

Applications

II

Boundary-Value problems in other coordinate systems.

: problems in Polar Coordinates

Laplacian in Polar Coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

these equations convert the two-dimensional
Laplacian of the function u ,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \text{ into polar coordinates}$$

using the Chain Rule (see the pdf-file) we

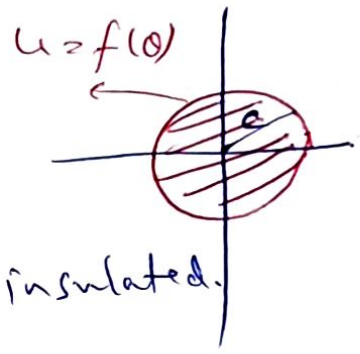
$$\text{get } \Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0} \quad (*)$$

we wish to solve the Laplace's eq (*) for
the steady-state temperature $u(r, \theta)$ in
a circular disk or plate of radius c
when the temperature of the circumference is

(2)

$$u(c, \theta) = f(\theta), \quad 0 < \theta < 2\pi.$$



It is assumed that the two faces of the plate are insulated.

ex. find the temperature $u(r, \theta)$ in a semi-circular plate by solving the following problem.

$$\left\{ \begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= 0, \quad 0 < \theta < \pi, \quad 0 < r < c. \end{aligned} \right. \quad (1)$$

$$u(c, \theta) = A, \quad 0 < \theta < \pi. \quad (2)$$

$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad 0 < r < c, \quad \text{where} \quad (3)$$

A is constant.

Sol. let $u(r, \theta) = R(r) \Theta(\theta)$

Substituting in (1), gives

$$R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0.$$

Divide by $\frac{1}{r^2} R \Theta$ and separating variables:

$$\frac{r^2 R'' + r R'}{R} = - \frac{\Theta''}{\Theta} = \lambda.$$

(3)

$$r^2 R'' + r R' - \lambda R = 0 \quad (4)$$

$$\Theta'' + \lambda \Theta = 0 \quad (5)$$

The homog. BC's give

$$u(r, 0) = 0 \Rightarrow \Theta(0) = 0$$

$$u(r, \pi) = 0 \Rightarrow \Theta(\pi) = 0$$

Now, consider the SLP:

$$\begin{cases} \Theta'' + \lambda \Theta = 0 \\ \Theta(0) = \Theta(\pi) = 0 \end{cases} \quad (6)$$

Case I: $\lambda = 0 \Rightarrow \Theta = 0$ trivial مستحيل

Case II: $\lambda = -\alpha^2 < 0 \Rightarrow \Theta = 0$ trivial

Case III: $\lambda = \alpha^2 > 0$

$$\Rightarrow \Theta(\theta) = C_1 \cos \alpha \theta + C_2 \sin \alpha \theta$$

$$\Theta(0) = C_1 = 0$$

$$\Theta(\pi) = C_2 \sin \alpha \pi = 0$$

$$\text{Let } C_2 \neq 0 \Rightarrow \alpha \pi = n\pi, n=1, 2, \dots$$

$$\Rightarrow \boxed{\alpha_n = n}, \quad \boxed{\lambda_n = n^2}$$

$$\therefore \boxed{\Theta_n(\theta) = C_2 \sin(n\theta), \quad n=1, 2, \dots}$$

Back to (4) with $\lambda_n = n^2$:

$$r^2 R'' + rR' - n^2 R = 0 \quad \text{which is Cauchy-Euler eq.}$$

the aux. eq. is $m^2 + (1-1)m - n^2 = 0$.
 $\Rightarrow m = \pm n$.

$$R_n(r) = c_3 r^n + c_4 r^{-n}$$

$$ax^2y'' + bxy' + cy = 0$$

the aux. eq.
 $am^2 + (b-a)m + c = 0$

Since we expect a solution u of the problem to be bounded at $r=0$, then we must take $c_4 = 0$ since r^{-n} is unbdd at $r=0$.

$$\therefore R_n(r) = c_3 r^n$$

therefore, $u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)$.

now, we use $u(c, \theta) = A$:

$$u(c, \theta) = \sum_{n=1}^{\infty} A_n c^n \sin(n\theta) = A$$

is Fourier-sine series

$$\Rightarrow A_n c^n = \frac{2}{\pi} \int_0^{\pi} A \cdot \sin(n\theta) d\theta$$

$$\Rightarrow A_n = \frac{2A}{\pi c^n} \left[\frac{1 - (-1)^n}{n} \right]$$

$$\Rightarrow u(r, \theta) = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \left(\frac{r}{c}\right)^n \sin(n\theta).$$

Ex 2 (H.W) Find the temperature $u(r, \theta)$ in the circular plate by solving the following problem:

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < 2\pi, 0 < r < 1 \\ u(1, \theta) = 5, \quad 0 < \theta < 2\pi \\ u(r, 0) = 0, \quad u(r, 2\pi) = 0, \quad 0 < r < 1. \end{array} \right.$$

Hint: See example 1 in the pdf file related to this section (14.1)

[6]

Problems in Cylindrical Coordinates

Radial Symmetry the two-dimensional heat and wave equations

$$k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} \quad \text{and}$$

$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2} \quad \text{expressed}$$

in polar coordinates are, in turn,

$$k \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) = \frac{\partial u}{\partial t} \quad \text{and}$$

$$a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) = \frac{\partial^2 u}{\partial t^2},$$

where $u = u(r, \theta, t)$.

Remark. To solve a BVP involving either of these equations by separation of variables we must define $u = R(r) \Theta(\theta) T(t)$. This assumption leads to a multiple infinite series. We shall consider the simpler case

when u is independent of $\theta \Rightarrow \frac{\partial^2 u}{\partial \theta^2} = 0$
 this is called radial symmetry and
 this case still important. Hence in the
 last wave & heat equations the solution
 $u = u(r, t)$.

Ex. Use separation of variables method to
 solve the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 2, \quad t > 0,$$

subject to the boundary conditions

$$\begin{cases} u(2, t) = 0, \quad t > 0 \\ u(r, 0) = 0, \quad 0 < r < 2 \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = \frac{3}{2}, \quad 0 < r < 2 \end{cases}$$

Also, $u(r, t)$ is bounded at $r=0$.

Solution. Let $u(r, t) = R(r)T(t)$. We find

$$R''T + \frac{1}{r}R'T = RT'' \Leftrightarrow$$

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$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T''}{T} = -\lambda \Leftrightarrow$$

$$rR'' + R' + \lambda rR = 0, \quad T'' + \lambda T = 0$$

$$u(z, t) = 0 \Rightarrow R(z) = 0$$

$$u(r, 0) = 0 \Rightarrow T(0) = 0. \text{ Thus, we have}$$

$$r^2 R'' + rR' + \lambda r^2 R = 0, \quad R(z) = 0$$

$$T'' + \lambda T = 0, \quad T(0) = 0$$

$$\text{Let } \lambda = \alpha^2. \text{ then } r^2 R'' + rR' + (\alpha^2 r^2)R = 0$$

is parametric Bessel eq. with $n=0$

$$\Rightarrow R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

we know $Y_0(\alpha r) \rightarrow -\infty$ as $r \rightarrow 0^+$, So,
we must ^{take} $c_2 = 0$ (since u is bdd).

$$\therefore R(r) = c_1 J_0(\alpha r)$$

$$\text{and } R(z) = c_1 J_0(2\alpha) = 0$$

$$\text{put } c_1 \neq 0 \Rightarrow J_0(2\alpha) = 0$$

$\Rightarrow \alpha_i$'s are the nonzero values such that

$$J_0(2\alpha_i) = 0 \Rightarrow \boxed{R = J_0(\alpha_i r)} \quad (9)$$

$$\boxed{\lambda = \alpha_i^2} \cdot T'' + \alpha_i^2 T = 0$$

$$\Rightarrow T_i = C_3 \cos(\alpha_i t) + C_4 \sin(\alpha_i t)$$

$$0 = T(0) = C_3 \Rightarrow \boxed{C_3 = 0}$$

$$\therefore \boxed{T_i = C_4 \sin(\alpha_i t)}$$

$$\text{Thus, } \boxed{u(r, t) = \sum_{i=1}^{\infty} A_i \sin(\alpha_i t) J_0(\alpha_i r)}$$

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{\infty} \alpha_i A_i \cos(\alpha_i t) J_0(\alpha_i r)$$

$$\text{Using } \left. \frac{\partial u}{\partial t} \right|_{t=0} = \frac{3}{2} \Rightarrow \boxed{\frac{3}{2} = \sum_{i=1}^{\infty} \alpha_i A_i J_0(\alpha_i r)}$$

here we have Bessel-Fourier Series

$$\text{case (i) } \underline{J_0(2\alpha) = 0}$$

$$C_i = \alpha_i A_i = \frac{2}{2^2 J_1^2(2\alpha_i)} \int_0^2 \frac{3}{2} \cdot r J_0(\alpha_i r) dr$$

$$\text{let } x = \alpha_i r$$

$$dx = \alpha_i dr$$

(10)

$$\alpha_i A_i = \frac{3}{4 J_1^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{x}{\alpha_i} J_0(x) \frac{dx}{\alpha_i}$$

$$=$$

$$= \frac{3}{4 \alpha_i^2 J_1^2(2\alpha_i)} \int_0^{2\alpha_i} x' J_0(x) dx$$

$$= \frac{3}{4 \alpha_i^2 J_1^2(2\alpha_i)} \left[x J_1(x) \right]_0^{2\alpha_i}$$

$$= \frac{3}{4 \alpha_i^2 J_1^2(2\alpha_i)} \cdot 2\alpha_i J_1(2\alpha_i)$$

$$\Rightarrow \alpha_i A_i = \frac{3}{2\alpha_i J_1(2\alpha_i)}$$

$$\text{or } A_i = \frac{3}{2\alpha_i^2 J_1(2\alpha_i)}$$

Hence,

$$u(r,t) = \frac{3}{2} \sum_{i=1}^{\infty} \frac{J_0(\alpha_i r)}{\alpha_i^2 J_1(2\alpha_i)} \sin(\alpha_i t)$$

(11)

(H.w) Consider a vibrating circular membrane governed by the problem:

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, & 0 < r < 1, t > 0 \\ u(1, t) = 0, & t > 0 \\ u(r, 0) = 0, \quad u_t(r, 0) = 1, & 0 < r < 1. \end{cases}$$

Use a separation of variables to find an expression for $u(r, t)$.

Ans.

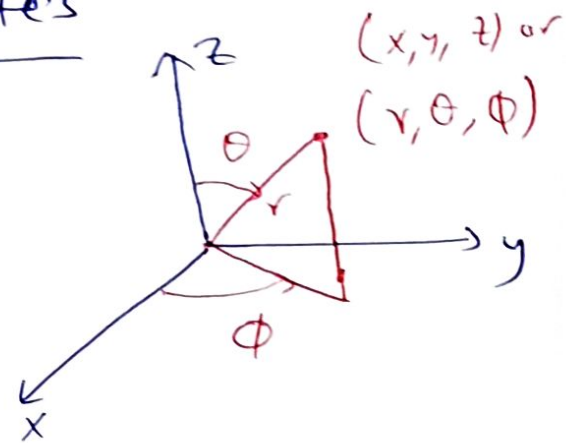
$$u(r, t) = 2 \sum_{i=1}^{\infty} \frac{\sin(\alpha_i t) J_0(\alpha_i r)}{\alpha_i^2 J_1(\alpha_i)}$$

(12)

problems in spherical coordinates

We are going to consider problems involving the heat, wave and Laplace's eq. in spherical coordinates

Laplacian in spherical coordinates



$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

By using these eqs, you can show that the Laplacian $\nabla^2 u$ in spherical coordinate system is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta}$$

In this section, we consider the problem where u is indep. of ϕ , i.e., $u = u(r, \theta)$.

(13)

Ex. Find the steady-state temperature $u(r, \theta)$ in a sphere of radius 2 by solving the problem

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \\ 0 < r < 2, 0 < \theta < \pi \\ u(2, \theta) = 1 - 2 \cos \theta, 0 < \theta < \pi. \end{cases}$$

Solution. Let $u(r, \theta) = R(r) \Theta(\theta)$. Then

$$R'' \Theta + \frac{2}{r} R' \Theta + \frac{1}{r^2} R \Theta'' + \frac{\cot \theta}{r^2} R \Theta' = 0$$

$$\Leftrightarrow \frac{r^2 R'' + 2r R'}{R} = - \frac{\Theta'' + \cot \theta \Theta'}{\Theta} = \lambda$$

$$\Rightarrow \boxed{\Theta'' + (\cot \theta) \Theta' + \lambda \Theta = 0} \quad \text{--- (1) and}$$

$$\boxed{r^2 R'' + 2r R' - \lambda R = 0} \quad \text{--- (2)}$$

Let $x = \cos \theta$ in eq (1) and using the chain rule, we get

$$\boxed{(1-x^2) \Theta''(x) - 2x \Theta'(x) + \lambda \Theta(x) = 0}$$

This eq. is Legendre eq. with $\lambda = n(n+1)$
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 $n = 0, 1, 2, \dots$

(14)

$$\Rightarrow \boxed{\Theta_n(x) = P_n(x)} ; \boxed{\Theta_n(\theta) = P_n(\cos \theta)}$$

Back to (2): $r^2 R'' + 2rR' - n(n+1)R = 0$ is Cauchy-Euler eq. the aux. eq. is

$$m^2 + (2-1)m - n(n+1) = 0$$

$$\Rightarrow m^2 + m - n(n+1) = 0$$

$$(m-n)(m+(n+1)) = 0$$

$$\Rightarrow \boxed{m_1 = n}, \boxed{m_2 = -(n+1)}$$

$$\boxed{R_n(r) = C_1 r^n + C_2 r^{-(n+1)}}$$

$u(r, \theta)$ is bounded $\Rightarrow C_2 = 0$ ($r^{-(n+1)}$ is unbounded at $r=0$)

$$\therefore \boxed{R_n(r) = C_1 r^n}$$

$$\Rightarrow \boxed{u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta)}$$

$$u(2, \theta) = 1 - 2 \cos \theta \Rightarrow$$

$$F(\theta) \boxed{1 - 2 \cos \theta} = \sum_{n=0}^{\infty} \boxed{A_n (2)^n} \overset{C_n}{P_n(\cos \theta)}$$

We use here the alternative form of

(15)

$$2^n A_n = \frac{2^{n+1}}{2} \int_0^\pi (1-2\cos\theta) P_n(\cos\theta) \sin\theta d\theta$$

$$\Rightarrow A_n = \frac{2^{n+1}}{2^{n+1}} \int_0^\pi P_0(\cos\theta) P_n(\cos\theta) \sin\theta d\theta$$

$$- \frac{2^{n+1}}{2^n} \int_0^\pi P_1(\cos\theta) P_n(\cos\theta) \sin\theta d\theta.$$

[here we use $1 = P_0(\cos\theta)$, $P_1(\cos\theta) = \cos\theta$]

By orthogonality $A_n = 0$, $n \neq 0$, $n \neq 1$.

$$A_0 = \frac{1}{2} \int_0^\pi P_0(\cos\theta) P_0(\cos\theta) \sin\theta d\theta - 0$$

$$= \frac{1}{2} \int_0^\pi 1^2 \sin\theta d\theta = \frac{1}{2} \cos\theta \Big|_0^\pi = 1.$$

$$A_1 = 0 - \frac{3}{2} \int_0^\pi P_1(\cos\theta) P_1(\cos\theta) \sin\theta d\theta$$

$$= -\frac{3}{2} \int_0^\pi \cos\theta \cdot \cos\theta \sin\theta d\theta.$$

$$= -\frac{3}{2} \int_1^{-1} x^2 \sin\theta dx \quad \text{let } x = \cos\theta$$

$$= \frac{3}{2} \int_1^{-1} x^2 dx = \frac{3}{2} \left. \frac{x^3}{3} \right|_1^{-1} = \frac{3}{2} \left(-\frac{2}{3} \right) = -1.$$

$$\therefore u(r, \theta) = A_0 r^0 P_0(\cos \theta) + A_1 r^1 P_1(\cos \theta) + \sum_{n=2}^{\infty} A_n r^n P_n(\cos \theta)$$

$$= 1 \cdot 1 + -1 r \cos \theta$$

$$\Rightarrow \boxed{u(r, \theta) = 1 - r \cos \theta}$$

(H.w)

Ex. Find the steady-state temperature $u(r, \theta)$ in a sphere of unit radius by solving the problem

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \\ \quad \quad \quad 0 < r < 1, \quad 0 < \theta < \pi \\ u(1, \theta) = \cos^2 \theta. \end{cases}$$

Ans: $u(r, \theta) = \frac{1}{3} + \frac{1}{3} r^2 (3 \cos^2 \theta - 1).$