

Chapter Five

5.1+5.2

Best Fit

given $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$ to Find the best Fitting Curve
i.e

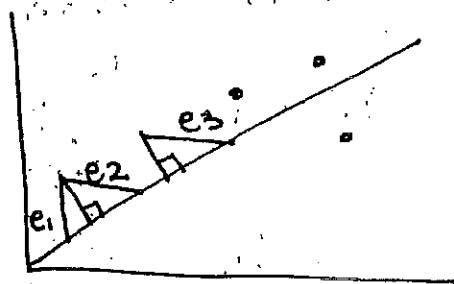
the curve with smallest distance to the given point

$$\text{if } e_k = f(x_k) - y_k$$

$$\text{max error } E_\infty(f) = \|f\|_\infty = \max_{0 \leq k \leq n} |e_k|$$

$$\text{Average error} = E_1(f) = \|f\|_1 = (\sum |e_k|) / n$$

$$\text{Root Mean Square error} = E_2(f) = \|f\|_2 = (\sum |e_k|^2 / n)^{1/2}$$



Example 5.1

Compare the max error, Average error and RMS error for the linear approximation $f(x) = -1.6x + 8.6$ to the data $(-1, 10), (0, 9), (1, 7), (2, 5), (3, 4), (4, 3), (5, 0), (6, -1)$

x_k	y_k	$f(x_k)$	$ e_k $	e_k^2
-1	10	10.2	0.2	0.04
0	9	8.6	0.4	0.16
1	7	7	0	0
2	5	5.4	0.4	0.16
3	4	3.8	0.2	0.04
4	3	2.2	0.8	0.64
5	0	0.6	0.6	0.36
6	-1	-1	0	0

$$\begin{aligned}\sum e_k &= 2.6 \\ E_\infty(f) &= 0.8 \\ E_1(f) &= \frac{\sum |e_k|}{n} \\ &= \frac{2.6}{8} \\ &= 0.325 \\ \sum e_k^2 &= 1.4 \\ E_2(f) &= \left(\frac{\sum e_k^2}{n} \right)^{1/2} \\ &= 0.42\end{aligned}$$

- to Find the best Fitting curve we need to minimize the least square error (RMS)

$$E_2(f) = \left(\frac{\sum_{k=1}^n |f(x_k) - y_k|^2}{n} \right)^{1/2}$$

$$n E_2^2(f) = \sum_{k=1}^n (f(x_k) - y_k)^2$$

$$E(f) = \left(\sum_{k=1}^n (f(x_k) - y_k)^2 \right)^{1/2}$$

1. To Find The best Fitting line $f(x) = Ax + B$

$$E(A, B) = \sum_{k=1}^n |Ax_k + B - y_k|^2$$

$$\frac{dE}{dA} = \sum_{k=1}^n 2 |Ax_k + B - y_k| \cdot x_k = 0 \quad \dots (1)$$

$$\frac{dE}{dB} = \sum_{k=1}^n 2 |Ax_k + B - y_k| \cdot 1 = 0 \quad \dots (2)$$

$$(1) \Rightarrow A \sum_{k=1}^n x_k^2 + B \sum_{k=1}^n x_k = \sum_{k=1}^n y_k x_k \quad \dots$$

$$(2) \Rightarrow A \sum_{k=1}^n x_k + nB = \sum_{k=1}^n y_k \quad \dots \rightarrow \text{Normal equations}$$

Example:-

Find the best Fitting line $F(x) = Ax + B$ For the data

$(-1, 10) (0, 9) (1, 7) (2, 5) (3, 4) (4, 3) (5, 0) (6, -1)$

<u>x_k</u>	<u>y_k</u>	<u>x_k^2</u>	<u>$x_k y_k$</u>
-1	10	1	-10
0	9	0	0
1	7	1	7
2	5	4	10
3	4	9	12
4	3	16	12
5	0	25	0
6	-1	36	-6
Σ 20	37	92	25

$$92A + 20B = 25$$

$$20A + 8B = 37$$

$$A = \frac{\begin{vmatrix} 25 & 20 \\ 37 & 8 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \approx -1.61$$

$$B = \frac{\begin{vmatrix} 92 & 25 \\ 20 & 37 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \approx 8.64$$

Example

For the following Data Find the best curve of the Form

$$y = Ax^2$$

$$E(A) = \sum_{k=1}^n (Ax_k^2 - y_k)^2$$

$$\frac{dE}{dA} = 2 \sum_{k=1}^n (Ax_k^2 - y_k) \cdot x_k^2 = 0$$

$$A = \frac{\sum_{k=1}^n y_k x_k^2}{\sum_{k=1}^n x_k^4}$$

التكملة
مثل
السابق

$$A = \frac{85}{2276} = 0.037346$$

Example

Find the best Fitting Parrabola $P(x) = Ax^2 + Bx + C$

$$E(A, B, C) = \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k]^2$$

$$\frac{dE}{dA} = 0 = 2 \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k] \cdot x_k^2$$

$$\frac{dE}{dB} = 0 = 2 \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k] \cdot x_k$$

$$\frac{dE}{dC} = 0 = 2 \sum_{k=1}^n [(Ax_k^2 + Bx_k + C) - y_k] \cdot 1$$

5.2

linearization

$$f(x) \rightarrow Ax+B$$

Example:-

Find the best Fitting curve of the form $f(x) = C e^{Dx}$ for the following table. (0, 1.5), (1, 2.5), (2, 3.5), (3, 5), (4, 7.5).

$$y = C e^{Dx}$$

$$\ln y = \ln C + Dx$$

$$\ln y = Dx + \ln C$$

$$Y = AX + B$$

$$Y = \ln y$$

$$X = x$$

$$D = A$$

$$C = e^B$$

x_k	y_k	X_k	$Y_k = \ln y_k$	x_k^2	$X_k Y_k$
0	1.5	0	0.405465	0	0
1	2.5	1	0.916291	1	0.916291
2	3.5	2	1.252788	4	2.505576
3	5	3	1.609438	9	4.828314
4	7.5	4	2.014903	16	8.059612
Σ		10	6.198860	30	16.309742

table 5.4 From the text book

$$30A + 10B = 16.309742$$

$$10A + 5B = 6.198860$$

$$A = \frac{\begin{vmatrix} 16.309742 & 10 \\ 6.198860 & 5 \end{vmatrix}}{\begin{vmatrix} 30 & 10 \\ 10 & 5 \end{vmatrix}} = 0.3912023$$

$$B = 0.457367$$

$$D = A \approx 0.39$$

$$C = e^B = e^{0.457367} \approx 1.58$$

$$f(x) = 1.58 e^{0.39x} = C e^{Dx}$$

Examples:-

1. $y = \frac{D}{x+c}$

$$\frac{1}{y} = \frac{x}{D} + \frac{c}{D}$$

$$Y = AX + B$$

$$Y = \frac{1}{y}, X = x, A = \frac{1}{D}, B = \frac{c}{D}$$

$\downarrow$ $\downarrow$

$$D = \frac{1}{A} \quad c = BD$$

$$y = \frac{D}{x+c}$$

$$y = \frac{1}{x} D + \frac{D}{c} \cdot \frac{1}{x}$$

$A = \frac{D}{c} \cdot \frac{1}{x}$

$A = \frac{D}{c} \cdot \frac{1}{x}$

$A = \frac{D}{c} \cdot \frac{1}{x}$

2. $y = \frac{x}{A+Bx}$

$$\frac{1}{y} = \frac{A}{x} + B$$

$$Y = AX + B$$

$$Y = \frac{1}{y}$$

$$X = \frac{1}{x}$$

$$A = A$$

$$B = B$$

3. $y = ce^{-Dx}$

$$\frac{y}{x} = ce^{-Dx}$$

$$\ln\left(\frac{y}{x}\right) = \ln c - Dx$$

$$\ln\left(\frac{y}{x}\right) = -Dx + \ln c$$

$$Y = AX + B$$

$$Y = \ln\left(\frac{y}{x}\right)$$

$$X = x$$

$$A = -D \rightarrow D = -A$$

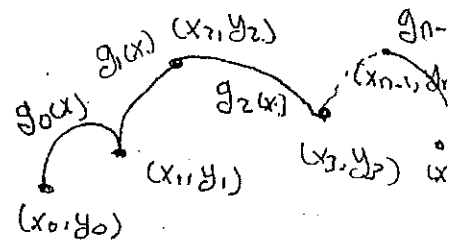
$$B = \ln c \rightarrow c = e^B$$

SECTION 5.3

Cubic spline

given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

The cubic spline is a function $g(x)$ such that it is a cubic polynomial between every two nodes and its of this form $g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$ on $[x_i, x_{i+1}]$ for $i=0, 1, \dots, n-1$ and that satisfies



$$1. g_i(x_i) = y_i \quad i=0, 1, \dots, n-1, \quad g_{n-1}(x_n) = y_n$$

$(n+1)$ conditions.

$$2. g_i(x_{i+1}) = g_{i+1}(x_{i+1}) \quad i=0, \dots, n-2$$

$(n-1)$ conditions

$$g_0(x_1) = g_1(x_1)$$

$$g_1(x_2) = g_2(x_2)$$

$$g_{n-2}(x_{n-1}) = g_{n-1}(x_{n-1})$$

$$3. g_i'(x_{i+1}) = g_{i+1}'(x_{i+1}) \quad i=0, \dots, n-2 \rightarrow (n-1) \text{ condition}$$

$$4. g_i''(x_{i+1}) = g_{i+1}''(x_{i+1}) \quad i=0, \dots, n-2 \rightarrow (n-1) \text{ condition}$$

so we have $(n+1) + (3(n-1)) = 4n-2$ conditions.

→ eq Since $g_i(x_i) = y_i \Rightarrow$
 $d_i = y_i$

→ equation (2) gives

$$y_{i+1} = g_{i+1}(x_{i+1}) = a_i(x_{i+1}-x_i)^3 + b_i(x_{i+1}-x_i)^2 + c_i(x_{i+1}-x_i) + d_i$$

$$= a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i$$

$$\text{where } h_i = (x_{i+1} - x_i)$$

$$g_i'(x) = 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i$$

$$g_c''(x) = 6a_c(x-x_c) + 2b_c$$

if $\boxed{S_c = g_c''(x_c)}$

substitute $\rightarrow \boxed{b_c = \frac{S_c}{2}}$

Using the same equation.

$$g_c''(x_{i+1}) = 6a_c(x_{i+1}-x_c) + 2b_c$$

$$S_{i+1} = 6a_c h_i + S_c$$

$$\boxed{a_c = \frac{S_{i+1} - S_c}{6h_i}}$$

substitute in *

$$C_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_c + h_i S_{i+1}}{6}$$

Considering the equation

$g_c''(x_i) = g_{i+1}''(x_i)$ we get

$$h_{i-1} S_{i-1} + 2(h_{i-1} + 2h_i) S_c + h_i S_{i+1} = 6 [f(x_i, x_{i+1}) - f(x_{i-1}, x_i)]$$

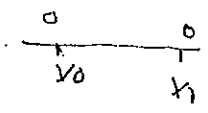
for $i=1, \dots, n-1$

$$\begin{bmatrix} S_0 & S_1 & S_2 & S_3 & \dots & S_{n-2} & S_{n-1} & S_n \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \dots & 0 & 0 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ & & & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} & 0 \\ & & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ \vdots \\ \vdots \\ S_n \end{bmatrix} = 6 \begin{bmatrix} f[x_1, x_2] - f[x_0, x_1] \\ f[x_2, x_3] - f[x_1, x_2] \\ f[x_3, x_4] - f[x_2, x_3] \\ \vdots \\ f[x_n, x_{n+1}] - f[x_{n-1}, x_n] \end{bmatrix}$$

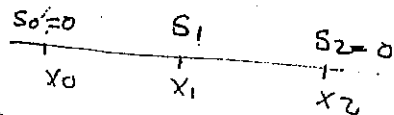
$(n-1)$ equations & $(n+1)$ unknown, we need two more condition.

1. Natural Spline $S_0 = S_n = 0$

we get $(n-1)$ equations with $(n-1)$ unknowns

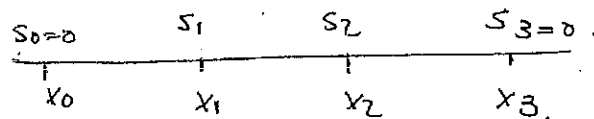
When $n=1$ لا يوجد matrix لأن يوجد 0 
 ما بين نقطتين لا يوجد معادلة واحدة.
 لا يوجد matrix لأن يوجد 0

When $n=2$



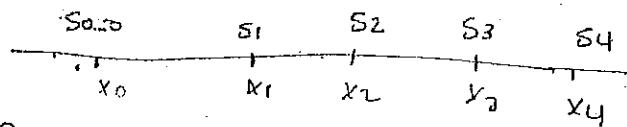
$$2(h_0 + h_1) S_1 = 6 [f[x_1, x_2] - f[x_0, x_1]]$$

When $n=3$



$$\begin{bmatrix} 2(h_0 + h_1) & h_1 \\ h_1 & 2(h_1 + h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} f[x_1, x_2] - f[x_0, x_1] \\ f[x_2, x_3] - f[x_1, x_2] \end{bmatrix}$$

When $n=4$



$$\begin{bmatrix} 2(h_0 + h_1) & h_1 & 0 \\ h_1 & 2(h_1 + h_2) & h_2 \\ 0 & h_2 & 2(h_2 + h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} f[x_1, x_2] - f[x_0, x_1] \\ f[x_2, x_3] - f[x_1, x_2] \\ f[x_3, x_4] - f[x_2, x_3] \end{bmatrix}$$

Example

Find the natural spline for the given table.

x_i	y_i
0	2
1	4.4366
1.5	6.7134
2.25	13.9130

$$h_0=1, h_1=0.5, h_2=0.75$$

$$f[0,1] = 2.4366$$

$$f[1,1.5] = 4.5536$$

$$f[1.5,2.5] = 9.5995$$

$$\begin{bmatrix} 2(1.5) & 0.5 \\ 0.5 & 2(1.25) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4.5536 - 2.4366 \\ 9.5995 - 4.5536 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0.5 \\ 0.5 & 2.5 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 12.7020 \\ 30.2754 \end{bmatrix}$$

$$s_1 = 2.292, s_2 = 11.6618$$

$$a_i = \frac{s_{i+1} - s_i}{6h_i}$$

$$a_0 = \frac{s_1 - s_0}{6h_0} = \frac{2.292 - 0}{6(1)} = 0.3820$$

$$a_1 = ??$$

$$a_2 = ??$$

$$b_i = \frac{s_i}{2}$$

$$b_0 = \frac{s_0}{2} = 0$$

$$b_1 = \frac{s_1}{2} = 1.146$$

$$c_i = \dots$$

$$c_0 = 2.0546$$

$$c_1 = 3.2005$$

$$c_2 = 6.6868$$

$$d_i = y_i$$

$$d_0 = 2$$

$$d_1 = 4.4215$$

$$d_2 = 6.7130$$

$$g_0(x) = 0.3820(x-0)^3 + 0(x-0)^2 + 2.054(x-0) + 2.000 \quad \text{on } [0, 1]$$

$$g_1(x) = 3.1199(x-1)^3 + 1.146(x-1)^2 + 5.205(x-1) + 4.4366 \quad \text{on } [1, 1.5]$$

$$g_2(x) = -2.5895(x-1.5)^3 + 5.8259(x-1.5)^2 + 6.6866(x-1.5) + 6.7134 \quad \text{on } [1.5, 2]$$

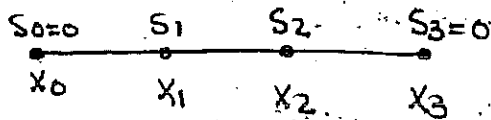
$$f(0.66) = 3.4659 \quad \text{Exact} = 3.34343$$

$$f(1.75) = 8.7087 \quad \text{Exact} = 8.4467$$

Natural Spline

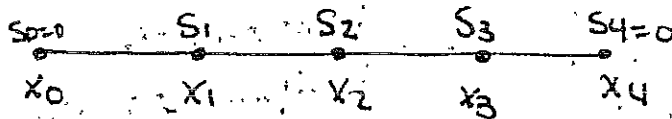
$$S_0 = 0 \quad S_n = 0$$

$$\underline{n=3}$$



$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} F(x_1, x_2) - F(x_0, x_1) \\ F(x_2, x_3) - F(x_1, x_2) \end{bmatrix}$$

$$\underline{n=4}$$



$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_1, x_2) - F(x_0, x_1) \\ F(x_2, x_3) - F(x_1, x_2) \\ F(x_3, x_4) - F(x_2, x_3) \end{bmatrix}$$

Cubed Spline

$$F'(x_0) = A$$

$$F'(x_n) = B$$

$$(1) \rightarrow 2h_0 s_0 + h_0 s_1 = 6 [F(x_0, x_1) - A]$$

$$(2) \rightarrow h_{n-1} s_{n-1} + 2h_{n-1} s_n = 6 [B - F(x_{n-1}, x_n)]$$

$$n=1$$

$$\begin{bmatrix} 2h_0 & h_0 \\ h_0 & 2h_0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} = 6 \begin{bmatrix} F(x_0, x_1) - A \\ B - F(x_0, x_1) \end{bmatrix}$$

$$g_0(x) = a_0 (x-x_0)^3 + b_0 (x-x_0)^2 + c_0 (x-x_0) + d_0$$

نحوض في النقاط وكذلك المشتقة عند الأطراف و بالتالي
نعرف المعاملات

$$n=2$$

$$\begin{bmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = 6 \begin{bmatrix} F(x_0, x_1) - A \\ F(x_1, x_2) - F(x_0, x_1) \\ B - F(x_1, x_2) \end{bmatrix}$$

$$g(x) = \begin{cases} a_0 (x-x_0)^3 + b_0 (x-x_0)^2 + c_0 (x-x_0) + d_0 & x_0 \leq x \leq x_1 \\ a_1 (x-x_1)^3 + b_1 (x-x_1)^2 + c_1 (x-x_1) + d_1 & x_1 \leq x \leq x_2 \end{cases}$$

$$f'(x_0) = A$$

$$f'(x_1) = B$$

$$f(x_0) = d_0 = y_0$$

$$f(x_1) = d_1 = y_1$$

$$g_0(x_1) = g_1(x_1)$$

$$g_0'(x_1) = g_1'(x_1)$$

$$g_0''(x_1) = g_1''(x_1)$$

$$g_1(x_2) = y_2$$

$$g_0(x_0) = y_0$$

$$g_1(x_1) = y_1$$

$$g_1(x_2) = y_2$$

For $n=3$

$$\begin{bmatrix} 2h_0 & h_0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \\ 0 & 0 & h_2 & 2h_2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_0, x_1) - A \\ F(x_1, x_2) - F(x_0, x_1) \\ F(x_2, x_3) - F(x_1, x_2) \\ B - F(x_2, x_3) \end{bmatrix}$$

Q2) Clamped spline

$(0,0) (1,1) (2,2)$

$S'(0)=1, S'(2)=1$

$$g(x) = \begin{cases} g_0(x) = a_0(x-0)^3 + b_0(x-0)^2 + c_0(x-0) + d_0 & \text{on } [0,1] \\ g_1(x) = a_1(x-1)^3 + b_1(x-1)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$$

$$g(x) = \begin{cases} g_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0 & \text{on } [0,1] \\ g_1(x) = a_1(x-1)^3 + b_1(x-2)^2 + c_1(x-1) + d_1 & \text{on } [1,2] \end{cases}$$

$$g_0(0) = d_0 = 0$$

$$g_1(2) = d_1 = 1$$

$$g'_0(x) = 3a_0x^2 + 2b_0x + c_0$$

$$g'_0(0) = c_0 = 1$$

$$g'_1(x) = 3a_1(x-1)^2 + 2b_1(x-1) + c_1$$

$$= 3a_1 + 2b_1 + c_1 = 2$$

$$g'_0(1) = g'_1(1)$$

$$3a_0x^2 + 2b_0x + c_0 = 3a_1(x-1)^2 + 2b_1(x-1) + c_1$$

$$3a_0 + 2b_0 + 1 = 3a_1(0) + 2b_1(0) + c_1$$

$$g_0''(1) = g_1''(1)$$

$$6a_0x + 2b_0 = 6a_1(x-1)^2 + 2b_1$$

$$6a_0 + 2b_0 = 2b_1$$

$$g_1(2) = 2$$

$$a_1 + b_1 + c_1 = 2 - 1$$

$$a_1 + b_1 + c_1 = 1$$

$$g_0'(1) = g_1'(1)$$

$$a_0 + b_0 + c_0 + d_0 = d_1$$

$$a_0 + b_0 + c_0 = 1$$

0	0	//	//1
1	1	1	//1
1	2	1	0

$$h_0 = 1$$

$$h_1 = 1$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$S_0 = 0$$

$$S_1 = 0$$

$$S_2 = 0$$

Q15 Cubic-poly [a,b]

$\begin{matrix} + & + \\ a & b \end{matrix}$

$f(x)$ its own clamped spline but it cannot be its own Free spline ?

$$f(a) =$$

Cubic $S_0, S_1 \neq \text{Zero}$

its not natural

المشتقة الثانية $\neq 0$
المشتقة الأولى $\neq 0$

$$a_3 \neq 0$$

$$g(x) = a_3x^3 + b_2x^2 + c_2x + d_2$$

$$g'(x) = 3a_3x^2 + 2b_2x + c_2$$

$$g''(x) = 6a_3x + 2b_2$$

مفرد عند الطرفين

$$= a_3/b_2 \quad \text{ولكن } a_3 = 0$$

4 - Unknowns

4 equ (condition)

$$y_0 = a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0$$

أربع نقاط مشتركة وبالتالي poly نفسياً

