14.7 Extreme Values and Saddle Points:

Def: Let $f(x,y)$ be defined on a Region R containing the point (a,b), then:

قمة (أعلى) في surface $z=f(x,y)$

1. $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$, $\forall (x,y) \in D$ in an open disk centered at (a,b).

قاع (أخفض) الواد

2. $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

Note: at the local maxima & local minima, the tangent planes, when they exist, are horizontal

Def: An Interior point of the domain of a function $f(x,y)$ where $f_x(a,b) = f_y(a,b) = 0$ or where one or both of $f_x(a,b)$ and $f_y(a,b)$ DNE is called a critical point.

If (a,b) is max. then $f_x(a,b) = f_y(a,b) = 0$

العكس غير صحيح

Def: A differential function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) (1) there are domain points (x,y) where $f(x,y) > f(a,b)$ (2) and domain points (x,y) where $f(x,y) < f(a,b)$.

The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface

saddle point.

Thm: First derivative Test for local extreme Values:

If $f(x,y)$ has a local maximum or minimum value at an Interior point (a,b) of its domain, and if the first derivatives exist, then $f_x(a,b) = 0$ & $f_y(a,b) = 0$

Relative Extreme

Thm: Second Derivative Test for local Extreme Values:

Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$

critical points

then:

1) f has a local maximum at (a,b) if

$f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

2) f has a local minimum at (a,b) if

$f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

3). f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

4). The test is inconclusive at (a,b) if

$f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b)

Note: The expression $f_{xx} f_{yy} - f_{xy}^2$ is called the discriminant or Hessian of f.

we can write it as: $f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

Example: Find local maximum and local minimum & saddle point for:

$\boxed{1}$ $f(x,y) = x^2 - 4xy + y^2 + 6y + 2.$

$f_x = 2x - 4y = 0 \implies x = 2y$

$f_y = -4x + 2y + 6 = 0 \implies y = 2x - 3$

then $(x,y) = (2,1).$

$f_{xx} = 2$, $f_{xx}(2,1) = 2 > 0$

$f_{yy} = 2$, $f_{yy}(2,1) = 2$

$f_{xy} = -4$, $f_{xy}(2,1) = -4$

$\therefore \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(2,1)} = (2)(2) - (-4)^2 = 4 - 16 = -12 < 0$

then f has a saddle point at $(2,1)$

$\boxed{2}$ $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$

$f_x = 3x^2 + 6x = 0 \implies 3x(x+2) = 0$

then $x = 0$ & $x = -2.$

$f_y = 3y^2 - 6y = 0 \implies 3y(y-2) = 0$

then $y = 0$ & $y = 2.$

Then critical points: $(0,0), (0,2), (-2,0), (-2,2)$.

Case (1): $(0,0)$

$f_{xx} = 6x + 6 \Rightarrow f_{xx}(0,0) = 6 > 0$

$f_{yy} = 6y - 6 \Longrightarrow f_{yy}(0,0) = -6$

$f_{xy} = 0$, then

$f_{xx} f_{yy} - f_{xy}^2 = (6)(-6) - 0 = -36 < 0$

Hence f has a saddle point at $(0,0)$

Case (2): $(0,2)$

$f_{xx}(0,2) = 6 > 0$

$f_{yy}(0,2) = 6 > 0$

$f_{xy}(0,2) = 0$

$\left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(0,2)} = (6)(6) - 0 = 36 > 0$

Hence f has a local minimum at $(0,2)$ which

is $f(0,2) = \boxed{-12}$

Case 3: $(-2,0)$

$f_{xx}(-2,0) = -6 < 0$, $f_{yy}(-2,0) = -6$, $f_{xy} = 0$

$$\left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(-2,0)} = (-6)(-6) - 0 = 36 > 0$$

then f has a local maximum at $(-2,0)$ which is

$f(-2,0) = \boxed{-4}$

Case 4: $(-2,2)$

$f_{xx}(-2,2) = -6 < 0$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$

$$\left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(-2,2)} = (-6)(6) - 0 = -36 < 0$$

Hence f has a saddle point at $(-2,2)$.

Absolute Maxima and Minima on closed Bounded Regions:

To find the absolute extreme of a continuous function

$f(x,y)$ on a closed and bounded region R

we have three steps:

1) Find all interior points of R where f has a local maxima and minima, then evaluate f at these points. (These are critical points).

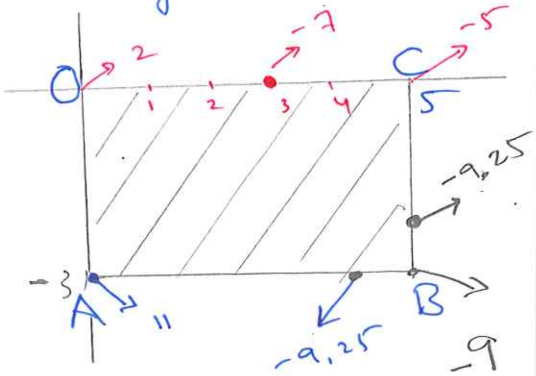
2) Find the Boundary points of R where f has local maxima and minima, then evaluate f at these points.

3.) Find Absolute max & Absolute min of (1) & (2).

Example: Find the absolute maxima and minima of

$$f(x,y) = x^2 + xy + y^2 - 6x + 2$$ on

the rectangle : $0 \le x \le 5$ & $-3 \le y \le 0$.



Boundary points :

$\boxed{1}$ $OC \Rightarrow y = 0$

$f(x,0) = x^2 - 6x + 2$

$f_x = 2x - 6 = 0 \Rightarrow x = 3$.

So we have critical point $(3,0)$

$f(3,0) = 9 - 18 + 2 = -7$

$f(0,0) = 2$

$f(5,0) = -3$

$\boxed{2}$ CB $\Rightarrow x = 5$

$f(5,y) = 25 + 5y + y^2 - 30 + 2 = y^2 + 5y - 3$

$f_y = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$

$\Rightarrow (5, -\frac{5}{2})$

$f(5, -\frac{5}{2}) = -\frac{37}{4}$

$f(5, -3) = -9$

$\boxed{3}$ AB $\Rightarrow y = -3$

$f(x,-3) = x^2 - 3x + 9 - 6x + 2 = x^2 - 9x + 11$

$f_x = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$

$\Rightarrow (\frac{9}{2}, -3)$

$f(\frac{9}{2}, -3) = -9.25$

$f(0,-3) = 0 - 0 + 11 = 11$

$\boxed{4}$ OA $\Rightarrow x = 0$

$f(0,y) = y^2 + 2$

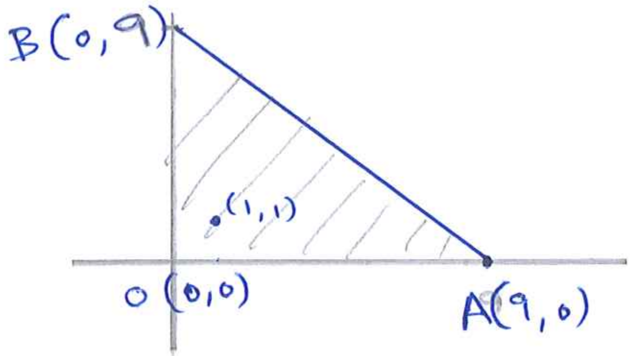
$f_y = 2y = 0 \Rightarrow y = 0$

$\Rightarrow (0,0)$

Example: Find the absolute Max. and Min. values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0, y = 9 - x$



sol: Interior points:

$f_x = 2 - 2x = 0 \Rightarrow \boxed{x = 1}$

$f_y = -2y + 2 = 0 \Rightarrow \boxed{y = 1}$

$\Rightarrow$ critical point ① $\boxed{(1,1)} \in R$

Boundary points:

$\overline{OA}$: $y = 0$, $0 \leq x \leq 9$.

$f(x,0) = 2 + 2x - x^2$

$f'(x) = 2 - 2x = 0 \Rightarrow \boxed{x = 1}$

So we have critical point ② $\boxed{(1,0)}$ on the Boundary $\overline{OA}$

& ③ $\boxed{(0,0)}$ & ④ $\boxed{(9,0)} \rightarrow$ end points

Now, on $\overline{OB}$: $x = 0$, $0 \leq y \leq 9$.

$f(0,y) = 2y - y^2$

$f'(y) = 2 - 2y = 0 \Rightarrow \boxed{y=1}$

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$\therefore$ critical point on the Boundary $\overline{OB}$ is $\boxed{(0,1)}$

repeated 6

& $(0,0)$ & $\boxed{(0,9)} \longrightarrow$ end points

Now, on $\overline{AB}$: $y = 9 - x$, $0 \leq x \leq 9$

$x = 0 \Rightarrow y = 9 \Rightarrow (0,9)$ repeated.

$x = 9 \Rightarrow y = 0 \Rightarrow (9,0)$ repeated.

Now, $f(x,y) = f(x, 9-x)$

$= 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$= -61 + 18x - 2x^2.$

$\therefore f'(x, 9-x) = 18 - 4x \overset{=0}{} \Rightarrow x = \frac{9}{2}$ & $y = \frac{9}{2}$

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$\therefore$ critical point $\left(\frac{9}{2}, \frac{9}{2}\right)$.

Summary:

$f(1,1) = 4$

$f(0,0) = 2$

$f(9,0) = -61$

$f(0,9) = -61$

$f(1,0) = 3$

$f(0,1) = 3$

$f\left(\frac{9}{2}, \frac{9}{2}\right) = \frac{-41}{2}$

$\therefore$ The Absolute Max. is 4 occurs at $(1,1)$

& the absolute Min is -61 occurs at $(0,9)$ & $(9,0)$.

5 Interior points :

$f_x = 2x + y - 6 = 0 \Rightarrow 2x + y = 6$
$f_y = x + 2y = 0 \Rightarrow 2x + 4y = 0$ } solve

so we have the point $(4, -2)$

$f(4, -2) = 16 - 8 + 4 - 24 + 2 = -10$.

Hence $f(x,y)$ has Absolute maxima at $(0,-3)$

which is $f(0,-3) = 11$

& $f(x,y)$ has Absolute minima at $(4,-2)$ which is -10

(53) Example: Find three numbers whose ~~three~~ sum is 9

and whose sum of squares is a minimum.

Sol: $x + y + z = 9 \Rightarrow z = 9 - x - y$.

Let $s(x,y,z) = x^2 + y^2 + z^2 = x^2 + y^2 + (9 - x - y)^2$.

$S_x = 2x + 2(9 - x - y)(-1) = 0$
$S_y = 2y + 2(9 - x - y)(-1) = 0$ } solve these two equations

$\Rightarrow x = 3, y = 3$ & $z = 3$.

$S_{xx}(3,3,3) = 4 > 0$, $S_{yy}(3,3,3) = 4$

& $S_{xy}(3,3,3) = 2 \Rightarrow S_{xx}S_{yy} - S_{xy}^2 = 12 > 0$

Hence S has a local minimum at $(3,3,3)$ (90)

Q23) (14.7) $f(x,y) = y\sin x$

Sol: $f_x = y\cos x = 0$ ... (1)

$f_y = \sin x = 0$ ... (2)

Eq(2) $\Rightarrow$ $x = 0, \pm\pi, \pm 2\pi, \ldots$

$x = n\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots$

Eq(1) $\Rightarrow$ $y \underbrace{\cos(n\pi)}_{0\neq} = 0 \Rightarrow y(-1)^n = 0 \Rightarrow y = 0$

$\therefore$ Critical points are $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$

$f_{xx} = -y\sin x \quad , \quad f_{yy} = 0 \quad , \quad f_{xy} = \cos x$

$\Delta(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -\cos^2 x$

$\therefore \Delta(n\pi, 0) = -\cos^2(n\pi) = -1 < 0$

$\therefore$ All $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$

are saddle points

Q24) $f(x,y) = e^{2x}\cos y$ $\quad (\frac{2}{3})(90)$

$f_x = \underbrace{2e^{2x}}_{0\neq}\cos y \Rightarrow \cos y = 0$

$f_y = -e^{2x}\sin y \Rightarrow \sin y = 0$

} There is No critical point $\Rightarrow$ No extreme
No saddle

Q30) (14.7) Let $f(x,y) = \ln(x+y) + x^2 - y$.

Find the extreme points.

Sol: $D = \{(x,y) : x+y > 0\}$

$$f_x = \frac{1}{x+y} + 2x = 0 \quad \dots (1)$$

$$f_y = \frac{1}{x+y} - 1 = 0 \quad \dots (2)$$

Eq(1) - Eq(2) : $2x + 1 = 0$

$\Rightarrow \boxed{x = -\frac{1}{2}}$

From eq. (2) : $\boxed{y = 3/2}$

$\therefore$ critical point is $\left(-\frac{1}{2}, \frac{3}{2}\right)$

$$f_{xx} = \frac{-1}{(x+y)^2} + 2 \quad , \quad f_{yy} = \frac{-1}{(x+y)^2}.$$

& $$f_{xy} = \frac{-1}{(x+y)^2}$$

$f_{xx}\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$, $f_{yy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1$, $f_{xy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1$

$$\therefore \Delta\left(-\frac{1}{2}, \frac{3}{2}\right) = f_{xx} f_{yy} - f_{xy}^2 \Big|_{\left(-\frac{1}{2}, \frac{3}{2}\right)} = -2 < 0$$

$\therefore f$ has a saddle point at $\left(-\frac{1}{2}, \frac{3}{2}\right)$.

$\left(\frac{3}{3}\right)$ (90)

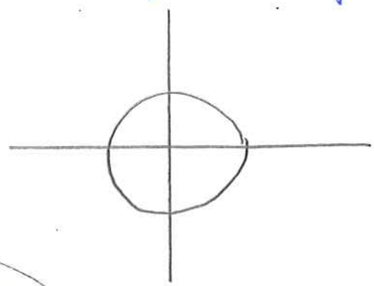
14.7 (41) A flat circular plate has the shape of the region

$x^2+y^2\le 1$. The plate, including the boundary where

$x^2+y^2=1$ is heated so that the tempreture at (x,y) is

$T(x,y)=x^2+2y^2-x$.

Find the tempreture of the hottest and Coldest points on plate.



Interior:

$T_x\big|_{(x,y)}=2x-1=0$, $T_y=4y=0$

$\therefore$ $x=\frac{1}{2}$, $y=0$ with $T(\frac{1}{2},0)=-\frac{1}{4}$

Boundary: $x^2+y^2=1 \Longrightarrow y^2=1-x^2$, substitute in T

Then $T(x)=-x^2-x+2$ on $-1\le x\le 1$

$T'(x)=-2x-1=0 \Longrightarrow x=-\frac{1}{2}$ & $y=\pm\frac{\sqrt{3}}{2}$

$T\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)=\frac{9}{4}$, $T\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)=\frac{9}{4}$

$T(-1,0)=2$, $T(1,0)=0$

The hottest is $\frac{9}{4}$, The clodest is $-\frac{1}{4}$

(45) show that $(0,0)$ is a critical point of $f(x,y) = x^2 + kxy + y^2$ no matter what value the Constant k has (Hint, $k=0$ & $k \neq 0$)

- If $k=0$, then $f(x,y) = x^2 + y^2$

$f_x = 2x = 0$ & $f_y = 2y = 0 \implies (0,0)$ is the only critical point

- If $k \neq 0$, then:

$f_x = 2x + ky = 0 \implies \boxed{y = -\frac{2}{k}x}$ (*)

$f_y = kx + 2y = 0 \implies kx + 2\left(\frac{-2}{k}x\right) = 0$

$\therefore \quad kx - \frac{4x}{k} = 0 \implies \left(k - \frac{4}{k}\right)x = 0$

$\implies x = 0$ or $k = \pm 2$

$\implies y = \left(\frac{-2}{k}\right)(0) = 0$, or $y = \pm x \implies (0,0)$ is critical point

$k = \pm 2$ from (*)

(56) Find the minimum distance from $z = \sqrt{x^2+y^2}$ to $(-6,4,0)$.

(x,y,z) to $(-6,4,0)$

$$d = \sqrt{(x+6)^2 + (y-4)^2 + (z-0)^2}$$

$$D(x,y,z) = d^2 = (x+6)^2 + (y-4)^2 + z^2.$$

$$= (x+6)^2 + (y-4)^2 + \left(\sqrt{x^2+y^2}\right)^2.$$

$$D = 2x^2 + 2y^2 + 12x - 8y + 52$$

$D_x = 4x + 12 = 0$, $D_y = 4y - 8 = 0 \implies$ Critical $(-3,2)$.

$z = \sqrt{13}$.

$D_{xx}\big|_{(-3,2)} = 4$, $D_{yy}\big|_{(-3,2)} = 4$, $D_{xy}\big|_{(-3,2)} = 0$

$\implies D_{xx}D_{yy} = 16 > 0$ & $D_{xx} > 0 \implies$ Local minimum

$d(-3,2,\sqrt{13}) = \sqrt{26}$ is the minimum distance

$\left(\frac{2}{2}\right)(91)$

14.8 Lagrange Multipliers.

Constrained Maxima and Minima.

Example: Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the Origin $(0,0,0)$.

Sol: The problem is to find the Min. value of the function

$$|\overline{OP}| = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

$d(x,y,z) = \sqrt{x^2+y^2+z^2}$ subject to $2x+y-z=5$

$z = 2x+y-5$

1) Find the Min. of $f(x,y,z) = x^2+y^2+z^2$

$$h(x,y) = x^2+y^2+(2x+y-5)^2$$

$$h_x = 2x + 2(2x+y-5)(2) = 0$$

$$\Leftrightarrow \quad 10x+4y = 20$$

$$\Leftrightarrow \quad 5x+2y = 10 \quad \cdots \text{①}$$

$$h_y = 2y + 2(2x+y-5)(1) = 0$$

$$\Leftrightarrow \quad 4x+4y = 10$$

$$2x+2y = 5 \quad \cdots \text{②}$$

Solving ① & ② $\Rightarrow$ $\boxed{x = \frac{5}{3}}$, $\boxed{y = \frac{5}{6}}$

$\Rightarrow$ The critical point : $\left(\frac{5}{3}, \frac{5}{6}\right)$

$h_{xx} = 10$, $h_{yy} = 4$, $h_{xy} = 2$.

Discriminant $= h_{xx} h_{yy} - (h_{xy})^2 = 40 - 4 = 36 > 0$

$\therefore$ Disc. $\left(\frac{5}{3}, \frac{5}{6}\right) = 36 > 0$
$h_{xx}\left(\frac{5}{3}, \frac{5}{6}\right) = 10 > 0$ $\Bigg\}$ $\Rightarrow$ h is Min. at $\left(\frac{5}{3}, \frac{5}{6}\right)$

$$Z = 2x + y - 5$$
$$= 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}$$

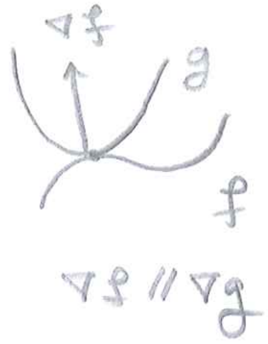
$\therefore$ The closest point is $P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$

& The closest Distance from $P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$

to the origin is

$$\sqrt{\frac{25}{9} + \frac{25}{36} + \frac{25}{36}} = \frac{5}{\sqrt{6}} .$$

The Method of Lagrange Multipliers



Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$. when $g(x,y,z)=0$. To find the local Maximum and minimum values of f subject to the Constraint $g(x,y,z)=0$ (If these exist), find the value of x, y, z and λ that simultaneously satisfy the equations $\nabla f = \lambda \nabla g$ & $g(x,y,z)=0$.

Example: Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1 \rightarrow$ Constraint.

Sol: $f(x,y) = xy$, $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

Let $\nabla f = \lambda \nabla g$

$$f_x \vec{i} + f_y \vec{j} = \lambda \left(g_x \vec{i} + g_y \vec{j} \right)$$

$$y \vec{i} + x \vec{j} = \lambda \left(\frac{x}{4} \vec{i} + y \vec{j} \right)$$

$$\Rightarrow \begin{cases} y = \lambda \left(\frac{x}{4} \right) & \ldots (1) \\ x = \lambda y & \ldots (2) \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 & \ldots (3) \end{cases}$$

substitute (2) into (1) : $y = \lambda \left(\frac{\lambda y}{4} \right) \Leftrightarrow 4y - \lambda^2 y = 0$

$\Rightarrow y(4 - \lambda^2) = 0 \Leftrightarrow y = 0$ or $\lambda = \pm 2$.

Case 1: If $y = 0 \overset{eq(2)}{\Rightarrow} x = 0$ $\left(\overset{(0,0)}{\text{reject}} : \frac{0^2}{8} + \frac{0^2}{2} \neq 1 \right)$

Case 2: $\lambda = 2 \overset{eq(2)}{\Longrightarrow} x = 2y$

$\overset{eq(3)}{\Longrightarrow} \frac{x^2}{8} + \frac{y^2}{2} = 1 \Leftrightarrow \frac{(2y)^2}{8} + \frac{y^2}{2} = 1$

$\Leftrightarrow \frac{4y^2}{8} + \frac{y^2}{2} = y^2 = 1 \Leftrightarrow y = \pm 1$.

If $y = +1 \Rightarrow x = 2 \Rightarrow (2,1)$

If $y = -1 \Rightarrow x = -2 \Rightarrow (-2,-1)$

Case 3: $\lambda = -2 \xRightarrow{eq(2)} x = -2y$

$$\xRightarrow{eq(3)} \frac{(-2y^2)}{8} + \frac{y^2}{2} = 1$$

$$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

If $y = 1 \Rightarrow x = -2 \Rightarrow (-2, 1)$

If $y = -1 \Rightarrow x = 2 \Rightarrow (2, -1)$

Therefore: $f(2,1) = 2$

$f(-2,-1) = 2$

$f(-2,1) = -2$

$f(2,-1) = -2$

$\therefore$ Absolute Max. value $= 2$ occurs at

$(2,1)$ & $(-2,-1)$

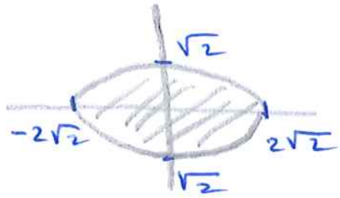
& Absolute Min. value $= -2$ occurs at

$(-2,1)$ & $(2,-1)$.

Example: Find the Maximum and Minimum of

$$f(x,y) = xy \quad \text{on} \quad \frac{x^2}{8} + \frac{y^2}{2} \leq 1.$$

Sol: Interior points:



$$\left.\begin{array}{l} f_x = y = 0 \\ f_y = x = 0 \end{array}\right\} \Rightarrow (0,0) \in \text{Region since } \frac{0^2}{8} + \frac{0^2}{2} \leq 1.$$

Boundary Points: $\left(\frac{x^2}{8} + \frac{y^2}{2} = 1 \right) \to$ Lagrange Multipliers.

i.e $\nabla f = \lambda \nabla g$

from previous example:

"$(2,1)$, $(-2,-1)$, $(2,-1)$, $(-2,1)$" "Boundary"

$\therefore$ $f(0,0) = 0$ (Interior)

$f(2,1) = 2$

$f(-2,-1) = 2$

$f(2,-1) = -2$

$f(-2,1) = -2$

Abs. Max. $= 2$ occurs at $(2,1)$ & $(-2,-1)$

Abs. Min $= -2$ " " $(2,-1)$ & $(-2,1)$.

Example: Find the Max. and Min. of

$$f(x,y,z) = x - 2y + 5z \quad \text{on the sphere}$$

$$x^2 + y^2 + z^2 = 30.$$

Sol: Let $g(x,y,z) = x^2 + y^2 + z^2 - 30 = 0$

& let $\nabla f = \lambda \nabla g$

$$\Rightarrow \quad \vec{i} - 2\vec{j} + 5\vec{k} = \lambda \left(2x\vec{i} + 2y\vec{j} + 2z\vec{k} \right)$$

$$\Rightarrow \begin{cases} 2x\lambda = 1 & \cdots (1) \\ 2y\lambda = -2 & \cdots (2) \\ 2z\lambda = 5 & \cdots (3) \\ x^2 + y^2 + z^2 = 30 & \cdots (4) \end{cases} \quad \begin{matrix} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{matrix} \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{-1}{\lambda} \\ z = \frac{5}{2\lambda} \end{cases}$$

Substitute x, y, z into eq. (4):

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

$$\Rightarrow \quad \frac{30}{4\lambda^2} = 30 \quad \Rightarrow \quad \lambda = \pm \frac{1}{2}$$

If $\lambda = \frac{1}{2} \Rightarrow x = \frac{1}{2(\frac{1}{2})} = 1$, $y = \frac{-1}{1/2} = -2$

$$z = \frac{5}{2(\frac{1}{2})} = 5.$$

$\therefore$ $P_1(1, -2, 5)$

If $\lambda = -\frac{1}{2}$, then $P_2(-1, 2, -5)$

$$f(P_1) = f(1,-2,5) = 1 - 2(-2) + 5(5) = 30$$

$$f(P_2) = f(-1,2,-5) = -1 - 2(2) + 5(-5) = -30$$

$\therefore$ Absolute Max. $= 30$ occurs at $(1,-2,5)$

Absolute Min $= -30$ occurs at $(-1,2,-5)$

Example: Find the extreme values of

$f(x,y,z) = xy + z^2$ subject to the

 Constraint : $x^2 + y^2 + z^2 = 1$

Sol: Let $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

Let $\nabla f = \lambda \nabla g.$

$$\Rightarrow \quad y\vec{i} + x\vec{j} + 2z\vec{k} = \lambda\left(2x\vec{i} + 2y\vec{j} + 2z\vec{k}\right)$$

$$\Rightarrow \begin{cases} y = 2x\lambda & \dots (1) \\ x = 2y\lambda & \dots (2) \\ 2z = 2z\lambda & \dots (3) \\ x^2 + y^2 + z^2 = 1 & \dots (4) \end{cases}$$

substitute (2) into (1) : $y = 2(2y\lambda)\lambda$

$$\Rightarrow y - 4y\lambda^2 = 0$$

$$\Rightarrow y(1 - 4\lambda^2) = 0$$

$\therefore \; y = 0$ or $\lambda = \pm \frac{1}{2}$

Case 1 : If $y = 0 \xRightarrow{eq.(2)} x = 0$

$$x = 0, \; y = 0 \xRightarrow{eq.(4)} z = \pm 1$$

$\therefore \; P_1(0,0,1)$ & $P_2(0,0,-1)$

Case 2 : $\lambda = -\frac{1}{2} \xRightarrow{eq(1) \text{ or } eq(2)} y = -x$

$$\xRightarrow{eq.(3)} 2z = -z \Rightarrow z = 0$$

$$\xrightarrow{eq(4)} \quad x^2+y^2+z^2=1$$

$$\Rightarrow \quad 2x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

If $x = +\frac{1}{\sqrt{2}}$, then $y = \frac{-1}{\sqrt{2}}$, $z=0$

$\therefore$ $P_3\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

If $x = \frac{-1}{\sqrt{2}}$, then $y = \frac{1}{\sqrt{2}}$, $z=0$

$\therefore$ $P_4\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

Case 3: If $\lambda = \frac{1}{2} \xrightarrow{eq(1) \text{ or } eq(2)} y = x$

$$\xrightarrow{eq(3)} \quad 2z = z \Rightarrow z = 0$$

$$\xrightarrow{eq(4)} \quad x^2+y^2+z^2=1$$

$$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

If $x = \frac{+1}{\sqrt{2}}$, $\Rightarrow y = \frac{+1}{\sqrt{2}}$, $z=0$

$P_5\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

If $x = \frac{-1}{\sqrt{2}}$, $y = \frac{-1}{\sqrt{2}}$, $z = 0$

$\therefore$ $P_6\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

$f(0,0,1) = 1$

$f(0,0,-1) = 1$

$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = \frac{-1}{2}$

$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{-1}{2}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2}$

$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = \frac{1}{2}$

$\therefore$ Absolute Max. $= 1$ occurs at $(0,0,1)$ & $(0,0,-1)$

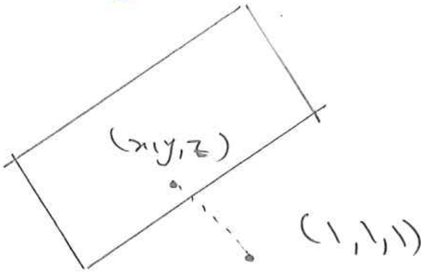
Absolute Min. $= \frac{-1}{2}$ occurs at $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

and $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Example (17): Find the point on the plane $\underbrace{x+2y+3z=13}_{g(x,y,z)=}$ closest to the point $(1,1,1)$.

We need to find the minimum value of d:

$$d = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$$



$$f(x,y,z) = d^2 = (x-1)^2+(y-1)^2+(z-1)^2$$

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$$

$$= 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k}$$

$$\nabla g = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\nabla f = \lambda \nabla g \qquad \& \quad g(x,y,z) = 0.$$

$$\begin{cases} 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k} = \lambda(\vec{i}+2\vec{j}+3\vec{k}) \\ x+2y+3z = 13. \end{cases}$$

$$\Longrightarrow \begin{cases} 2(x-1) = \lambda & \dots (1) \\ 2(y-1) = 2\lambda & \dots (2) \\ 2(z-1) = 3\lambda & \dots (3) \\ x+2y+3z = 13 & \dots (4) \end{cases}$$

- substitute (1) in (2):

$$2(y-1) = 2(2(x-1)) \Rightarrow \quad y-1 = 2(x-1).$$

$$\therefore \quad y+1 = 2x \quad \Rightarrow \quad x = \frac{y+1}{2}$$

- Substitute (2) in (3) $\quad y-1 = \lambda$

$$2(z-1) = 3(y-1) \Rightarrow \quad 2z = 3y-1 \Rightarrow z = \frac{3y-1}{2}$$

but (4) : $x + 2y + 3z = 13$

$$\Rightarrow \quad \frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) = 12$$

$$\therefore \quad 7y = 14$$

$$y = 2 \quad , \quad x = \frac{3}{2} \quad , \quad z = \frac{5}{2}$$

So the point is $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$..

14.8 17 Find the point on $x+2y+3z=13$ closest to $(1,1,1)$.

$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$

$g(x,y,z) = x+2y+3z$

$\nabla f = \lambda \nabla g$ & $x+2y+3z-13=0$

$\nabla f = 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k}$.

$\nabla g = 1\vec{i} + 2\vec{j} + 3\vec{k}$.

$\therefore$ $2(x-1) = \lambda$, $2(y-1) = 2\lambda$, $2(z-1) = 3\lambda$

$\therefore$ $2(x-1) = (y-1)$ & $3(x-1) = (z-1)$

$\therefore$ $y = 2x-1$ & $z = 3x-2$

Using: $g(x,y,z) = 13 \Rightarrow$

$13 = x + 2(2x-1) + 3(3x-2) \Longrightarrow x = \frac{3}{2}, y = 2, z = \frac{5}{2}$

23 Find the maximum and minimum values of

$f(x,y,z) = x-2y+5z$ on sphere $x^2+y^2+z^2=30$.

$\nabla f = \vec{i} - 2\vec{j} + 5\vec{k}$, $\nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\therefore$ $1 = 2\lambda x$ & $-2 = 2\lambda y$ & $5 = 2\lambda z$

& $x^2+y^2+z^2 = 30$ ...(*)

$\Rightarrow \nabla f = \lambda \nabla g \Rightarrow i-2j+5k = \lambda(2x)i + \lambda(2y)j + \lambda(2z)\vec{k}$

$\Rightarrow 1 = 2x\lambda, \ -2 = 2y\lambda$ & $5 = 2z\lambda$

$\Rightarrow x = \frac{1}{2\lambda}, \ y = \frac{-1}{\lambda} = -2x, \ z = \frac{5}{2\lambda} = 5x$ .. subst. in (*)

$\Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow \boxed{x = \pm 1}$

$\Rightarrow f(1,-2,5) = 30$ is maximum & $f(-1,2,-5) = -30$ is minimum