

14.7 Extreme Values and Saddle Points:

Def: Let $f(x,y)$ be defined on a region R containing the point (a,b) , then:

$z = f(x,y)$ surface $z = f(x,y)$

1. $f(a,b)$ is a local maximum value of f if

$f(a,b) \geq f(x,y)$, $\forall (x,y) \in D$ in an open disk centered at (a,b) .

2. $f(a,b)$ is a local minimum value of f if

$f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b) .

Note: at the local maxima & local minima, the tangent planes, when they exist, are horizontal.

Def: An Interior point of the domain of a function

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$f(x,y)$ where $f_x(a,b) = f_y(a,b) = 0$

or where one or both of $f_x(a,b)$ and $f_y(a,b)$

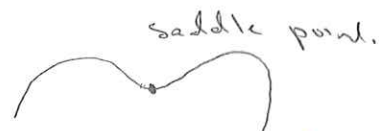
DNE is called a critical point.

If (a,b) is max. then $f_x(a,b) = f_y(a,b) = 0$

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Def: A differential function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) ^① there are domain points (x,y) where $f(x,y) > f(a,b)$ ^② and domain points (x,y) where $f(x,y) < f(a,b)$.

The corresponding point $(a,b, f(a,b))$ on the surface $Z = f(x,y)$ is called a saddle point of the surface



Thm: First derivative Test for local extreme values:

If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain, and if the first derivatives exist, then $f_x(a,b) = 0$ & $f_y(a,b) = 0$

Relative Extreme

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Thm: Second Derivative Test for local extreme values:

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Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$ critical points then:

1.) f has a local maximum at (a, b) if

$$f_{xx} < 0 \quad \text{and} \quad f_{xx} f_{yy} - f_{xy}^2 > 0 \quad \text{at } (a, b).$$

2.) f has a local minimum at (a, b) if

$$f_{xx} > 0 \quad \text{and} \quad f_{xx} f_{yy} - f_{xy}^2 > 0 \quad \text{at } (a, b).$$

3.) f has a saddle point at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$
at (a, b)

4.) The test is inconclusive at (a, b) if

$$f_{xx} f_{yy} - f_{xy}^2 = 0 \quad \text{at } (a, b)$$

Note: The expression $f_{xx} f_{yy} - f_{xy}^2$ is called

the discriminant or Hessian of f .

we can write it as: $f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

Example: Find local maximum and local minimum
& saddle point for:

$$\boxed{1} \quad f(x, y) = x^2 - 4xy + y^2 + 6y + 2.$$

$$f_x = 2x - 4y = 0 \quad \Rightarrow \quad x = 2y$$

$$f_y = -4x + 2y + 6 = 0 \quad \Rightarrow \quad y = 2x - 3$$

$$\text{then } (x, y) = (2, 1).$$

$$f_{xx} = 2, \quad f_{xx}(2, 1) = 2 > 0$$

$$f_{yy} = 2, \quad f_{yy}(2, 1) = 2$$

$$f_{xy} = -4, \quad f_{xy}(2, 1) = -4$$

$$\therefore \left(f_{xx} f_{yy} - f_{xy}^2 \right) \Big|_{(2,1)} = (2)(2) - (-4)^2 = 4 - 16 = -12 < 0$$

then f has a saddle point at $(2, 1)$

$$\boxed{2} \quad f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

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$$f_x = 3x^2 + 6x = 0 \quad \Rightarrow \quad 3x(x+2) = 0$$

$$\text{then } x = 0 \quad \& \quad x = -2.$$

$$f_y = 3y^2 - 6y = 0 \quad \Rightarrow \quad 3y(y-2) = 0$$

$$\text{then } y = 0 \quad \& \quad y = 2.$$

Then critical points: $(0,0), (0,2), (-2,0), (-2,2)$.

Case (1): $(0,0)$

$$f_{xx} = 6x + 6 \Rightarrow f_{xx}(0,0) = 6 > 0$$

$$f_{yy} = 6y - 6 \Rightarrow f_{yy}(0,0) = -6$$

$$f_{xy} = 0 \quad , \quad \text{then}$$

$$f_{xx} f_{yy} - f_{xy}^2 = (6)(-6) - 0 = -36 < 0$$

Hence f has a saddle point at $(0,0)$

Case (2): $(0,2)$

$$f_{xx}(0,2) = 6 > 0$$

$$f_{yy}(0,2) = 6 > 0$$

$$f_{xy}(0,2) = 0$$

$$(f_{xx} f_{yy} - f_{xy}^2) \Big|_{(0,2)} = (6)(6) - 0 = 36 > 0$$

Hence f has a local minimum at $(0,2)$ which

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$$\text{is } f(0,2) = \boxed{-12}$$

Case 3: $(-2, 0)$

$$f_{xx}(-2, 0) = -6 < 0, \quad f_{yy}(-2, 0) = -6, \quad f_{xy} = 0$$

$$(f_{xx} f_{yy} - f_{xy}^2) \Big|_{(-2, 0)} = (-6)(-6) - 0 = 36 > 0$$

then f has a local maximum at $(-2, 0)$ which is

$$f(-2, 0) = \boxed{-4}$$

Case 4: $(-2, 2)$

$$f_{xx}(-2, 2) = -6 < 0, \quad f_{yy}(-2, 2) = 6, \quad f_{xy}(-2, 2) = 0$$

$$(f_{xx} f_{yy} - f_{xy}^2) \Big|_{(-2, 2)} = (-6)(6) - 0 = -36 < 0$$

Hence f has a saddle point at $(-2, 2)$.

Absolute Maxima and Minima on closed Bounded Regions:

To find the absolute extreme of a continuous function

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$f(x, y)$ on a closed and bounded region R

we have three steps:

1) Find all interior points of R where f has a local maxima and minima, then evaluate f at these points. (These are critical points).

2) Find the Boundary points of R where f has local maxima and minima, then evaluate f at these points.

3) Find Absolute max & Absolute min of (1) & (2).

Example: Find the absolute maxima and minima of

$$f(x, y) = x^2 + xy + y^2 - 6x + 2 \quad \text{on}$$

the rectangle: $0 \leq x \leq 5$ & $-3 \leq y \leq 0$.

Boundary points :

II $OC \Rightarrow y = 0$

$$f(x, 0) = x^2 - 6x + 2$$

$$f'_x = 2x - 6 = 0 \Rightarrow x = 3$$

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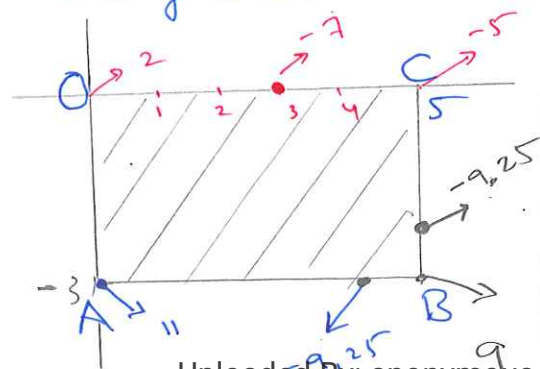
critical point

so we have $(3, 0)$

$$f(3, 0) = 9 - 18 + 2 = -7$$

$$f(0, 0) = 2$$

$$f(5, 0) = -3$$



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$$\boxed{2} \quad CB \Rightarrow x = 5$$

$$f(5, y) = 25 + 5y + y^2 - 30 + 2 = y^2 + 5y - 3$$

$$f_y = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$$

$$\Rightarrow (5, -\frac{5}{2})$$

$$f(5, -\frac{5}{2}) = -\frac{37}{4}$$

$$f(5, -3) = -9$$

$$\boxed{3} \quad AB \Rightarrow y = -3$$

$$f(x, -3) = x^2 - 3x + 9 - 6x + 2 = x^2 - 9x + 11$$

$$f_x = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$$

$$\Rightarrow (\frac{9}{2}, -3)$$

$$f(\frac{9}{2}, -3) = -9.25$$

$$f(0, -3) = 0 - 0 + 11 = 11.$$

$$\boxed{4} \quad OA \Rightarrow x = 0$$

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$$f(0, y) = y^2 + 2$$

$$f_y = 2y = 0 \Rightarrow y = 0$$

$$\Rightarrow (0, 0)$$

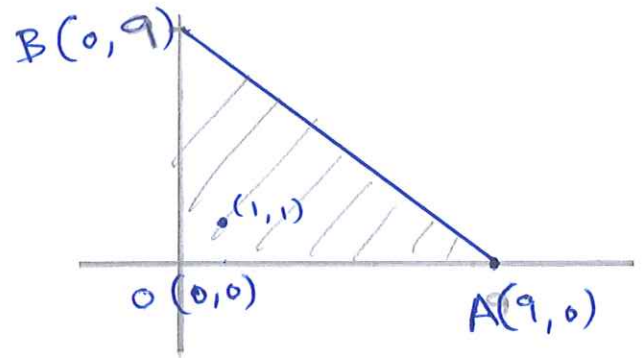
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Example: Find the absolute Max. and Min. values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant

bounded by the lines $x=0, y=0, y=9-x$



sol: Interior points:

$$f_x = 2 - 2x = 0 \Rightarrow \boxed{x=1}$$

$$f_y = -2y + 2 = 0 \Rightarrow \boxed{y=1}$$

$\Rightarrow$ critical point ^① $\boxed{(1,1)} \in R$

Boundary points:

$\overline{OA} : y=0, 0 \leq x \leq 9.$

$$f(x,0) = 2 + 2x - x^2$$

$$f'(x) = 2 - 2x = 0 \Rightarrow \boxed{x=1}$$

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$\therefore$ we have critical point ^② $\boxed{(1,0)}$ on the Boundary

& ^③ $\boxed{(0,0)}$ & ^④ $\boxed{(9,0)}$ $\rightarrow$ end points

Now, on $\overline{OB} : x=0, 0 \leq y \leq 9.$

$$f(0, y) = 2y - y^2$$

$$f'(y) = 2 - 2y = 0 \Rightarrow \boxed{y = 1}$$

$\therefore$ critical point on the Boundary $\overline{OB}$ is $\boxed{(0, 1)}$ ^⑤

& $(0, 0)$ & $\boxed{(0, 9)}$ ^⑥ $\rightarrow$ end points

Now, on $\overline{AB}$: $y = 9 - x$, $0 \leq x \leq 9$

$x = 0 \Rightarrow y = 9 \Rightarrow (0, 9)$ repeated

$x = 9 \Rightarrow y = 0 \Rightarrow (9, 0)$ repeated

$$\text{Now, } f(x, y) = f(x, 9 - x)$$

$$= 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2$$

$$= -61 + 18x - 2x^2$$

$$\therefore f'(x, 9 - x) = 18 - 4x = 0 \Rightarrow x = \frac{9}{2} \text{ \& } y = \frac{9}{2}$$

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$\therefore$ critical point ^⑦ $\left(\frac{9}{2}, \frac{9}{2}\right)$.

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$\left(\frac{3}{4}\right)(89)$

Summary:

$$f(1,1) = 4$$

$$f(0,0) = 2$$

$$f(9,0) = -61$$

$$f(0,9) = -61$$

$$f(1,0) = 3$$

$$f(0,1) = 3$$

$$f\left(\frac{9}{2}, \frac{9}{2}\right) = \frac{-41}{2}$$

∴ The Absolute Max. is 4 occurs at (1,1)

& the absolute Min is -61 occurs at
(0,9) & (9,0).

5 Interior points :

$$\left. \begin{aligned} f_x &= 2x + y - 6 = 0 \Rightarrow 2x + y = 6 \\ f_y &= x + 2y = 0 \Rightarrow 2x + 4y = 0 \end{aligned} \right\} \text{ solve}$$

so we have the point $(4, -2)$

$$f(4, -2) = 16 - 8 + 4 - 24 + 2 = -10$$

Hence $f(x, y)$ has Absolute maxima at $(0, -3)$

$$\text{which is } f(0, -3) = 11$$

& $f(x, y)$ has Absolute minima at $(4, -2)$ which is -10

Example (53) Find three numbers whose ~~there~~ sum is 9

and whose sum of squares is a minimum.

$$\text{Sol: } x + y + z = 9 \Rightarrow z = 9 - x - y$$

$$\text{Let } S(x, y, z) = x^2 + y^2 + z^2 = x^2 + y^2 + (9 - x - y)^2$$

$$\left. \begin{aligned} S_x &= 2x + 2(9 - x - y)(-1) = 0 \\ S_y &= 2y + 2(9 - x - y)(-1) = 0 \end{aligned} \right\} \text{ solve these two equations}$$

$$\Rightarrow x = 3, y = 3 \quad \& \quad z = 3$$

$$S_{xx}(3, 3, 3) = 4 > 0, \quad S_{yy}(3, 3, 3) = 4$$

$$\& \quad S_{xy}(3, 3, 3) = 2 \Rightarrow S_{xx} S_{yy} - S_{xy}^2 = 12 > 0$$

Hence S has a local minimum at $(3, 3, 3)$ $(\frac{1}{3})(90)$

Q23) (14.7) $f(x, y) = y \sin x$

Sol: $f_x = y \cos x = 0 \quad \dots (1)$

$f_y = \sin x = 0 \quad \dots (2)$

$E_q(2) \Rightarrow x = 0, \pm\pi, \pm 2\pi, \dots$

$x = n\pi, \quad n = 0, \pm 1, \pm 2, \dots$

$E_q(1) \Rightarrow y \underbrace{\cos(n\pi)}_{0 \neq} = 0 \Rightarrow y(-1)^n = 0 \Rightarrow y = 0$

$\therefore$ Critical points are $(n\pi, 0), \quad n = 0, \pm 1, \pm 2, \dots$

$f_{xx} = -y \sin x, \quad f_{yy} = 0, \quad f_{xy} = \cos x$

$\Delta(x, y) = f_{xx} f_{yy} - f_{xy}^2 = -\cos^2 x$

$\therefore \Delta(n\pi, 0) = -\cos^2(n\pi) = -1 < 0$

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$\therefore$ All $(n\pi, 0), \quad n = 0, \pm 1, \pm 2, \dots$

are saddle points

Q24) $f(x, y) = e^{2x} \cos y$

(2/3)(90)

$f_x = \frac{2e^{2x}}{e^{2x}} \cos y \Rightarrow \cos y = 0$
 $f_y = -e^{2x} \sin y \Rightarrow \sin y = 0$

} There is No critical point $\Rightarrow$ No extreme No saddle

Q30)(14.7) Let $f(x, y) = \ln(x+y) + x^2 - y$.

Find the extreme points.

so $D = \{(x, y) : x+y > 0\}$

$$\begin{aligned} f_x &= \frac{1}{x+y} + 2x = 0 \quad \text{--- (1)} \\ f_y &= \frac{1}{x+y} - 1 = 0 \quad \text{--- (2)} \end{aligned} \quad \left. \begin{array}{l} \text{Eq (1)} - \text{Eq (2)} : \\ \Rightarrow \end{array} \right\} \begin{array}{l} 2x+1=0 \\ \boxed{x = -\frac{1}{2}} \end{array}$$

From eq. (2) : $\boxed{y = \frac{3}{2}}$

$\therefore$ critical point is $(-\frac{1}{2}, \frac{3}{2})$

$$f_{xx} = \frac{-1}{(x+y)^2} + 2, \quad f_{yy} = \frac{-1}{(x+y)^2}$$

$$\& f_{xy} = \frac{-1}{(x+y)^2}$$

$$f_{xx}\left(-\frac{1}{2}, \frac{3}{2}\right) = 1, \quad f_{yy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1, \quad f_{xy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1$$

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$$\therefore \Delta\left(-\frac{1}{2}, \frac{3}{2}\right) = f_{xx}f_{yy} - f_{xy}^2 \Big|_{(-\frac{1}{2}, \frac{3}{2})} = -2 < 0$$

$\therefore f$ has a saddle point at $(-\frac{1}{2}, \frac{3}{2})$.

(41) A flat circular plate has the shape of the region

$x^2 + y^2 \leq 1$. The plate, including the boundary where

$x^2 + y^2 = 1$ is heated so that the temperature at (x, y) is

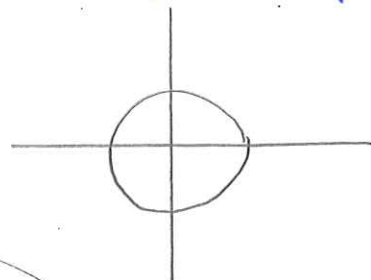
$$T(x, y) = x^2 + 2y^2 - x.$$

Find the temperature of the hottest and coldest points on plate.

Interior:

$$T_x \Big|_{(x,y)} = 2x - 1 = 0, \quad T_y = 4y = 0$$

$$\therefore \underline{x = \frac{1}{2}, y = 0} \text{ with } T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$$



Boundary: $x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2$, substitute in T

$$\text{Then } T(x) = -x^2 - x + 2 \text{ on } -1 \leq x \leq 1$$

$$T'(x) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2} \quad \& \quad y = \pm \frac{\sqrt{3}}{2}$$

$$T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4}, \quad T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$$

$$T(-1, 0) = 2, \quad T(1, 0) = 0$$

The hottest is $\frac{9}{4}$, The coldest is $-\frac{1}{4}$

(45) show that $(0,0)$ is a critical point of

$$f(x,y) = x^2 + kxy + y^2 \text{ no matter what value the}$$

Constant k has (Hint, $k=0$ & $k \neq 0$)

• If $k=0$, then $f(x,y) = x^2 + y^2$

$$f_x = 2x = 0 \quad \& \quad f_y = 2y = 0 \Rightarrow (0,0) \text{ is the only critical point}$$

• If $k \neq 0$, then:

$$f_x = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x \quad (*)$$

$$f_y = kx + 2y = 0 \Rightarrow kx + 2\left(-\frac{2}{k}x\right) = 0$$

$$\therefore kx - \frac{4x}{k} = 0 \Rightarrow \left(k - \frac{4}{k}\right)x = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad k = \pm 2$$

$$\Rightarrow y = \left(-\frac{2}{k}\right)(0) = 0, \text{ or } k = \pm 2 \text{ from } (*) \Rightarrow y = \pm x \Rightarrow (0,0) \text{ is critical point}$$

(56) Find the minimum distance from $z = \sqrt{x^2 + y^2}$ to $(-6, 4, 0)$.

$$d = \sqrt{(x+6)^2 + (y-4)^2 + (z-0)^2}$$

$$(x,y,z) \text{ to } (-6, 4, 0)$$

$$D(x,y,z) = d^2 = (x+6)^2 + (y-4)^2 + z^2$$

$$= (x+6)^2 + (y-4)^2 + (\sqrt{x^2+y^2})^2$$

$$D = 2x^2 + 2y^2 + 12x - 8y + 52$$

$$D_x = 4x + 12 = 0, \quad D_y = 4y - 8 = 0 \Rightarrow \text{Critical } (-3, 2)$$

$$z = \sqrt{13}$$

$$D_{xx}|_{(-3,2)} = 4, \quad D_{yy}|_{(-3,2)} = 4, \quad D_{xy}|_{(-3,2)} = 0$$

$$d(-3, 2, \sqrt{13}) = \sqrt{26}$$

is the minimum distance

$$\Rightarrow D_{xx} D_{yy} = 16 > 0 \quad \& \quad D_{xx} > 0 \Rightarrow \text{Local minimum } \left(\frac{3}{2}\right)(91)$$

14.8 Lagrange Multipliers.

Constrained Maxima and Minima.

Example: Find the point $P(x, y, z)$ on the plane

$2x + y - z - 5 = 0$ that is closest to the origin $(0, 0, 0)$.

sol: The problem is to find the Min. value of the function

$$|\overline{OP}| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2} \quad \text{subject to } 2x + y - z = 5$$

$z = 2x + y - 5$

1) Find the Min. of $f(x, y, z) = x^2 + y^2 + z^2$

$$h(x, y) = x^2 + y^2 + (2x + y - 5)^2$$

$$h_x = 2x + 2(2x + y - 5)(2) = 0$$

$$\Leftrightarrow 10x + 4y = 20$$

$\Leftrightarrow$

$$5x + 2y = 10 \quad \dots \textcircled{1}$$

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$$h_y = 2y + 2(2x + y - 5)(1) = 0$$

$$\Leftrightarrow 4x + 4y = 10$$

$$2x + 2y = 5 \quad \dots \textcircled{2}$$

$$\text{Solving } ① \text{ \& } ② \Rightarrow \boxed{x = \frac{5}{3}}, \boxed{y = \frac{5}{6}}$$

$\Rightarrow$ The critical point: $(\frac{5}{3}, \frac{5}{6})$

$$h_{xx} = 10, \quad h_{yy} = 4, \quad h_{xy} = 2.$$

$$\text{Discriminant} = h_{xx}h_{yy} - (h_{xy})^2 = 40 - 4 = 36 > 0$$

$$\therefore \left. \begin{array}{l} \text{Discr. } (\frac{5}{3}, \frac{5}{6}) = 36 > 0 \\ h_{xx}(\frac{5}{3}, \frac{5}{6}) = 10 > 0 \end{array} \right\} \Rightarrow h \text{ is Min. at } (\frac{5}{3}, \frac{5}{6})$$

$$Z = 2x + y - 5$$

$$= 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}$$

$\therefore$ The closest point is $P(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6})$

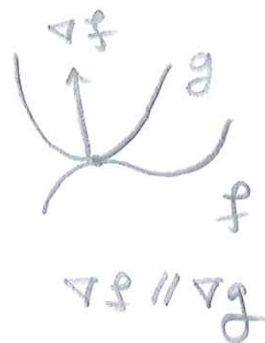
STUDENTS-HUB.com & the closest Distance from $P(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6})$ Uploaded By: anonymous

to the origin is

$$\sqrt{\frac{25}{9} + \frac{25}{36} + \frac{25}{36}} = \frac{5}{\sqrt{6}}$$

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when



$g(x, y, z) = 0$. To find the local Maximum and minimum values of f subject to the Constraint

$g(x, y, z) = 0$ (If these exist), find the value

of x, y, z and λ that simultaneously satisfy

the equations $\nabla f = \lambda \nabla g$ & $g(x, y, z) = 0$.

Example: Find the greatest and smallest values that the function $f(x, y) = xy$ takes on

the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1 \rightarrow$ Constraint.



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So let $f(x, y) = xy$, $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

Let $\nabla f = \lambda \nabla g$

$$x \vec{i} + y \vec{j} = \lambda (x \vec{i} + y \vec{j})$$

$$y \vec{i} + x \vec{j} = \lambda \left(\frac{x}{4} \vec{i} + y \vec{j} \right)$$

$$\Rightarrow \begin{cases} y = \lambda \left(\frac{x}{4} \right) & \dots (1) \\ x = \lambda y & \dots (2) \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 & \dots (3) \end{cases}$$

substitute (2) into (1): $y = \lambda \left(\frac{\lambda y}{4} \right) \Leftrightarrow 4y - \lambda^2 y = 0$

$$\Rightarrow y(4 - \lambda^2) = 0 \Leftrightarrow y = 0 \text{ or } \lambda = \pm 2.$$

Case 1: If $y = 0 \xRightarrow{\text{eq (2)}} x = 0$ $\left(\text{reject : } \frac{0^2}{8} + \frac{0^2}{2} \neq 1 \right)$

Case 2: $\lambda = 2 \xRightarrow{\text{eq (2)}} x = 2y$
 $\xRightarrow{\text{eq (3)}} \frac{x^2}{8} + \frac{y^2}{2} = 1 \Leftrightarrow \frac{(2y)^2}{8} + \frac{y^2}{2} = 1$

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$$\Leftrightarrow \frac{4y^2}{8} + \frac{y^2}{2} = y^2 = 1 \Leftrightarrow y = \pm 1.$$

If $y = +1 \Rightarrow x = 2 \Rightarrow (2, 1)$

If $y = -1 \Rightarrow x = -2 \Rightarrow (-2, 1)$

Case 3: $\lambda = -2 \xRightarrow{\text{eq (2)}} x = -2y$

$\xRightarrow{\text{eq (1)}} \frac{(-2y^2)}{8} + \frac{y^2}{2} = 1$

$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

If $y = 1 \Rightarrow x = -2 \Rightarrow (-2, 1)$

If $y = -1 \Rightarrow x = 2 \Rightarrow (2, -1)$

Therefore: $f(2, 1) = 2$

$f(-2, -1) = 2$

$f(-2, 1) = -2$

$f(2, -1) = -2$

$\therefore$ Absolute Max. value = 2 occurs at

$(2, 1)$ & $(-2, -1)$

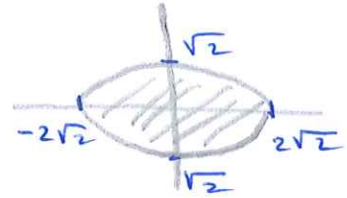
& Absolute Min. value = -2 occurs at

$(-2, 1)$ & $(2, -1)$,

Example: Find the Maximum and Minimum of

$$f(x, y) = xy \text{ on } \frac{x^2}{8} + \frac{y^2}{2} \leq 1.$$

Sol: Interior points:



$$\left. \begin{array}{l} f_x = y = 0 \\ f_y = x = 0 \end{array} \right\} \Rightarrow (0, 0) \in \text{Region since } \frac{0^2}{8} + \frac{0^2}{2} \leq 1.$$

Boundary Points: $\left(\frac{x^2}{8} + \frac{y^2}{2} = 1 \right) \rightarrow$ Lagrange Multiplier.

$$\text{i.e. } \nabla f = \lambda \nabla g$$

From previous example:

" $(2, 1), (-2, -1), (2, -1), (-2, 1)$ " Boundary"

$$\therefore f(0, 0) = 0 \quad (\text{Interior})$$

$$f(2, 1) = 2$$

$$f(-2, -1) = 2$$

$$f(2, -1) = -2$$

$$f(-2, 1) = -2$$

Abs. Max. = 2 occurs at $(2, 1)$ & $(-2, -1)$

Abs. Min = -2 = $(2, -1)$ & $(-2, 1)$.

Example! Find the Max. and Min. of

$f(x, y, z) = x - 2y + 5z$ on the sphere

$$x^2 + y^2 + z^2 = 30.$$

Sol: Let $g(x, y, z) = x^2 + y^2 + z^2 - 30 = 0$

$$\& \text{ let } \nabla f = \lambda \nabla g$$

$$\Rightarrow \vec{i} - 2\vec{j} + 5\vec{k} = \lambda (2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$\Rightarrow \begin{cases} 2x\lambda = 1 & \dots (1) \\ 2y\lambda = -2 & \dots (2) \\ 2z\lambda = 5 & \dots (3) \\ x^2 + y^2 + z^2 = 30 & \dots (4) \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{-1}{\lambda} \\ z = \frac{5}{2\lambda} \end{cases}$$

Substitute x, y, z into eq. (4):

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$$

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$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

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$$\Rightarrow \frac{30}{4\lambda^2} = 30 \Rightarrow \lambda = \pm \frac{1}{2}$$

$$\text{If } \lambda = \frac{1}{2} \Rightarrow x = \frac{1}{2(\frac{1}{2})} = 1, y = \frac{-1}{\frac{1}{2}} = -2$$

$$z = \frac{5}{2(\frac{1}{2})} = 5.$$

$$\therefore P_1(1, -2, 5)$$

$$\text{If } \lambda = -\frac{1}{2}, \text{ then } P_2(-1, 2, -5)$$

$$f(P_1) = f(1, -2, 5) = 1 - 2(-2) + 5(5) = 30$$

$$f(P_2) = f(-1, 2, -5) = -1 - 2(2) + 5(-5) = -30$$

$\therefore$ Absolute Max. = 30 occurs at $(1, -2, 5)$

Absolute Min = -30 occurs at $(-1, 2, -5)$

Example: Find the extreme values of

$$f(x, y, z) = xy + z^2 \text{ subject to the}$$

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$$\text{constraint: } x^2 + y^2 + z^2 = 1$$

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Sol: Let $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

$$\text{Let } \nabla f = \lambda \nabla g.$$

$$\Rightarrow y \vec{i} + x \vec{j} + 2z \vec{k} = \lambda (2x \vec{i} + 2y \vec{j} + 2z \vec{k})$$

$$\Rightarrow \begin{cases} y = 2x\lambda & \dots (1) \\ x = 2y\lambda & \dots (2) \\ 2z = 2z\lambda & \dots (3) \\ x^2 + y^2 + z^2 = 1 & \dots (4) \end{cases}$$

substitute (2) into (1) : $y = 2(2y\lambda)\lambda$

$$\Rightarrow y - 4y\lambda^2 = 0$$

$$\Rightarrow y(1 - 4\lambda^2) = 0$$

$$\therefore y = 0 \quad \text{or} \quad \lambda = \pm \frac{1}{2}$$

Case 1: If $y = 0$ $\xRightarrow{\text{eq. (2)}} x = 0$

$$x = 0, y = 0 \xRightarrow{\text{eq. (4)}} z = \pm 1$$

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$$\therefore P_1(0, 0, 1) \quad \& \quad P_2(0, 0, -1)$$

Case 2: $\lambda = -\frac{1}{2} \xRightarrow{\text{eq. (1) or eq. (2)}} y = -x$

$$\xRightarrow{\text{eq. (3)}} 2z = -z \Rightarrow z = 0$$

$$\stackrel{\text{eq (4)}}{\Rightarrow} x^2 + y^2 + z^2 = 1$$

$$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$$\text{If } x = +\frac{1}{\sqrt{2}}, \text{ then } y = -\frac{1}{\sqrt{2}}, z = 0$$

$$\therefore P_3 \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right)$$

$$\text{If } x = -\frac{1}{\sqrt{2}}, \text{ then } y = \frac{1}{\sqrt{2}}, z = 0$$

$$\therefore P_4 \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\text{Case 3: If } \lambda = \frac{1}{2} \stackrel{\text{eq (1) or eq (2)}}{\Rightarrow} y = x$$

$$\stackrel{\text{eq (3)}}{\Rightarrow} 2z = z \Rightarrow z = 0$$

$$\stackrel{\text{eq (4)}}{\Rightarrow} x^2 + y^2 + z^2 = 1$$

$$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

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$$\text{If } x = +\frac{1}{\sqrt{2}}, \Rightarrow y = +\frac{1}{\sqrt{2}}, z = 0$$

$$P_5 \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\text{If } x = \frac{-1}{\sqrt{2}}, y = \frac{-1}{\sqrt{2}}, z = 0$$

$$\therefore P_6 \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right)$$

$$f(0, 0, 1) = 1$$

$$f(0, 0, -1) = 1$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = -\frac{1}{2}$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2}$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = \frac{1}{2}$$

$\therefore$ Absolute Max. = 1 occurs at $(0, 0, 1)$ & $(0, 0, -1)$

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Absolute Min. = $-\frac{1}{2}$ occurs at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

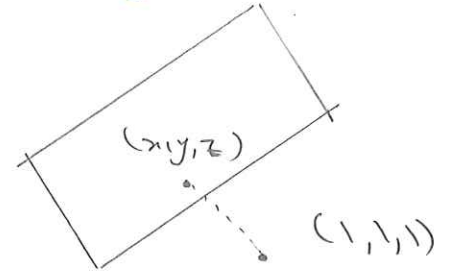
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and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Example: ⁽¹⁷⁾ Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$. $g(x, y, z) =$

We need to find the minimum value of d :

$$d = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$$



$$f(x, y, z) = d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$$

$$\begin{aligned} \nabla f &= f_x \vec{i} + f_y \vec{j} + f_z \vec{k} \\ &= 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k} \end{aligned}$$

$$\nabla g = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\nabla f = \lambda \nabla g \quad \& \quad g(x, y, z) = 0$$

$$\begin{cases} 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k} = \lambda(\vec{i} + 2\vec{j} + 3\vec{k}) \\ x + 2y + 3z = 13 \end{cases}$$

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$$\Rightarrow \begin{cases} 2(x-1) = \lambda & \dots (1) \\ 2(y-1) = 2\lambda & \dots (2) \\ 2(z-1) = 3\lambda & \dots (3) \\ x + 2y + 3z = 13 & \dots (4) \end{cases}$$

• substitute (1) in (2):

$$2(y-1) = 2(2x-1) \Rightarrow y-1 = 2(x-1),$$

$$\therefore y+1 = 2x \Rightarrow x = \frac{y+1}{2}$$

• substitute (2) in (3) $y-1 = x$

$$2(z-1) = 3(y-1) \Rightarrow 2z = 3y-1 \Rightarrow z = \frac{3y-1}{2}$$

but (4): $x + 2y + 3z = 13$

$$\Rightarrow \frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) = 12$$

$$\therefore 7y = 14$$

$$y = 2, \quad x = \frac{3}{2}, \quad z = \frac{5}{2}$$

So the point is $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$.

(14.8) (17) Find the point on $x+2y+3z=13$ closest to $(1,1,1)$.

$$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$$

$$g(x,y,z) = x+2y+3z$$

$$\nabla f = \lambda \nabla g \quad \& \quad x+2y+3z-13=0$$

$$\nabla f = 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k}$$

$$\nabla g = 1\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\therefore 2(x-1) = \lambda, \quad 2(y-1) = 2\lambda, \quad 2(z-1) = 3\lambda$$

$$\therefore 2(x-1) = (y-1) \quad \& \quad 3(x-1) = (z-1)$$

$$\therefore y = 2x - 1 \quad \& \quad z = 3x - 2$$

$$\text{Using: } g(x,y,z) = 13 \Rightarrow$$

$$13 = x + 2(2x-1) + 3(3x-2) \Rightarrow x = \frac{3}{2}, y=2, z=\frac{5}{2}$$

(23) Find the maximum and minimum values of

$$f(x,y,z) = x-2y+5z \text{ on sphere } x^2+y^2+z^2=30$$

$$\nabla f = \vec{i} - 2\vec{j} + 5\vec{k}, \quad \nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\text{STUDENTS-HUB.com} \quad \& \quad -2 = 2\lambda y \quad \& \quad 5 = 2\lambda z$$

$$\& \quad x^2 + y^2 + z^2 = 30 \quad \dots (*)$$

$$\Rightarrow \nabla f = \lambda \nabla g \Rightarrow \vec{i} - 2\vec{j} + 5\vec{k} = \lambda(2x)\vec{i} + \lambda(2y)\vec{j} + \lambda(2z)\vec{k}$$

$$\Rightarrow 1 = 2x\lambda, \quad -2 = 2y\lambda \quad \& \quad 5 = 2z\lambda$$

$$\Rightarrow x = \frac{1}{2\lambda}, \quad y = \frac{-1}{\lambda} = -2x, \quad z = \frac{5}{2\lambda} = 5x \quad \dots \text{substitute in (*)}$$

$$\Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow \boxed{x = \pm 1}$$

$$\Rightarrow f(1, -2, 5) = 30 \text{ is maximum} \quad \& \quad f(-1, 2, -5) = -30 \text{ is minimum}$$

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