq 0$, i.e., $F \cdot T \geq 0$. Hence by Stokes's Theorem and Exercise 5.1.4,

$$\int_{\partial S} F \cdot T\, ds = \iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma = 0$$

implies $F \cdot T = 0$ everywhere on ∂S. We conclude that F and T are orthogonal.

b) Let (ϕ, I) be a piecewise smooth parameterization of ∂S. By the Cauchy–Schwarz Inequality and hypothesis, $F_k(\phi(t)) \cdot \phi(t) \to F(\phi(t)) \cdot \phi(t)$ uniformly on I, as $k \to \infty$. Hence by Stokes's Theorem and Theorem 7.10,

$$\begin{aligned}\lim_{k\to\infty} \iint_S \operatorname{curl} F_k \cdot \mathbf{n}\, d\sigma &= \lim_{k\to\infty} \int_{\partial S} F_k \cdot T\, ds \\ &= \int_{\partial S} F \cdot T\, ds = \iint_S \operatorname{curl} F \cdot \mathbf{n}\, d\sigma.\end{aligned}$$

13.6.8. Suppose i) holds. Since $F = \nabla f$, f must be $\mathcal{C}^2$ on E, hence $Q_x = f_{yx} = f_{xy} = P_y$ on E.

Suppose ii) holds and let C be any piecewise smooth curve of the type described in condition iii). Then by Green's Theorem,

$$\int_C F \cdot T\, ds = \iint_\Omega (Q_x - P_y)\, dA = 0.$$

Suppose iii) holds and let $\mathbf{x}_0 \in E^0$. By Green's Theorem, $\int_{\partial B_r(\mathbf{x}_0)} (Q_x - P_y)\, dA = 0$ for all $r > 0$ sufficiently small, hence by Exercise 12.2.3,

$$(Q_x - P_y)(\mathbf{x}_0) = \lim_{r\to 0} \frac{1}{A(B_r(\mathbf{x}_0))} \int_{B_r(\mathbf{x}_0)} (Q_x - P_y)\, dA = 0.$$

Since $Q_x - P_y$ is continuous on E, it follows that $Q_x - P_y = 0$ everywhere on E.

To find an f such that $\nabla f = F$, we must solve $f_x = P$ and $f_y = Q$. Since $f = \int_0^y Q(x, v)\, dv + h(x)$, we have by differentiating under the integral sign that

$$f_x = \int_0^y Q_x(x, v)\, dv + h'(x) = \int_0^y P_v(x, v)\, dv + h'(x).$$

Hence by the Fundamental Theorem of Calculus,

$$f_x = \int_0^y P_v(x, v)\, dv + h'(x) = P(x, y) - P(x, 0) + h'(x).$$

Thus set $h'(x) = P(x, 0)$, i.e., $h(x) = \int_0^x P(u, 0)\, du$. In particular,

$$f(x, y) = \int_0^y Q(x, v)\, dv + \int_0^x P(u, 0)\, du.$$

13.6.9. The proof that i) implies ii) and ii) implies iii) is similar to the proof of Theorem 13.61. Thus it remains to prove that iii) implies i).

Suppose $\operatorname{div} F = 0$ everywhere on Ω. Let $G = (P, Q, 0)$, $F = (p, q, r)$, and suppose $\nabla \times G = F$. Then $Q_z = -p$, $P_z = q$, and $Q_x - P_y = r$. Integrating, we obtain

$$Q = -\int_0^z p(x, y, v)\, dv + g(x, y) \quad \text{and} \quad P = \int_0^z q(x, y, v)\, dv + h(x, y).$$

Differentiating under the integral sign, we have

$$Q_x = -\int_0^z p_x(x, y, v)\, dv + g_x(x, y) \quad \text{and} \quad P_y = \int_0^z q_y(x, y, v)\, dv + h_y(x, y).$$

Thus by hypothesis iii),

$$r = Q_x - P_y = \int_0^z r_z(x, y, v)\, dv + g_x - h_y = r(x, y, z) - r(x, y, 0) + g_x - h_y.$$

Hence set $h_y = r(x,y,0)$ and $g = 0$. In particular,

$$P = \int_0^z q(x,y,v)\,dv + \int_0^y r(x,u,0)\,du \quad \text{and} \quad Q = -\int_0^z p_x(x,y,v)\,dv.$$

13.6.10. a) Let C_1 be the "outside" curve. By Green's Theorem and hypothesis,

$$\int_{C_1} F \cdot T\,ds - \int_{C_2} F \cdot T\,ds = \iint_E (Q_x - P_y)\,dA = 0.$$

b) Since $Q_x = (y^2 - x^2)/(x^2+y^2)^2 = P_y$, we can replace ∂E with any simple closed curve which surrounds $(0,0)$ and is disjoint from ∂E. $x^2 + y^2 = r^2$ is such a curve for r sufficiently small. Let $\phi(t) = (r\cos t, r\sin t)$. Then,

$$F(\phi(t)) \cdot \phi'(t) = \left(\frac{-\sin t}{r}, \frac{\cos t}{r} \right) \cdot (-r\sin t, r\cos t) = \sin^2 t + \cos^2 t = 1.$$

Therefore,

$$\int_{\partial E} F \cdot T\,ds = \int_0^{2\pi} dt = 2\pi.$$

c) If S_1 and S_2 are disjoint "concentric" surfaces which do not contain the origin, and the normals of S_1 and S_2 both point away from the origin, then

$$\iint_{S_1} F \cdot \mathbf{n}\,d\sigma = \iint_{S_2} F \cdot \mathbf{n}\,d\sigma$$

for all $\mathcal{C}^1$ functions F which satisfy $F = \operatorname{curl} G$ for some $\mathcal{C}^2$ function G on $\mathbf{R}^3 \setminus \{(0,0,0)\}$.

14.1 Introduction.

14.1.1. a) Since $x^2 \sin kx$ is odd, $b_k(x^2) = 0$ for $k = 1, 2, \ldots$ Since $x^2 \cos kx$ is even, we can integrate by parts twice to verify

$$a_k(x^2) = \frac{2}{\pi}\int_0^\pi x^2 \cos kx\,dx = -\frac{4}{k\pi}\int_0^\pi x \sin kx\,dx = \frac{4(-1)^k}{k^2}$$

for $k \neq 0$. Finally,

$$a_0(x^2) = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}.$$

b) Since $\cos^2 x = (1 + \cos 2x)/2$, we have by orthogonality that $a_0(\cos^2 x) = 1/2$, $a_2(\cos^2 x) = 1/2$, and all other Fourier coefficients of $\cos^2 x$ are zero.

14.1.2. By definition and a sum angle formula,

$$\begin{aligned}(S_N f)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \\ &\quad + \sum_{k=0}^{N}\left(\frac{\cos kx}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt + \frac{\sin kx}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\right) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=0}^{N}(\cos kx\cos kt + \sin kx \sin kt)\right)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) D_N(x-t)\,dt.\end{aligned}$$

14.1.3. These formulas follow easily from the linear properties of integration. For example,

$$\begin{aligned}a_k(f+g) &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(t)+g(t))\cos kt\,dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt + \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos kt\,dt = a_k(f) + a_k(g).\end{aligned}$$

14.1.4. Integrating by parts, we have

$$\begin{aligned}a_k(f') &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos kt\,dt \\ &= \frac{1}{\pi}\left(f(t)\cos kt\,\Big|_{-\pi}^{\pi} + k\int_{-\pi}^{\pi} f(t)\sin kt\,dt\right) \\ &= 0 + kb_k(f)\end{aligned}$$

since f is periodic. A similar argument establishes $b_k(f') = -ka_k(f)$.

14.1.5. a) Since $f_N(x) - f(x) \to 0$ uniformly on $[-\pi, \pi]$, it follows from Theorem 7.10 that

$$|a_k(f_N) - a_k(f)| \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f_N(t) - f(t)|\,|\cos kt|\,dt \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f_N(t) - f(t)|\,dt$$

converges to zero as $N \to \infty$.

b) The proof of part a) also proves this statement.

14.1.6. a) Since $f(x)\cos kx$ is odd, $a_k(f) = 0$ for $k = 0, 1, \ldots$ Since $f(x)\sin kx$ is even, we have

$$b_k(f) = \frac{2}{\pi}\int_0^\pi \frac{x}{|x|}\sin kx\,dx = \frac{2}{\pi}\left(\frac{-\cos kx}{k}\,\Big|_0^\pi = \frac{2}{k\pi}((-1)^{k+1} + 1).\right.$$

Thus $b_k(f) = 4/(k\pi)$ when k is odd and 0 when k is even.

b) By part a) and the Fundamental Theorem of Calculus,

$$(S_{2N}f)(x) = \frac{4}{\pi}\sum_{k=1}^{N-1}\frac{\sin(2k+1)x}{2k+1} = \frac{4}{\pi}\int_0^x \sum_{k=1}^{N-1}\cos(2k+1)t\,dt.$$

Since by a sum angle formula and telescoping we have

$$2\sin t\sum_{k=1}^{N-1}\cos(2k+1)t = \sum_{k=1}^{N-1}(\sin 2kt - \sin(2k-2)t) = \sin 2Nt,$$

it follows that

$$(S_{2N}f)(x) = \frac{4}{\pi}\int_0^x \frac{\sin 2Nt}{2\sin t}\,dt = \frac{2}{\pi}\int_0^x \frac{\sin 2Nt}{\sin t}\,dt.$$

c) By part b) and a change of variables,

$$(*)\qquad (S_{2N}f)(\frac{\pi}{2N}) = \frac{2}{\pi}\int_0^{\pi/2N}\frac{\sin 2Nt}{\sin t}\,dt = \frac{2}{\pi}\int_0^{\pi}\frac{\sin u}{2N\sin(u/2N)}\,du.$$

Fix $u \in [0,\pi]$ and set $g(x) = x\sin(u/x)$ for $x > 0$. Since $\tan(u/x) \ge u/x$ for $x > 2u/\pi$ (see (1) in Appendix B), and $f'(x) = \sin(u/x) - (u/x)\cos(u/x)$, we see that $f(x)$ is increasing for $x > 2u/\pi$. Therefore, $2N\sin(u/2N) \uparrow u$ as $N \to \infty$ for each $u \in [0,\pi]$. In particular, it follows from Theorem 9.41 and (*) that

$$\lim_{N\to\infty}(S_{2N}f)(\frac{\pi}{2N}) = \frac{2}{\pi}\int_0^{\pi}\lim_{N\to\infty}\frac{\sin u}{2N\sin(u/2N)}\,du = \frac{2}{\pi}\int_0^{\pi}\frac{\sin u}{u}\,du.$$

Using either power series or Simpson's Rule, we can show that this last integral is approximately 1.179.

14.2 Summability of Fourier Series.

14.2.1. Let $\epsilon > 0$ and set $S_k(x) := \sum_{j=0}^k f_j(x)$ for $k \ge 0$ and $x \in E$. Since $S_k \to f$ uniformly on E, choose $N_1 \in \mathbf{N}$ such that $k \ge N_1$ implies $|S_k(x) - f(x)| < \epsilon/2$ for all $x \in E$. Since f_k and f are bounded on E, choose $N_2 \in \mathbf{N}$ such that $N_2 > N_1$ and $\sum_{k=0}^{N_1}|S_k(x) - f(x)| < \epsilon N_2/2$ for all $x \in E$. If $N > N_2$ then

$$\begin{aligned}|\sigma_N(x) - f(x)| &= \frac{|(S_0(x) - f(x)) + \cdots + (S_N(x) - f(x))}{N+1} \\ &\le \frac{1}{N+1}\sum_{k=0}^{N_1}|S_k(x) - f(x)| + \frac{\epsilon}{2}\left(\frac{N-N_1}{N+1}\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

14.2.2. Since any Riemann integrable function is bounded, it follows from (8) and (9) that

$$|(\sigma_N f)(x)| \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|\,|K_N(x-t)|\,dt \le \frac{M}{\pi}\int_{-\pi}^{\pi}K_N(x-t)\,dt = M.$$

14.2.3. If S is the Fourier series of a continuous, periodic function f then $\sigma_N = \sigma_N f \to f$ uniformly on $\mathbf{R}$ by Corollary 14.15.

Conversely, suppose $\sigma_N \to f$ uniformly on $\mathbf{R}$. Fix $k \in \mathbf{N}$ and observe by Theorem 7.10 and orthogonality that

$$a_k(f) = \lim_{N\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}\sigma_N(x)\cos kx\,dx = \lim_{N\to\infty}\left(1 - \frac{k}{N+1}\right)a_k = a_k.$$

Similarly, $a_0(f) = a_0$ and $b_k(f) = b_k$ for $k \in \mathbf{N}$. Therefore, S is the Fourier series of f.

14.2.4. a) If $(Sf)(x_0)$ converges to M, then by Remark 14.6, $(\sigma_N f)(x_0) \to M$ as $N \to \infty$, i.e., $M = L$.

b) Any continuous function f can be extended from a compact subset K of $(0, 2\pi)$ to be continuous and periodic on $[0, 2\pi]$. Indeed, choose a, b such that $K \subseteq [a, b] \subset (0, 2\pi)$ and define $\widetilde{f}$ on $(0, 2\pi) \setminus [a, b]$ to be linear, i.e., its graph is a straight line from $(b, f(b))$ to $(a + 2\pi, f(a))$ and it is extended by periodicity to all of $\mathbf{R}$. Since $f(x) := \pi\sqrt{2}\cos\sqrt{2}x$ is continuous on $\mathbf{R}$, it follows that its Fourier series must be uniformly Cesàro summable to f on compact subsets of $(0, 2\pi)$. Since the given series is uniformly convergent by the Weierstrass M-Test, it remains to show that this series is the Fourier series of f.

Since $f(x)$ is even, $b_k(f) = 0$ for $k \in \mathbf{N}$. Clearly,

$$a_0(f) = \frac{\pi\sqrt{2}}{\pi}\int_{-\pi}^{\pi}\cos\sqrt{2}x\,dx = 2\sin\sqrt{2}\pi.$$

And, by a sum angle formula,

$$\begin{aligned}
a_k(f) &= \frac{\pi\sqrt{2}}{\pi}\int_{-\pi}^{\pi}\cos\sqrt{2}x\cos kx\,dx \\
&= \frac{\sqrt{2}}{2}\int_{-\pi}^{\pi}(\cos(\sqrt{2}+k)x + \cos(\sqrt{2}-k)x)\,dx \\
&= \sqrt{2}\left(\frac{\sin(\sqrt{2}+k)\pi}{\sqrt{2}+k} + \frac{\sin(\sqrt{2}-k)\pi}{\sqrt{2}-k}\right) \\
&= \sqrt{2}\left(\frac{\sin\sqrt{2}\pi\cos k\pi}{\sqrt{2}+k} + \frac{\sin\sqrt{2}\pi\cos k\pi}{\sqrt{2}-k}\right) \\
&= \sqrt{2}\left(\frac{2\sqrt{2}(-1)^k\sin\sqrt{2}\pi}{2-k^2}\right) = \frac{4(-1)^k\sin\sqrt{2}\pi}{2-k^2}.
\end{aligned}$$

14.2.5. a) If $P(x) = \sum_{k=0}^{n} a_k x^k$ then

$$\int_a^b P(x)f(x)\,dx = \sum_{k=0}^{n} a_k\int_a^b x^k f(x)\,dx = 0.$$

b) By Exercise 10.7.6d, choose polynomials P_N which converge to f uniformly on $[a, b]$ as $N \to \infty$. Then $P_N(x)f(x) \to f^2(x)$ uniformly on $[a, b]$ as $N \to \infty$ and we have by part a) and Theorem 7.10 that

$$0 = \lim_{N\to\infty}\int_a^b P_N(x)f(x)\,dx = \int_a^b f^2(x)\,dx.$$

c) By part b) and Exercise 5.1.4, $f^2(x) = 0$ for all $x \in [a, b]$, thus $f(x) = 0$ for all $x \in [a, b]$.

14.2.6. Let

$$\Delta_N(x) = \int_0^{2\pi} f(x-t)\phi_N(t)\,dt - f(x).$$

Since the integral of ϕ_N is 1, notice that

$$\Delta_N(x) = \int_0^{2\pi}(f(x-t) - f(x))\phi_N(t)\,dt.$$

Let $\epsilon > 0$. Since f is continuous and periodic, choose $\delta \in (0, 2\pi)$ such that

$$t \in E_\delta := [0, \delta] \cup [2\pi - \delta, 2\pi] \quad \text{implies} \quad |f(x-t) - f(x)| < \epsilon$$

for all $x \in \mathbf{R}$. If $C := \sup_{x\in\mathbf{R}}|f(x)|$, then

$$\begin{aligned}
|\Delta_N(x)| &\le \epsilon\int_{E_\delta}|\phi_N(t)|\,dt + \int_\delta^{2\pi-\delta}|f(x-t) - f(x)|\,|\phi_N(t)|\,dt \\
&\le \epsilon M + 2C\int_\delta^{2\pi-\delta}|\phi_N(t)|\,dt,
\end{aligned}$$

i.e., $\limsup_{N\to\infty} |\Delta_N(x)| \le \epsilon M$ for all $x \in \mathbf{R}$. Therefore, $\Delta_N(x) \to 0$ uniformly in x, as $N \to \infty$.

14.2.7. a) Suppose $P(x) \equiv a_n x^n + \cdots + a_1 x + a_0$ is a polynomial on $\mathbf{R}$ and $\varepsilon > 0$. Let $M := \sup\{|x|^k : x \in [a,b], k = 0,1,\dots,n\}$, and choose rationals b_k such that $|a_k - b_k| < \varepsilon/(n+1)M$ for $0 \le k \le n$. Then $Q(x) := b_n x^n + \cdots + b_1 x + b_0$ is a polynomial with rational coefficients which satisfies

$$|P(x) - Q(x)| \le |a_n - b_n||x|^n + \cdots + |a_0 - b_0| < \varepsilon$$

for all $x \in [a,b]$.

b) Let $\varepsilon > 0$. Clearly, the set of polynomials with rational coefficients is countable. Therefore, it suffices to prove that given any continuous f on $[a,b]$, there is a polynomial Q, with rational coefficients, such that $|f(x)-Q(x)| < \varepsilon$ for all $x \in [a,b]$.

Choose (by Exercise 10.7.6d) a polynomial P such that $|f(x) - P(x)| < \varepsilon/2$ for all $x \in [a,b]$, and (by part b) a polynomial Q, with rational coefficients, such that $|Q(x) - P(x)| < \varepsilon/2$ for all $x \in [a,b]$. Then by the triangle inequality, $|f(x) - Q(x)| < \varepsilon$ for all $x \in [a,b]$.

14.2.8. By Theorem 9.49, f is continuous almost everywhere, hence by Fejér's Theorem, $\sigma_N f \to f$ almost everywhere as $N \to \infty$.

14.3 Growth of Fourier Coefficients.

14.3.1. Since $\sin(k+\alpha)x = \sin kx \cos \alpha x + \cos kx \sin \alpha x$, we have

$$\int_{-\pi}^{\pi} f(x)\sin(k+\alpha)x\,dx = \pi b_k(f(x)\cos\alpha x) + \pi a_k(f(x)\sin\alpha x)$$

for all $k \in \mathbf{N}$. Since $f(x)\cos\alpha x$ and $f(x)\sin\alpha x$ are integrable on $[-\pi,\pi]$, it follows from the Riemann-Lebesgue Lemma that the integral converges to zero as $k \to \infty$.

14.3.2. If f were continuous and $|a_k(f)| \ge 1/\sqrt{k}$, then

$$\sum_{k=1}^{\infty} |a_k(f)|^2 \ge \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

which contradicts Bessel's Inequality.

14.3.3. By Theorem 14.23, $k^2 a_k(f) \to 0$ as $k \to \infty$, in particular, $|a_k(f)| \le 1/k^2$ for k large. Similarly, $|b_k(f)| \le 1/k^2$ for k large. Therefore, Sf converges uniformly on $\mathbf{R}$ by the Weierstrass M-Test and absolutely on $\mathbf{R}$ by the Comparison Test.

14.3.4. Fix $j \in \mathbf{N}$. Since

$$|d^j/dx^j(\cos kx)| = \begin{cases} k^j|\sin kx| & \text{when } j \text{ is odd} \\ k^j|\cos kx| & \text{when } j \text{ is even,} \end{cases}$$

it follows from Theorem 14.23 that $|d^j/dx^j(a_k(f)\cos kx)| \le 1/k^2$ for k large. A similar estimate holds for the sine terms. Hence by the Weierstrass M-Test, the j-th term by term derivative of Sf converges uniformly on $\mathbf{R}$. Since f is continuous and periodic, we have by Exercise 14.2.4 that $S_N f \to f$ uniformly on $\mathbf{R}$. We conclude by Theorem 7.12 that

$$\frac{d^j f}{dx^j}(x) = \sum_{k=1}^{\infty} \frac{d^j}{dx^j}(a_k(f)\cos kx + b_k(f)\sin kx).$$

14.3.5. a) Let $k \ge j \ge 0$. Since the coefficients are nonnegative,

$$(S_k f)(0) = \frac{a_0(f)}{2} + \sum_{\ell=1}^{k} a_\ell(f) \ge \frac{a_0(f)}{2} + \sum_{\ell=1}^{j} a_\ell(f) = (S_j f)(0).$$

b) By part a)

$$\begin{aligned}(S_N f)(0) &\le \frac{(S_N f)(0) + \cdots + (S_{2N} f)(0)}{N+1} \\ &\le \frac{(S_0 f)(0) + \cdots + (S_{2N} f)(0)}{N+1} \le 2(\sigma_{2N} f)(0).\end{aligned}$$

c) By Exercise 14.2.2, $\sigma_{2N}f$ is uniformly bounded on $\mathbf{R}$, hence by part b), $\sum_{k=1}^{\infty}|a_k(f)|$ converges.

d) If f is even, then $b_k(f)=0$ for $k\in\mathbf{N}$. Hence,

$$Sf=\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}a_k(f)\cos kx$$

which converges uniformly and absolutely on $\mathbf{R}$ by the Weierstrass M-Test and part c). In particular, f must be continuous by Theorem 7.9.

14.3.6. a) Using the change of variables $t=u+\pi/k$, $dt=du$, and a sum angle formula, we have

$$\begin{aligned}a_k(f)&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos kt\,dt\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(u+\frac{\pi}{k})(\cos ku\cos\pi-\sin ku\sin\pi)\,du=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(u+\frac{\pi}{k})\cos ku\,du.\end{aligned}$$

Thus

$$a_k(f)=\frac{a_k(f)+a_k(f)}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(u)-f(u+\frac{\pi}{k}))\cos ku\,du.$$

b) By part a),

$$|a_k(f)|\le\omega(f,\frac{\pi}{k})\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos ku|\,du\le\omega(f,\frac{\pi}{k}).$$

A similar argument shows $|b_k(f)|\le\omega(f,\pi/k)$.

c) If f is continuous on $\mathbf{R}$ then f is uniformly continuous on $[-\pi,\pi]$. Therefore, $\omega(f,\pi/k)\to 0$ as $k\to\infty$ and it follows from part b) that $a_k(f)$ and $b_k(f)$ converge to zero as $k\to\infty$.

14.3.7. a) Since $f(x)=x$ is odd, it is clear that $a_k(f)=0$ for $k=0,1,\ldots.$ On the other hand, it is easy to see by parts that

$$b_k(f)=\frac{1}{\pi}\int_{\pi}^{\pi}x\sin kx\,dx=-\frac{2}{k}$$

for $k\in\mathbf{N}$.

b) Since

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\int_{-\pi}^{\pi}x^2\,dx=\frac{2\pi^2}{3},$$

it follows from part a) and Parseval's Formula that

$$\sum_{k=1}^{\infty}\frac{4}{k^2}=\frac{2\pi^2}{3}$$

as promised.

14.4 Convergence of Fourier Series.

14.4.1. Define $g(x)=f(x)$ for $x\in[0,2\pi)$ and $g(2\pi)=f(0)$. Then g is periodic and of bounded variation on $[-\pi,\pi]$ and continuous on any interval $[a,b]\subset(-\pi,\pi)$. Hence by Theorem 14.29, Sg converges to g uniformly on $[a,b]$ and pointwise on $(-\pi,\pi)$. Since $f=g$ on $(-\pi,\pi)$ implies $Sf=Sg$, we conclude that Sf converges to f uniformly on $[a,b]\subset(-\pi,\pi)$ and pointwise on $(-\pi,\pi)$.

14.4.2. a) By Example 14.8, this is the Fourier series of x. Hence by Theorem 14.29 and Exercise 14.4.1, this series must converge to x uniformly on $[a,b]\subset(-\pi,\pi)$ and pointwise on $(-\pi,\pi)$.

b) By Example 14.9, this is the Fourier series of $|x|$. Since $|x|$ is periodic and continuous on $[-\pi,\pi]$, it follows from Theorem 14.29 that this series converges to $|x|$ uniformly on $[-\pi,\pi]$.

c) By part b),

$$0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2},$$

hence $\sum_{k=1}^\infty 1/(2k-1)^2 = (\pi/2)(\pi/4) = \pi^2/8$.

14.4.3. By the proof of Corollary 14.27, if $a_0(f) = 0$ then $F(x) := \int_0^x f(t)\,dt$ is continuous, periodic, of bounded variation, $a_k(F) = b_k(f)/k$, and $b_k(F) = -a_k(f)/k = 0$ for $k \in \mathbf{N}$. Thus

$$(SF)(x) = \frac{a_0(F)}{2} + \sum_{k=1}^\infty \frac{b_k(f)}{k} \cos kx.$$

By Theorem 14.29, SF converges to F uniformly on $\mathbf{R}$, in particular, at $x = 0$. Therefore, $\sum_{k=1}^\infty b_k(f)/k$ converges.

14.4.4. a) Fix $N \in \mathbf{N}$ and $r \in (0,1)$. By Abel's Transform,

$$\sum_{k=0}^N a_k r^k = S_N r^N + (1-r)\sum_{k=0}^{N-1} S_k r^k.$$

If $\sum_{k=0}^\infty S_k r^k$ converges, then $S_N r^N \to 0$ as $N \to \infty$, and we have verified the first identity. To show this is also the case when $\sum_{k=0}^\infty a_k \rho^k$ converges for all $\rho \in (0,1)$, fix $r < \rho < 1$ and observe since $|a_k \rho^k| \le C$ for all $k \ge 0$ that

$$|S_N r^N| = |\sum_{j=0}^N a_j r^N| \le C \sum_{j=0}^N \frac{r^N}{\rho^j} \le C\frac{r^N}{\rho^N(1-\rho)}$$

for all $N \in \mathbf{N}$. Since this last quotient converges to zero as $N \to \infty$, it follows that $S_N r^N \to 0$ as $N \to \infty$. This verifies the first identity.

Similarly, since $(k+1)\sigma_k = S_0 + \cdots + S_k$, we can prove

$$(1-r)\sum_{k=0}^\infty S_k r^k = (1-r)^2 \sum_{k=0}^\infty (k+1)\sigma_k r^k.$$

b) Let $\epsilon > 0$ and choose $N \in \mathbf{N}$ such that $k \ge N$ implies $|\sigma_k - L| < \epsilon$. It is easy to see that

$$1 = (1-r)^2 \sum_{k=0}^\infty (k+1) r^k.$$

(Either apply Abel's Transformation to the Geometric Series, or use the techniques introduced in Section 7.3.) Thus by part a),

$$\begin{aligned} |\sum_{k=0}^\infty a_k r^k - L| &= (1-r)^2 |\sum_{k=0}^\infty (k+1)(\sigma_k - L) r^k| \\ &\le (1-r)^2 \sum_{k=0}^{N-1} (k+1)|\sigma_k - L| r^k + \epsilon(1-r)^2 \sum_{k=N}^\infty (k+1) r^k \\ &\le (1-r)^2 \sum_{k=0}^{N-1} (k+1)|\sigma_k - L| r^k \ + \epsilon. \end{aligned}$$

Since N is fixed, it follows that

$$\limsup_{r\to 1-} |\sum_{k=0}^\infty a_k r^k - L| \le \epsilon.$$

Taking the limit of this inequality as $\epsilon \to 0$, we conclude that $\sum_{k=0}^\infty a_k r^k \to L$ as $r \to 1-$.

c) By Fejér's Theorem, $\sigma_N f \to f$ uniformly. The estimates in part b) can be made uniform if σ_k converges uniformly. Thus it follows that Sf is uniformly Abel summable to f.

d) If $\sum_{k=0}^\infty a_k$ does not converge, then $\sum_{k=0}^\infty a_k = \infty$. Thus given $M > 0$, choose N so large that $S_N \ge M$. Then it follows from part a) that

$$\sum_{k=0}^\infty a_k r^k \ge (1-r)\sum_{k=N}^\infty S_k r^k \ge (1-r) M \sum_{k=N}^\infty r^k = M r^N.$$

Taking the limit of this inequality as $r \to 1-$, we conclude that $L \geq M$ for all $M > 0$, i.e., $L = \infty$, a contradiction.

14.4.5. a) Fix $h \in \mathbf{R}$ and $k \in \mathbf{N}$. By using the change of variables $u = x + h$, $du = dx$, and a sum angle formula, we have

$$\begin{aligned} a_k(f(x+h)) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+h)\cos kx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)(\cos ku\cos kh + \sin ku \sin kh)\,du \\ &= a_k(f)\cos kh + b_k(f)\sin kh. \end{aligned}$$

Similarly, $a_k(f(x-h)) = a_k(f)\cos kh - b_k(f)\sin kh$. Thus $a_k(f(x+h) - f(x-h)) = 2b_k(f)\sin kh$ for $k \in \mathbf{N}$.

In the same way, we can prove that $a_0(f(x+h) - f(x-h)) = 0$ and $b_k(f(x+h) - f(x-h)) = -2a_k(f)\sin kh$ for $k \in \mathbf{N}$. Hence it follows from Parseval's Identity that

$$4\sum_{k=1}^{\infty}(a_k^2(f) + b_k^2(f))\sin^2 kh = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+h) - f(x-h)|^2\,dx$$

for each $h \in \mathbf{R}$.

b) Since $\sin^2 hx$ is increasing on $[2^{n-1}, 2^n]$,

$$\sin^2 kh = \sin^2\frac{k\pi}{2^{n+1}} \geq \sin^2\frac{\pi}{4} = \frac{1}{2}$$

for $k \in [2^{n-1}, 2^n]$.

c) Let $h = \pi/2^{n+1}$. By part b),

$$\sum_{k=2^{n-1}}^{2^n-1}(a_k^2(f) + b_k^2(f)) \leq 2\sum_{k=2^{n-1}}^{2^n-1}(a_k^2(f) + b_k^2(f))\sin^2 kh.$$

Since f belongs to $Lip\,\alpha$, it follows from part a) that

$$\begin{aligned} \sum_{k=2^{n-1}}^{2^n-1}(a_k^2(f) + b_k^2(f)) &\leq 2\sum_{k=2^{n-1}}^{2^n-1}(a_k^2(f) + b_k^2(f))\sin^2 kh \\ &\leq 2\sum_{n=1}^{\infty}\sum_{k=2^{n-1}}^{2^n-1}(a_k^2(f) + b_k^2(f))\sin^2 kh \\ &= 2\sum_{k=1}^{\infty}(a_k^2(f) + b_k^2(f))\sin^2 kh \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h) - f(x-h)|^2\,dx \\ &\leq \frac{M^2}{2\pi}\int_{-\pi}^{\pi}|h|^{2\alpha}\,dx = M^2|h|^{2\alpha}. \end{aligned}$$

d) By the given inequality and part c),

$$\begin{aligned} \sum_{k=1}^{\infty}(|a_k(f)| + |b_k(f)|) &= \sum_{n=1}^{\infty}\sum_{k=2^{n-1}}^{2^n-1}(|a_k(f)| + |b_k(f)|) \\ &\leq \sum_{n=1}^{\infty}2^{n/2}\left(\sum_{k=2^{n-1}}^{2^n-1}(a_k^2(f) + b_k^2(f))\right)^{1/2} \\ &\leq \frac{M\pi^\alpha}{2^\alpha}\sum_{n=1}^{\infty}2^{n(1/2-\alpha)}. \end{aligned}$$

This last series converges since $\alpha > 1/2$. Therefore, Sf converges absolutely and uniformly by the Weierstrass M-Test.

e) If f is periodic and continuously differentiable, then $|f'(c)| \le M < \infty$ for all $c \in \mathbf{R}$. Hence by the Mean Value Theorem,

$$|f(x+h) - f(x-h)| \le |f(x+h) - f(x)| + |f(x) - f(x-h)| = |h|\,|f'(c_1)| + |h|\,|f'(c_2)| \le 2M|h|$$

for some c_j's and all $h \in \mathbf{R}$. Thus f belongs to $Lip\,1$.

14.4.6. By Theorem 9.49, f is almost everywhere continuous on $[-\pi, \pi]$. Hence by Theorem 14.29, Sf converges to f almost everywhere on $[-\pi, \pi]$.

14.5 Uniqueness.

14.5.1. If $F(x_0)$ is a local minimum, then $F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0) \ge 0$ for all $h \in \mathbf{R}$. Thus $D_2F(x_0) \ge 0$. A similar argument establishes the opposite inequality for local maxima.

14.5.2. Since the coefficients of the second formal integral are dominated by M/k^2, it follows from the Weierstrass M-Test that this series converges uniformly on $\mathbf{R}$.

14.5.3. If $S_N \to f$ and $T_N \to f$ as $N \to \infty$, then $S - T$ is a trigonometric series which converges to zero. Hence by Cantor's Theorem, the coefficients of $S - T$ must be zero, i.e., S and T are the same series.

14.5.4. Let $g(x) = (f(x+) - f(x-))/2$ for each $x \in \mathbf{R}$. By Theorem 14.29, $S - Sg$ converges everywhere to zero. Hence by Cantor's Theorem, $S = Sg$. But f and g differ at at most finitely many points in $[-\pi, \pi]$. Therefore, $Sf = Sg$, i.e., S is the Fourier series of f.

14.5.5. Suppose F is not convex. Then there exist points $a < c < x < d < b$ such that $f(x)$ lies above the chord through $(c, F(c))$ and $(d, F(d))$. Since F is continuous on $[c, d]$, choose $x_0 \in [c, d]$ such that $F(x_0) \ge F(u)$ for all $u \in [c, d]$. Since $f(x)$ lies above the chord, the maximum of F occurs at the endpoints. Hence $x_0 \in (c, d)$. Let $\epsilon > 0$ be so small that $[x_0 - \epsilon, x_0 + \epsilon] \subset (c, d)$. Then

$$\frac{F(x_0 + 2h) + F(x_0 - 2h) - 2F(x_0)}{4h^2} < 0$$

for all $|h| < \epsilon$. Taking the limit of this inequality as $h \to 0$, we conclude that $D_2F(x_0) \le 0$, a contradiction.

These are solutions to the exercises from Chapter 15 of the 3rd edition.

Chapter 15

15.1 Differentiable Forms on R^n.

15.1.1. a)
$$\begin{aligned}3(\,dx+dy)\,dz+2(\,dx+dz)\,dy &= 3\,dx\,dz+3\,dy\,dz+2\,dx\,dy-2\,dy\,dz\\ &= dy\,dz-3\,dz\,dx+2\,dx\,dy.\end{aligned}$$

b)
$$\begin{aligned}(x\,dy-y\,dx)(x\,dz-z\,dy) &= x^2\,dy\,dz-xz\,dy\,dy-xy\,dx\,dz+yz\,dx\,dy\\ &= x^2\,dy\,dz+xy\,dz\,dx+yz\,dx\,dy.\end{aligned}$$

$$\begin{aligned}&(x^2\,dx\,dy-\cos x\,dy\,dz)\,(y^2\,dy+\cos x\,dw)\\ &\qquad-(x^3\,dy\,dz-\sin x\,dy\,dw)(y^3\,dy+\sin x\,dz)\\ &= x^2\cos x\,dx\,dy\,dw-\cos^2 x\,dy\,dz\,dw+\sin^2 x\,dy\,dw\,dz\\ &= -dy\,dz\,dw+x^2\cos x\,dx\,dy\,dw.\end{aligned}$$

15.1.2. a) By definition

$$d\omega=(2x\,dx+0\,dy)\,dy+(0\,dx-2y\,dy)\,dx=(2x+2y)\,dx\,dy.$$

b) By definition

$$\begin{aligned}d\omega &= (y\cos(xy)\,dx+x\cos(xy)\,dy)\,dz\,dw+(-w\sin(zw)\,dz-z\sin(zw)\,dw)\,dx\,dy\\ &= y\cos(xy)\,dx\,dz\,dw+x\cos(xy)\,dy\,dz\,dw-w\sin(zw)\,dx\,dy\,dz-z\sin(zw)\,dx\,dy\,dw.\end{aligned}$$

c) Let $\rho=\sqrt{x^2+y^2}$. By definition,

$$d\omega=\frac{x}{\rho}\,dx\,dy\,dz-\frac{y}{\rho}\,dy\,dx\,dz=\frac{x+y}{\sqrt{x^2+y^2}}\,dx\,dy\,dz.$$

d) By the Product Rule,

$$\begin{aligned}d\omega &= d(e^{xy}\,dz+e^{yz}\,dx)(\sin x\,dy+\cos y\,dx)+(e^{xy}\,dz+e^{yz}\,dx)d(\sin x\,dy+\cos y\,dx)\\ &= (ye^{xy}\,dx\,dz+xe^{xy}\,dy\,dz+ze^{yz}\,dy\,dx+ye^{yz}\,dz\,dx)(\sin x\,dy+\cos y\,dx)\\ &\qquad-(e^{xy}\,dz+e^{yz}\,dx)(\cos x\,dx\,dy-\sin y\,dy\,dx)\\ &= y\sin xe^{xy}\,dx\,dz\,dy+y\sin xe^{yz}\,dz\,dx\,dy+x\cos ye^{xy}\,dy\,dz\,dx-\cos xe^{xy}\,dz\,dx\,dy\\ &= (y\sin xe^{yz}-y\sin xe^{xy}+x\cos ye^{xy}-\cos xe^{xy})\,dx\,dy\,dz.\end{aligned}$$

15.1.3. a) If ω is decomposable, then

$$\omega^2=(f\,dx_{i_1}\dots dx_{i_r})(f\,dx_{i_1}\dots dx_{i_r})=f^2\cdot 0=0.$$

If ω is odd, then $\omega=\sum_{j=1}^N\omega_j$, where each ω_j is odd and decomposable. Thus $\omega_j\omega_k=\omega_j^2=0$ when $j=k$, and by anticommutativity, $\omega_j\omega_k=-\omega_k\omega_j$ when $j\neq k$. Hence,

$$\omega^2=\left(\sum_{j=1}^N\omega_j\right)\left(\sum_{k=1}^N\omega_k\right)=\sum_{\substack{j,k=1\\ j\neq k}}^N\omega_j\omega_k=0.$$

b) If ω is even then $\omega_j^2 = 0$ when $j = k$, and $\omega_j\omega_k = \omega_k\omega_j$ when $j \neq k$. Therefore,

$$\omega^2 = \left(\sum_{j=1}^{N} \omega_j\right)\left(\sum_{k=1}^{N} \omega_k\right) = 2\sum_{\substack{j,k=1\\ j<k}}^{N} \omega_j\omega_k.$$

15.1.4. a) By definition,

$$d\omega = f_x\,dx + f_y\,dy + f_z\,dz = \operatorname{grad} f \cdot (dx, dy, dz).$$

b) By definition,

$$\begin{aligned} d\omega &= (P_x\,dx + P_y\,dy + P_z\,dz)\,dx + (Q_x\,dx + Q_y\,dy + Q_z\,dz)\,dy \\ &\quad + (R_x\,dx + R_y\,dy + R_z\,dz)\,dz \\ &= (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy \\ &= \operatorname{curl} F \cdot (dy\,dz, dz\,dx, dx\,dy), \end{aligned}$$

and

$$d\eta = P_x\,dx\,dy\,dz + Q_y\,dy\,dz\,dx + R_z\,dz\,dx\,dy = \operatorname{div} F \cdot dx\,dy\,dz.$$

15.1.5. a) By the Fundamental Theorem of Differential Transforms,

$$\phi^*(\omega) = (P \circ \phi) \cdot \phi_1' + (Q \circ \phi) \cdot \phi_2' = (F \circ \phi) \cdot \phi'.$$

Therefore,

$$\int_C F \cdot T\,ds = \int_I F(\phi(t)) \cdot \phi'(t)\,dt = \int_I \phi^*(\omega).$$

b) By the Fundamental Theorem of Differential Transforms and the proof of Theorem 13.36,

$$\phi^*(\eta) = (P \circ \phi)\frac{\partial(\phi_2, \phi_3)}{\partial(u, v)} + (Q \circ \phi)\frac{\partial(\phi_3, \phi_1)}{\partial(u, v)} + (Q \circ \phi)\frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} = (F \circ \phi) \cdot N_\phi.$$

Therefore,

$$\iint_S F \cdot \mathbf{n}\,d\sigma = \int_E F(\phi(u, v)) \cdot N_\phi(u, v)\,d(u, v) = \iint_E \phi^*(\eta).$$

15.2 Differentiable Manifolds.

15.2.1. a) Let V_α be open in M and set $V = \cup_{\alpha \in A} V_\alpha$. Let (U, g) be a chart of M. Then

$$g(V \cap U) = \bigcup_{\alpha \in A} g(V_\alpha \cap U)$$

is open. Hence V is open in M by definition.

b) Let V_j be open in M and set $V = \cap_{j=1}^{N} V_j$. Let (U, g) be a chart of M. Then

$$g(V \cap U) = \bigcap_{j=1}^{N} g(V_j \cap U)$$

is open. Hence V is open in M by definition.

15.2.2. The reflexive and symmetric properties are obvious by definition. The transitive property is easy to see since the composition of $\mathcal{C}^p$ functions is a $\mathcal{C}^p$ function, and

$$\Delta_{\phi \circ \psi} = (\Delta_\phi \circ \psi) \cdot \Delta_\psi.$$

For example, if $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{O}$ are $\mathcal{C}^p$ compatible atlases of M, if (U, g) is a chart of $\mathcal{A}$, (V, h) is a chart of $\mathcal{B}$, and (Ω, σ) is a chart of $\mathcal{O}$, then $g \circ \sigma^{-1} = g \circ h^{-1} \circ h \circ \sigma^{-1}$ is a $\mathcal{C}^p$ function and

$$\Delta_{g \circ \sigma^{-1}} = (\Delta_{g \circ h^{-1}} \circ \sigma^{-1}) \cdot \Delta_{h \circ \sigma^{-1}} > 0.$$

15.2.3. Given a manifold $(M,\mathcal{A})$ with boundary, let

$$(V,h) \in \mathcal{B} := \{(V,h) : V \cap \partial M \neq \emptyset\}.$$

Since $h : V \to \mathcal{H}$ for some half space $\mathcal{H}$ "in variable j", define $\widetilde{h} : \partial\mathcal{H} \cap V \to \mathbf{R}^{n-1}$ by

$$\widetilde{h}(x_1,\dots,x_m) = (h_1(\mathbf{x}),\dots,\widehat{h_j(\mathbf{x})},\dots,h_n(\mathbf{x})).$$

Then $\widetilde{h}$ takes $\partial M \cap V$ onto an open subset of $\mathbf{R}^{n-1}$ and $\widetilde{h} \circ \widetilde{g}^{-1}$ is as smooth as $h \circ g^{-1}$. Thus ∂M is an $n-1$-dimensional manifold.

15.2.4. a) Notice that σ is a $\mathcal{C}^\infty$ function and $\Delta_\sigma > 0$. Thus

$$(\sigma \circ h) \circ g^{-1} = \sigma \circ (h \circ g^{-1}) \quad \text{and} \quad g \circ (\sigma \circ h)^{-1} = (g \circ h^{-1}) \circ \sigma^{-1}$$

are $\mathcal{C}^p$ when $h \circ g^{-1}$ and $g \circ h^{-1}$ are, and the Jacobians

$$\Delta_{(\sigma\circ h)\circ g^{-1}} = \Delta_\sigma \Delta_{h\circ g^{-1}} \quad \text{and} \quad \Delta_{g\circ(\sigma\circ h)^{-1}} = \Delta_{g\circ h^{-1}}\Delta_{\sigma^{-1}}$$

are positive when $\mathcal{A}$ is oriented. Thus $\mathcal{B}$ is orientation compatible with $\mathcal{A}$.

b) Let $(U,g) \in \mathcal{A}$ be a chart at $\mathbf{x}$. Since $g(U)$ is an open set containing $h(\mathbf{x})$, we can choose a $\delta > 0$ such that $B_\delta(h(\mathbf{x})) \subset g(U)$. Let $\sigma(\mathbf{y}) = (\mathbf{y} - g(\mathbf{x}))/\delta$, $h = \sigma \circ g$, and $V = h^{-1}(B_\delta(h(\mathbf{x})))$. Then σ takes $B_\delta(g(\mathbf{x}))$ onto $B_1(0)$, and by part a), (V,h) belongs to $\mathcal{A}$. Thus (V,h) is a chart in $\mathcal{A}$ which satisfies $h(V) = B_1(0)$ and $h(\mathbf{x}) = 0$.

15.2.5. a) Suppose (U,g) and (V,h) are charts from different atlases of M. Then

$$f \circ g^{-1} = (f \circ h^{-1}) \circ h \circ g^{-1}$$

is a $\mathcal{C}^p$ function if and only if $f \circ h^{-1}$ is $\mathcal{C}^p$. Since these two atlases are compatible, the definition of $\mathcal{C}^p$ functions on M does not change from one atlas to another.

b) If $f : M \to \mathbf{R}^k$ is $\mathcal{C}^p$ and $G : \mathbf{R}^k \to \mathbf{R}^\ell$ is $\mathcal{C}^p$, then so is

$$(G \circ f) \circ h^{-1} = G \circ (f \circ h^{-1})$$

for every chart (h,V) of M. Thus $G \circ f$ is $\mathcal{C}^p$ on M by definition.

c) Since $h \circ g^{-1}$ is $\mathcal{C}^\infty$ for each chart (U,g) of M, it follows from definition that h is a $\mathcal{C}^\infty$ function on M for each chart (V,h) of M.

15.2.6. Let $U = \{\mathbf{x} : x_1 > -a\}$, $V = \{\mathbf{x} : x_1 < a\}$,

$$g(x_1,\dots,x_n) = \left(\frac{x_2}{a+x_1},\dots,\frac{x_n}{a+x_1}\right) \quad \text{and} \quad h(x_1,\dots,x_n) = \left(\frac{x_2}{a-x_1},\dots,\frac{x_n}{a-x_1}\right).$$

Then (U,g), (V,h) are $(n-1)$–dimensional charts which cover the sphere. Notice that

$$g(U\cap V) = h(U \cap V) = \{\mathbf{u} : \mathbf{u} = (x_2,\dots,x_n) \neq 0\}$$

is open. Also notice that

$$\sum_{k=1}^n |g_k(\mathbf{x})|^2 = \frac{x_2^2 + \cdots + x_n^2}{(a+x_1)^2} = \frac{a^2 - x_1^2}{(a+x_1)^2} = \frac{a-x_1}{a+x_1}.$$

Hence for each j,

$$h_j(\mathbf{x}) = \frac{x_j}{a-x_1} = \frac{x_j}{a+x_1}\frac{a+x_1}{a-x_1} = \frac{g_j(\mathbf{x})}{\sum_{k=1}^n |g_k(\mathbf{x})|^2}.$$

It follows that

$$h \circ g^{-1}(\mathbf{u}) = \frac{\mathbf{u}}{\|\mathbf{u}\|^2}.$$

In particular, the transitions are $\mathcal{C}^p$ on $g(U \cap V)$.

15.3 Stokes's Theorem on Manifolds.

15.3.1. Since $d(x^3\,dy\,dz\,dw + y^2\,dx\,dz\,dw) = (3x^2 - 2y)\,dx\,dy\,dz\,dw$, we have by Stokes's Theorem and spherical coordinates that

$$\begin{aligned}
&\int_{\partial B_a(0,0,0,0)} (x^3\,dy\,dz\,dw + y^2\,dx\,dz\,dw) \\
&\quad = \int_{B_a(0,0,0,0)} (3x^2 - 2y)\,d(x,y,z,w) \\
&\quad = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{a} (3\rho^2\cos^2\varphi - 2\rho\sin\varphi\cos\psi)\rho^3\sin^2\varphi\sin\psi\,d\rho\,d\varphi\,d\psi\,d\theta \\
&\quad = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{a} (3\rho^5\sin^2\varphi\cos^2\varphi\sin\psi - 2\rho^4\sin^3\varphi\cos\psi\sin\psi)\,d\rho\,d\varphi\,d\psi\,d\theta \\
&\quad = 2\pi\left(\frac{a^6}{2}\int_0^{\pi}\sin^2\varphi\cos^2\varphi\,d\varphi\int_0^{\pi}\sin\psi\,d\psi - \frac{2a^5}{5}\int_0^{\pi}\sin^3\varphi\,d\varphi\int_0^{\pi}\cos\psi\sin\psi\,d\psi\right) \\
&\quad = 2\pi a^6\int_0^{\pi}\sin^2\varphi\cos^2\varphi\,d\varphi = \frac{a^6\pi^2}{4}.
\end{aligned}$$

15.3.2. Since

$$d(\sum_{j=1}^n x_j^2\,dx_1\ldots\widehat{dx_j}\ldots dx_n) = 2\sum_{j=1}^n(-1)^{j-1}x_j\,dx_1\ldots dx_n$$

we have by Stokes's Theorem that

$$\int_{\partial Q}\omega = 2\sum_{j=1}^n\int_Q(-1)^{j-1}x_j\,dx_1\ldots dx_n = \sum_{j=1}^n(-1)^{j-1}a_1\ldots\widehat{a_j}\ldots a_n\cdot a_j^2$$

is 1 if n is odd, and 0 if n is even.

15.3.3. Since

$$d(\sum_{j=1}^n dx_1\ldots\widehat{dx_j}\ldots dx_n) = \sum_{j=1}^n(-1)^{j-1}\,dx_1\ldots dx_n = \begin{cases} dx_1\ldots dx_n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}$$

we have by Stokes's Theorem that

$$\int_{\partial E}\omega = \sum_{j=1}^n\int_E(-1)^{j-1}\,dx_1\ldots dx_n = \sum_{j=1}^n(-1)^{j-1}\mathrm{Vol}\,(E).$$

In particular, the integral is $\mathrm{Vol}\,(E)$ if n is odd and 0 if n is even.

15.3.4. By Stokes's Theorem, the Product Rule, the Poincaré Lemma, and the substitution $d\eta = \omega$ we have

$$\int_{\partial M}\eta\omega = \int_M d(\eta\omega) = \int_M d\eta\cdot\omega + (-1)^r\eta d\omega = \int_M d\eta\cdot\omega = \int_M\omega^2.$$