

Chapter 6

6.1

Th:- Central difference Formula of order $O(h^2)$ (f_1).

assum that $f \in C^2[a, b]$, and $x-h, x, x+h \in [a, b]$ then

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

furthermore there exists a number $c \in [a, b]$ such that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

where the error then $-\frac{h^2 f^{(3)}(c)}{6}$ is called the truncation error and is denoted by

$$E_{\text{trunc}}(f, h) \neq 0, \text{ i.e. } E_{\text{trunc}}(f, h) = \frac{h^2 f^{(3)}(c)}{6}$$

Let

t	d
0.1	13.21
0.2	20.55
0.3	24.12
0.4	29.79

$$V(0.2) = \frac{d(0.3) - d(0.1)}{2(0.1)} = \frac{24.12 - 13.21}{0.2} = 54.55$$

$$\begin{aligned} \text{error} &= C(0.1)^2 \\ &= C(0.01) \rightarrow \text{error in the 4th digit.} \end{aligned}$$

$$V(0.3) = \frac{d(0.4) - d(0.2)}{0.02}$$

$$V(0.4) = \text{أعرف}$$

$$V(0.1) = \text{أعرف}$$

$$f(x) = \cos x$$

$$f'(0.8) = ??$$

$$h = 0.01$$

$$f'(0.8) \approx \frac{f(0.8+0.01) - f(0.8-0.01)}{2(0.01)} \approx \frac{\cos(0.81) - \cos(0.79)}{0.02}$$

$$\approx \frac{0.689498933 - 0.7303895326}{0.02} = -0.717344160$$

$$\text{Exact} = f'(0.8) = \sin(0.8) = -0.717356091$$

by Theorem $C(h^2)^2 = \text{Error} = C(0.01)^2 = C(0.0001)$

أبغ منازل صحيحة وفقاً للمطلوب

Derivation

Using Taylor expansion at x ,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c_1), \quad c_1 \in (x, x+h)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(c_2), \quad c_2 \in (x-h, x)$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6} (f'''(c_1) - f'''(c_2))$$

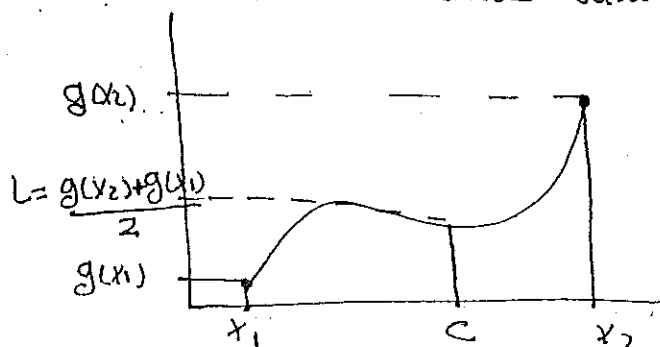
$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6} (2f'''(c)) \quad c \in (c_1, c_2)$$

$$\frac{f(x+h) - f(x-h)}{2h} \approx \frac{h^2 f'''(c)}{6} = f'(x)$$

F_1

Error.

IUP (Intermediate Value property)



$\Rightarrow \exists c \in (x_1, x_2)$ such that $g(c) = \frac{g(x_1) + g(x_2)}{2}$

Section 6.1

Central difference Formula of $O(h^4)$

assume $f \in C^5[a, b]$ and $x-2h, x-h, x+h, x+2h \in [a, b]$ then

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

with error

$$E_{\text{trunc}}(f, h) = \frac{h^4 f^{(5)}(c)}{30} \approx Ch^4$$

Example 1

Let

<u>t</u>	<u>d</u>
0.1	13.25
0.2	18.53
0.3	21.25
0.4	24.30
0.5	27.12

فقط بيوت
استخدام
عند 0.3

$$\begin{aligned} V(0.3) &= \frac{-d(0.5) + 8d(0.4) - 8d(0.2) + d(0.1)}{12(0.1)} \\ &= \frac{-27.12 + 8(24.30) - 8(18.53) + 13}{12} \end{aligned}$$

Example 2

$$f(x) = \cos x$$

$f'(0.8)$ using $h = 0.01$

$$f'(0.8) = \frac{-\cos(0.82) + 8\cos(0.81) - 8\cos(0.79) + \cos(0.78)}{0.12}$$

$$f'(0.8) = -0.717356108$$

Compare to exact $-\sin(0.8) = -0.717356091$

$$\text{error } C(0.01)^4 = C(10^{-8})$$

• Derivation

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) + \frac{h^5}{5!} f^{(5)}(x)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) - \frac{h^5}{5!} f^{(5)}(x)$$

نفر

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!} f'''(x) + \frac{2h^5}{5!} f^{(5)}(x)$$

$$-8(f(x+h) - f(x-h)) = 16hf'(x) + \frac{16h^3}{3!} f'''(x) + \frac{16h^5}{5!} f^{(5)}(x)$$

$$2) \dots f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16h^3}{3!} f'''(x) + \frac{64h^5}{5!} f^{(5)}(x)$$

نفر (1) و (2)

$$-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h) = 12f'(x) - \frac{48h^5}{120} f^{(5)}(x)$$

$$\frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{1}{30} h^4 f^{(5)}(x) = f'(x)$$

F2

Etrac (f, h)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

when h is smaller we get best estimation for f'(x): ?

Example

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f'(1) = e$$

$$f'(1) \approx \frac{f(1+h) - f(1)}{h} = \frac{e^{1+h} - e^1}{h} \xrightarrow{??} e$$

هل هذا صحيح يقرب
من e عندما $h \rightarrow 0$

h	$D_n = e'' - e/h$
0.1	2.858841960
0.01	2.731918700
0.001	2.719642000
0.0001	2.718420000
10^{-5}	2.718300000 → the best h
10^{-6}	2.719000000
10^{-7}	
10^{-10}	00000000

• Notation

$$f(x+h) = y_1 + e_1$$

$$f(x-h) = y_{-1} + e_{-1}$$

$$\vdots$$

$$f(x+kh) = y_k + e_k$$

$$f(x+h) = \cos(0.81) = \underbrace{0.689498433}_{y_1} \text{ (is not exact (have error))}$$

$$= y_1 + e_1$$

$$|e_1| < 0.5 \times 10^{-10}$$

$$< 0.5 \times 10^{-9}$$

$$F_1 = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

$$= \frac{(y_1 + e_1) - (y_{-1} + e_{-1})}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

$$= \underbrace{\frac{y_1 - y_{-1}}{2h}}_{\text{Round off error}} + \underbrace{\frac{e_1 - e_{-1}}{2h}}_{\text{E}_{\text{round}}(f, h)} - \underbrace{\frac{h^2 f^{(3)}(c)}{6}}_{\text{truncation error}}$$

Round off error

truncation error

$E_{\text{round}}(f, h)$

$E_{\text{trunc}}(f, h)$

$$\text{Total error} = E_{\text{tot}}(f, h) = E_{\text{round}}(f, h) + E_{\text{trunc}}(f, h)$$

$$= \frac{e_1 - e_{-1}}{2h} - \frac{h^2 f^{(3)}(c)}{6}$$

$$|E_{\text{tot}}(f, h)| = \left| \frac{e_1 - e_{-1}}{2h} \right| + \left| \frac{h^2 f^{(3)}(c)}{6} \right| \quad \text{if } |e_k| < \epsilon$$

$$\leq \frac{2\epsilon}{2h} + \frac{h^2 M_3}{6}$$

$$M_3 = \max |f^{(3)}(x)|$$

$g(h)$

$$g(h) = \frac{\epsilon}{h} + \frac{h^2}{6} M$$

$$g'(h) = -\frac{\epsilon}{h^2} + \frac{h}{3} M = 0$$

$$\frac{h}{3} M = \frac{\epsilon}{h^2}$$

$$h^3 = \frac{3\epsilon}{M}$$

$$h = \left(\frac{3\epsilon}{M} \right)^{1/3} \text{ best } h$$

• $f(x) = \cos x$, $\epsilon = 0.5 \times 10^{-9}$

$$h = \left(\frac{3 \times 0.5 \times 10^{-9}}{M} \right)^{1/3} = 0.001144714$$

① max for $|f^{(3)}(x)|$

$$h = 0.001 \text{ best } h$$

• Find best h for F_2 .

F_2

$$F_2'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$E_{\text{tot}} = \frac{-e_2 + 8e_1 - 8e_{-1} + e_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

$$|E(f, h)| \leq \frac{|e_2| + 8|e_1| + 8|e_{-1}| + |e_{-2}|}{12h} + \frac{h^4 M}{30}$$

$$M = \max |f^{(5)}(x)| \quad a \leq x \leq b$$

$$\leq \frac{18\epsilon}{12h} + \frac{h^4 M}{30} = \frac{3\epsilon}{2h} + \frac{h^4 M}{30} = g(h)$$

$$|e_k| < \epsilon$$

$$g'(h) = -\frac{3E}{2h^2} + \frac{4h^3 M}{30} = 0$$

$$\frac{2}{15} h^3 M = \frac{3E}{2h^2}$$

$$h^5 = \frac{45E}{4M}$$

$$\text{Optimal } h = \left(\frac{45E}{4M} \right)^{1/5}$$

$$- f(x) = \cos x$$

$$E = 0.5 \times 10^{-9}$$

$$M = 1$$

$$h = \left(\frac{45 \times 0.5 \times 10^{-9}}{4 \times 1} \right)^{1/5} = 0.022 \dots$$

$$\text{Optimal } h = 0.01$$

Section 6.2

High order derivations

$O(h^2)$

$$1. f''(x) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

$$f_k = f(x + kh)$$

$$2. f'''(x) \approx \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3}$$

$O(h^4)$

$$1. f^{(4)}(x) \approx \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2}$$

$$2. f^{(4)}(x) \approx \dots$$

$$3. f^{(4)}(x) \approx \dots$$

حيث المعلومات نحدد فاذا نستخدم

$$f_1 = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x)$$

$$f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x)$$

$$f_1 + f_{-1} = 2f_0 + h^2 f''(x) + \frac{h^4}{12} f^{(4)}(x)$$

where $f_0 = f(x)$

$$\underbrace{\frac{f_1 - 2f_0 + f_{-1}}{h^2}}_{\text{Formula}} - \underbrace{\frac{h^2}{12} f^{(4)}(x)}_{\text{truncation error}} = f''(x)$$

Formula

truncation
error

Best h:-

$$E_{\text{tot}}(f, h) = E_{\text{round}}(f, h) + E_{\text{trunc}}(f, h)$$

$$E_{\text{tot}}(f, h) = \frac{e_1 - 2e_0 + e_{-1}}{h^2} - \frac{h^2}{12} f^{(4)}(c)$$

if $|e_n| < \epsilon$, and $M = \max_{a \leq x \leq b} |f^{(4)}(x)|$

then

$$|E_{\text{tot}}| \leq \frac{4\epsilon}{h^2} + \frac{h^2 M}{12} = g(h)$$

$$g'(h) = -\frac{8\epsilon}{h^3} + \frac{hM}{6} = 0$$

$$\frac{hM}{6} = \frac{8\epsilon}{h^3}$$

$$h^4 = \frac{48\epsilon M}{M}$$

$$h = \left(\frac{48\epsilon}{M}\right)^{1/4}$$

- Example

$$f(x) = \cos x$$

$f''(0.8)$ using $h=0.01$.

$$f''(0.8) \approx \frac{\cos(0.81) - 2\cos(0.8) + \cos(0.79)}{(0.01)^2} \approx -0.696690006$$

$$\text{Exact} = -\cos(0.8) = -0.697067$$

EXAMPLE

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.987480

$$V(0) = ??$$

$$V(0.1) \approx \checkmark$$

$$V(0.2) = \checkmark$$

$$V(0.3) = ??$$

$$a(0) = ??$$

$$a(0.1) = \frac{d(0.2) - 2d(0.1) + d(0.0)}{(0.1)^2}$$

$$= \frac{0.998295 - 2(0.999135) + 0.989992}{0.01}$$

$$a(0.2) = \checkmark = \frac{d(0.3) - 2d(0.2) + d(0.1)}{(0.1)^2}$$

- Forward difference Formula's of $O(h^2)$

$$f'(x) \approx \frac{-3f_0 + 4f_1 - f_2}{2h}$$

$$f''(x) \approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2}$$

- Backward difference Formula's of $O(h^2)$

$$f'(x_0) \approx \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

$$f''(x_0) \approx \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$$

$$f'(x_2) = \frac{3f_2 - 4f_1 + f_0}{2h}$$

$$f''(x_2) \approx \frac{2f_2 - 5f_1 + 4f_0 - f_{-1}}{h^2}$$

Example

$$f(x) = \cos x$$

$$h = 0.01$$

• Forward

$$f'(0.8) = \frac{-3\cos(0.8) + 4\cos(0.81) - \cos(0.82)}{2(0.01)}$$

• Backward

$$f'(0.8) = \frac{3\cos(0.8) - 4\cos(0.79) + \cos(0.78)}{2(0.01)}$$

• Forward

$$f''(0.8) = \frac{2\cos(0.8) - 5\cos(0.81) + 4\cos(0.82) - \cos(0.83)}{(0.01)^2}$$

• Backward

$$f''(0.8) = \frac{2\cos(0.8) - 5\cos(0.79) + 4\cos(0.78) - \cos(0.77)}{(0.01)^2}$$

• Using the table

$$V(0) = \frac{-3d(0) + 4d(0.1) - d(0.2)}{2(0.1)}$$

• Forward استخدام

$$+ V(0.1) = \frac{-3d(0.1) + 4d(0.2) - d(0.3)}{2(0.1)}$$

central

$$V(0.2) \quad \text{central} \quad \text{استخدام}$$

Forward

$$V(0.3) = \frac{3d(0.3) - 4d(0.2) + d(0.1)}{2(0.1)} \quad \text{backward}$$

t	d
0.0	0.989992
0.1	0.999135
0.2	0.998295
0.3	0.998...

$$a(0) \approx \frac{2d(0) - 5d(0.1) + 4d(0.2) - d(0.3)}{(0.1)^2}$$

$$a(0.1) = \text{Central}$$

$$a(0.2) = \text{Central}$$

$$a(0.3) = \frac{2d(0.3) - 5d(0.2) + 4d(0.1) - d(0)}{(0.1)^2}$$

• derive $f'(x_2) = \frac{3f_2 - 4f_1 + f_0}{2h} \quad O(h^2)$

$$f_1 = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c)$$

$$f_2 = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{6} h^3 f'''(c)$$

$$3f_2 = 3f(x) + 6hf'(x) + 6h^2 f''(x) + 4h^3 f'''(c)$$

$$4f_1 = 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{4}{3} h^3 f'''(c)$$

$$3f_2 - 4f_1 = -f(x) + 2hf'(x)$$

$$-f_{-1} = f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(c)$$

$$f_{-2} = f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{8}{6} h^3 f'''(c)$$

$$-4f_{-1} = -4f_0 + 4hf'(x) - 2h^2 f''(x) + \frac{4}{6} h^3 f'''(c)$$

$$-4f_{-1} + f_{-2} = -3f_0 + 2hf'(x) + 0 - \frac{4}{6} h^3 f'''(c)$$

$$\frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{2}{3} h^2 f'''(c) = f'(x)$$

Formula

Error