chapter 2: Multivariable distribution.

2.1: Distribution of Two Random variables.

exp: X_1, X_2 Random variable.

(X_1, X_2)	$(0,0)$	$(0,1)$	$(1,1)$	$(1,2)$	$(2,2)$	$(2,3)$
$Pr((X_1,X_2)=(x_1,x_2))$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

1. $Pr(X_1=0) = \sum_{x_2} Pr\left((X_1,X_2)=(0,x_2)\right) = Pr\left((X_1,X_2)=(0,0)\right) + Pr\left((X_1,X_2)=(0,1)\right)$

$= \frac{1}{8} + \frac{1}{8}$

$= \frac{2}{8}$

2. $Pr(X_1=1) = \frac{2}{8} + \frac{2}{8} = \frac{4}{8}$

3. $Pr(X_1=2) = \frac{2}{8}$

4. $Pr(X_2=0) = \frac{1}{8}$

5. $Pr(X_2=1) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$

6. $Pr(X_2=2) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$

7. $Pr(X_2=3) = \frac{1}{8}$

$\sum Pr(X_2) = 1$

Cont. exp: • $A = \{(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)\}$.

A = space of the Random variables X_1 and X_2.

• $A_1 = \{0,1,2\}$

A_1 = space of the Random variable X_1.

• $A_2 = \{0,1,2,3\}$

A_2 = space of the Random variable X_2.

$X_2 \backslash X_1$	0	1	2	$f_2(x_2)$
0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	0	$\frac{3}{8}$
2	0	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
3	0	0	$\frac{1}{8}$	$\frac{1}{8}$
$f_1(x_1)$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	1

احتمالات X_2

احتمالات X_1

$\approx$ summation.

Def: $Pr((X_1,X_2)=(x_1,x_2)) = \underline{f(x_1,x_2)}$ joint density "joint p.d.f"

Def: $Pr(X_1=x_1) = \underline{f_1(x_1)}$ marginal p.d.f of X_1.

Def: $Pr(X_2=x_2) = f_2(x_2)$ marginal p.d.f of X_2.

Def 1: Given a Random experement with a sample space $\mathcal{C}$, consider two Random variable X_1 and X_2 which assign to each element $c \in \mathcal{C}$ one and only one ordered pair of numbers $X_1(c) = x_1$, $X_2(c) = x_2$. The space of X_1 and X_2 is the set of ordered pairs $\mathcal{A} = \{(x_1, x_2) : x_1 = X_1(c), x_2 = X_2(c), c \in \mathcal{C}\}$

Def:

- $P_r((X_1, X_2) \in A) = P(C)$.

  $C = \{c : c \in \mathcal{C}, (X_1(c), X_2(c)) \in A\}$

- $P_{r_{X_1, X_2}}(A) = P_r((X_1, X_2) \in A)$.

- we may write P instead of P_{X_1, X_2} if it obvious from the that its P of $\mathcal{A}$

Def: let X and Y be Random variables:

$$A \subset \mathcal{A} \subset \mathbb{R}^2 \rightsquigarrow P_r(A) = P_r((X, Y) \in A) = \begin{cases} \sum_A \sum_A f(x, y), & X, Y \text{ discrete r.v} \\ \iint_A f(x, y)\,dx\,dy, & X, Y \text{ continuous r.v.} \end{cases}$$

where $f(x, y)$ is the joint p.d.f of X and Y.

$\rightarrow$

$\star$ $f(x,y)$ satisfies the following :

① $f(x,y) \geq 0 \qquad \forall (x,y) \in \mathbb{R}^2$.

$f(x,y) > 0 \qquad \forall (x,y) \in A$

$f(x,y) = 0 \qquad \forall (x,y) \in A^c$.

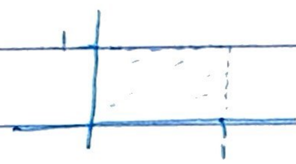
②

- $\sum\sum_{A} f(x,y) = \sum\sum_{\mathbb{R}^2} f(x,y) = 1$, x,y discrete Random variable .

- $\iint_{A} f(x,y)\,dx\,dy = \iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$, x,y continuous Random variable .

Exp :

$$f(x,y) = \begin{cases} 6x^2y & , 0<x<1 , 0<y<1 \\ 0 & , \text{elsewhere} \end{cases}$$



$$P_r\left(0<x<\tfrac{3}{4} , \tfrac{1}{3}<y<2\right) = \int_{1/3}^{1}\int_{0}^{3/4} 6x^2y\,dx\,dy + \int_{1}^{2}\int_{0}^{3/4} 0\,dx\,dy$$

$\int_{1/3}^{2}\int_{0}^{3/4} f(x,y)\,dx\,dy$

$$= \frac{3}{8} + 0$$

$$= \frac{3}{8}$$

Def: let X, y be Random variable's, we defined the c.d.f of X, y as follows:

- $F(x,y) = Pr(X \le x, Y \le y)$.

- $F(x,y) = \sum\limits_{v \le y} \sum\limits_{u \le x} f(u,v)$: X, y discrete Random variable's.

- $F(x,y) = \int\limits_{-\infty}^{y} \int\limits_{-\infty}^{x} f(u,v)\,du\,dv$: X, y continuous Random variables.

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proposition: X, Y continuous Random variables.

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y) \text{ at points of continuity of } f(x,y).$$

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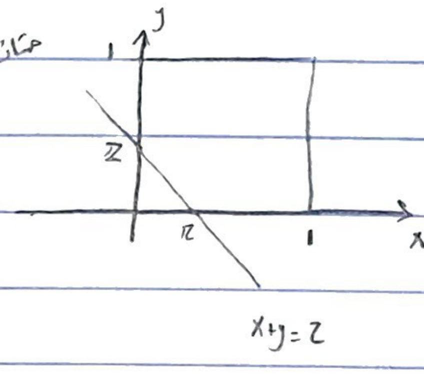
exp:

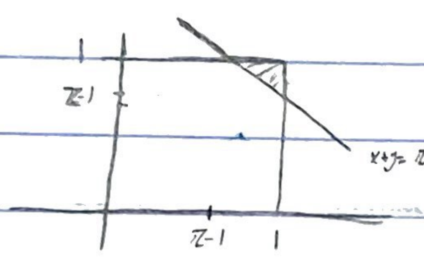
$$f(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & \text{elswhere} \end{cases}$$

$f(x,y)$ is a p.d.f for r.v's X and y. Define $Z = X + Y$ Find the distribution of Z.

Define $G(z) :=$ c.d.f of Z

مكان أختصر حدود التكامل الأول





التكامل الثاني

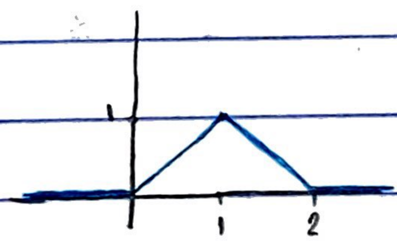
$$G(z) = \begin{cases} 0 & , \quad z < 0 \\ \int_0^z \int_0^{z-x} 1 \, dy \, dx = \frac{z^2}{2} & , \quad 0 \leq z < 1 \\ 1 - \int_{z-1}^{1} \int_{z-x}^{1} 1 \, dy \, dx = \frac{1}{2}(z-2)^2 & , \quad 1 \leq z < 2 \\ 1 & , \quad 2 \leq z \end{cases}$$

Recall : $G(z) = \Pr(Z \leq z) = \iint_A f(x,y) \, dy \, dx$

$A = \{ (x,y) \in \mathbb{R}^2 : 0<x<1 , 0<y<1 , x+y \leq z$

$g(z) = G'(z)$ for all z where g is cont.

$$\Rightarrow \quad g(z) = \begin{cases} z & , \quad 0 < z \leq 1 \\ \frac{-2}{2}(z-2) = 2-z & , \quad 1 < z < 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$



example 2: $\sqcup$ 6- chips.

chip: (1,1), (2,1), (3,1), (1,2), (2,2), (3,2).

freq.: 1 1 2 1 2 3

→ Tabular Form for joint and Marginal p.d.f's:

x_2 \ x_1	1	2	3	$f_2(x_2)$
1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{6}{10}$
$f_1(x_1)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	1

→ Functional Form for joint and Marginal p.d.f's.

→ joint p.d.f X_1 and X_2.

$$f(x_1,x_2) = \begin{cases} \frac{1}{10}, & (x_1,x_2) = (1,1), (1,2), (2,1) \\ \frac{2}{10}, & (x_1,x_2) = (3,1), (2,2). \\ \frac{3}{10}, & (x_1,x_2) = (3,2) \\ 0, & \text{elsewhere.} \end{cases}$$

→ Marginal p.d.f for X_1:

$$f_1(x_1) = \begin{cases} \frac{2}{10}, & x_1 = 1 \\ \frac{3}{10}, & x_1 = 2 \\ \frac{5}{10}, & x_1 = 3 \\ 0, & \text{elsewhere} \end{cases}$$

→ Marginal p.d.f for X_2:

$$f_2(x_2) = \begin{cases} \frac{4}{10}, & x_2 = 1 \\ \frac{6}{10}, & x_2 = 2 \\ 0, & \text{elsewhere.} \end{cases}$$

continue exp2 :

$Pr(X_1 + X_2 = 3) = Pr((X_1, X_2) \in A)$.

$\cdot\ A = \{ (x_1, x_2) : x_1 + x_2 = 3 \} \subseteq \mathcal{A}$

$A = \{ (1,2), (2,1) \}$.

$\Rightarrow Pr(X_1 + X_2 = 3) = P(A) = \sum\sum_{(x_1,x_2)\in A} f(x_1, x_2) = f(1,2) + f(2,1)$

$= \frac{1}{10} + \frac{1}{10}$

$= \frac{2}{10}$.

Remark : X_1 and X_2 discrete Random variables with joint p.d.f $f(x_1, x_2)$

1. $f_1(x_1) = \sum_{x_2} f(x_1, x_2)$.

2. $f_2(x_2) = \sum_{x_1} f(x_1, x_2)$.

3. $Pr((X_1, X_2) \in A) = \sum\sum_{(x_1,x_2)\in A} f(x_1, x_2)$

4. $Pr(X_1 \in A_1) = \sum_{x_1\in A_1} f_1(x_1) = \sum_{x_1\in A_1}\sum_{x_2\in A_2} f(x_1, x_2)$

5. $Pr(X_2 \in A_2) = \sum_{x_2\in A_2} f_2(x_2) = \sum_{x_2\in A_2}\sum_{x_1\in A_1} f(x_1, x_2)$.

Remark :

1. Marginal p.d.f of X_1 when X_1 and X_2 continuse with joint p.d.f $f(x_1, x_2)$:

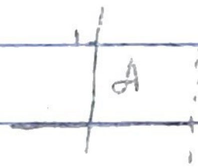
$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 .$$

2. Marginal p.d.f of X_2 when X_1, X_2 continuous with joint p.d.f $f(x_1, x_2)$:

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 .$$

example 3 : X_1 and X_2 Random variables with p.d.f

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & , \ 0 < x_1 < 1 \ , \ 0 < x_2 < 1 \\ 0 & , \ \text{elsewhere} . \end{cases}$$



1. Find marginal p.d.f of X_1.

$$f_1(x_1) = \int_0^1 (x_1 + x_2)\, dx_2 = x_1 x_2 \Big|_0^1 + \frac{x_2^2}{2} \Big|_0^1 \qquad 0 < x_1 < 1$$

$$= x_1 + \frac{1}{2} \qquad 0 < x_1 < 1 .$$

$$\Rightarrow f_1(x_1) = \begin{cases} x_1 + \frac{1}{2} & , \ 0 < x_1 < 1 \\ 0 & , \ \text{elsewhere} . \end{cases}$$

2. Find Marginal p.d.f for X_2:

$$f_2(x_2) = \int_0^1 (x_1 + x_2)\, dx_1 = x_1 x_2 \Big|_0^1 + \frac{x_1^2}{2}\Big|_0^1 = x_2 + \frac{1}{2}, \quad 0 < x_2 < 1.$$

$$\rightarrow f_2(x_2) = \begin{cases} x_2 + \frac{1}{2}, & 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Remarks: X_1 and X_2 continuous.

1. $Pr(X_1 \in A_1) = \int_{A_1} f_1(x_1)\, dx_1 = \int_{A_1} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2\, dx_1$.

2. $Pr(X_2 \in A_2) = \int_{A_2} f_2(x_2)\, dx_2 = \int_{A_2} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2$.

3. $Pr((X_1, X_2) \in A) = \iint_A f(x_1, x_2)\, dx_1\, dx_2$.

Back to exp.3:

3. Find $Pr(X_1 \leq \frac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f_1(x_1)\, dx_1 = \int_0^{\frac{1}{2}} x_1 + \frac{1}{2}\, dx_1 = \frac{x_1^2}{2} + \frac{1}{2}x_1 \Big|_0^{\frac{1}{2}}$

$$= \frac{1}{8} + \frac{1}{4}$$

$$= \frac{3}{8}.$$

Another sol.:

$$Pr(X_1 \leq \frac{1}{2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{1}{2}} f(x_1, x_2)\, dx_1\, dx_2 = \underbrace{\int_0^1 \int_0^{\frac{1}{2}} (x_1 + x_2)\, dx_1\, dx_2}_{\text{حلها}} = \frac{3}{8}$$

4. $Pr(X_1 + X_2 \leq 1) = \int_0^1 \int_0^{1-x_2} (x_1 + x_2)\, dx_1\, dx_2 = \int_0^1 \int_0^{1-x_1} (x_1 + x_2)\, dx_2\, dx_1 = \frac{1}{3}$

$$\rightarrow \underbrace{\int_0^1 \int_0^{1-x_2} (x_1 + x_2)\, dx_1\, dx_2}_{} = \int_0^1 \left[\frac{x_1^2}{2} + x_1 x_2 \Big|_0^{1-x_2} \right] dx_2$$

$$= \int_0^1 \frac{(1-x_2)^2}{2} + (1-x_2)x_2 \quad dx_2$$

$$= \int_0^1 \left(\frac{1}{2} - \cancel{\frac{2x_2}{2}} + \frac{x_2^2}{2} + \cancel{x_2} - x_2^2 \right) dx_1$$

$$= \int_0^1 \frac{1}{2}(1 - x_2^2)\, dx_2$$

$$= \frac{1}{2}\left[x^2 - \frac{x_2^3}{3} \Big|_0^1 \right] = \frac{1}{2}\left[1 - \frac{1}{3} \right]$$

$$= \frac{1}{3} \quad \checkmark$$

$$\rightarrow \underbrace{\int_0^1 \int_0^{1-x_1} (x_1 + x_2)\, dx_2\, dx_1}_{} = \int_0^1 \left(x_1 x_2 + \frac{x_2^2}{2} \Big|_0^{1-x_1} \right) dx_1$$

$$= \int_0^1 x_1(1-x_1) + \frac{(1-x_1)^2}{2} \quad dx_1$$

$$= \int_0^1 \cancel{x_1} - x_1^2 + \frac{1}{2} - \cancel{\frac{2x_1}{2}} + \frac{x_1^2}{2} \quad dx_1$$

$$= \int_0^1 \frac{1}{2}(1 - x_1^2)\, dx_1$$

$$= \frac{1}{2}\left[x_1 - \frac{x_1^3}{3} \Big|_0^1 \right] = \frac{1}{2}\left[1 - \frac{1}{3} \right]$$

$$= \frac{1}{3} \quad \checkmark$$