- **Laplace and Inverse Laplace Transform**

$$f(t) = e^{-3t}\cos \pi t$$

```
syms s t                      % Define s and t as symbol variables
f=exp(-3*t)*cos(pi*t);        % Write an expression for f(t)
F=laplace(f)                  % Perform Laplace Transform

F =

(s + 3)/((s + 3)^2 + pi^2)
```

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}$$

```
syms s t                      % Define s and t as symbol variables
F=(5*(s+2))/(s^2*(s+1)*(s+2));  % Write an expression for F(s)
f=ilaplace(F)                   % Perform Laplace Transform

f =

5*t + 5*exp(-t) - 5
```

- **Partial Fraction Expansion**

$$F(s) = \frac{2s^2+4s+5}{s(s+1)} = \frac{2s^2+4s+5}{s^2+s}$$

```
a=[2 4 5];                    % Coefficients of numerator
b=[1 1 0];                    % Coefficients of denominator
[r,p,k]=residue(a,b)          % Function that performs PFE

r =

  -3
   5


p =

  -1
   0
```

```
k =

    2
```

This corresponds to : $F(s) = 2 + \frac{-3}{s+1} + \frac{5}{s}$

- **Solving a Linear Ordinary Differential Equation using Laplace in Matlab**

```
syms s t Y                 % Define s t and Y as symbols
f=exp(-t)+5*dirac(t-2);    % Define the RHS of the equation
F=laplace(f);             % Take the Laplace transform of the RHS
Y1=s*Y;                    % Laplace of y'(t)
Y2=s^2*Y;                  % Laplace of y''(t)
sol=solve(Y2+2*Y1+2*Y-F,Y) % Solve the equation

sol =

(5*exp(-2*s) + 1/(s + 1))/(s^2 + 2*s + 2)

y=ilaplace(sol)          % Take the inverse Laplace of the solution

y =

exp(-t) - exp(-t)*cos(t) + 5*heaviside(t - 2)*sin(t - 2)*exp(2 - t)
```

Solution: $y(t) = e^{-t} - e^{-t}\cos(t) + 5u(t-2)\sin(t-2)e^{2-t}$

- **Defining a Transfer Function**

$$G(s) = \frac{2s + 10}{s^2 + 2s + 10}$$

```
>> num = [2 10];          % Coefficients of the numerator
>> den = [1 2 10];        % Coefficients of the denominator
>> sys = tf(num,den)      % Create the Transfer Function

sys =

     2 s + 10
  --------------
  s^2 + 2 s + 10

Continuous-time transfer function.
```

- Simple Step Input

```
>> num = [2 10];
>> den = [1 2 10];
>> sys = tf(num,den)

sys =

     2 s + 10
  --------------
  s^2 + 2 s + 10

Continuous-time transfer function.
>> step(sys)                % Find the step response
```

- Example on the Step Response

Consider the mechanical system shown in Figure 4–8. The system is at rest initially. The displacements x and y are measured from their respective equilibrium positions. Assuming that $p(t)$ is a step force input and the displacement $x(t)$ is the output, obtain the transfer function of the system. Then, assuming that $m = 0.1$ kg, $b_2 = 0.4$ N-s/m, $k_1 = 6$ N/m, $k_2 = 4$ N/m, and $p(t)$ is a step force of magnitude 10 N, obtain an analytical solution $x(t)$.

The equations of motion for the system are

$$m\ddot{x} + k_1 x + k_2(x - y) = p$$
$$k_2(x - y) = b_2\dot{y}$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$(ms^2 + k_1 + k_2)X(s) = k_2 Y(s) + P(s) \qquad (4\text{–}9)$$
$$k_2 X(s) = (k_2 + b_2 s)Y(s) \qquad (4\text{–}10)$$

Solving Equation (4–10) for $Y(s)$ and substituting the result into Equation (4–9), we get

$$(ms^2 + k_1 + k_2)X(s) = \frac{k_2{}^2}{k_2 + b_2 s}X(s) + P(s)$$

or

$$[(ms^2 + k_1 + k_2)(k_2 + b_2 s) - k_2^2]X(s) = (k_2 + b_2 s)P(s)$$

from which we obtain the transfer function

$$\frac{X(s)}{P(s)} = \frac{b_2 s + k_2}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2)b_2 s + k_1 k_2} \qquad (4\text{–}11)$$

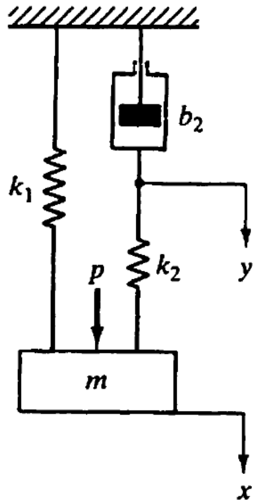


Figure 4–8 Mechanical system.

```
% This code is to simulate the step response of the system in the example
clear
clc
close


% Parameters:
m = 0.1;          % Mass in kg
b2 = 0.4;         % Damping Ratio in N-s/m
k1 = 6;           % Spring Stiffness Coefficient in N/m
k2 = 4;           % Spring Stiffness Coefficient in N/m


n = 10*[b2 k2];
d = [m*b2 m*k2 (k1+k2)*b2 k1*k2];


sys = tf(n,d);


t = [0:0.01:5];
x = step(sys,t);


plot(t,x)
xlabel('Time (s)')
ylabel('x(t)')
title('Unit-Step Response')
```

- Example on the Impulse Response

Consider the mechanical system shown in Figure 4–14. A bullet of mass m is shot into a block of mass M (where $M \gg m$). Assume that when the bullet hits the block, it becomes embedded there. Determine the response (displacement x) of the block after it is hit by the bullet. The displacement x of the block is measured from the equilibrium position before the bullet hits it. Suppose that the bullet is shot at $t = 0-$ and that the initial velocity of the bullet is $v(0-)$. Assuming the following numerical values for M, m, b, k, and $v(0-)$, draw a curve $x(t)$ versus t:

$$M = 50 \text{ kg}, \qquad m = 0.01 \text{ kg}, \qquad b = 100 \text{ N-s/m},$$
$$k = 2500 \text{ N/m}, \qquad v(0-) = 800 \text{ m/s}$$

The input to the system in this case can be considered an impulse, the magnitude of which is equal to the rate of change of momentum of the bullet. At the instant the bullet hits the block, the velocity of the bullet becomes the same as that of the block, since the bullet is assumed to be embedded in it. As a result, there is a sudden change in the velocity of the bullet. [See Figure 4–15(a).] Since the change in the velocity of the bullet occurs instantaneously, $\dot{v}$ has the form of an impulse. (Note that $\dot{v}$ is negative.)

For $t > 0$, the block and the bullet move as a combined mass $M + m$. The equation of motion for the system is

$$(M + m)\ddot{x} + b\dot{x} + kx = F(t) \tag{4–22}$$

$$(M + m)\ddot{x} + b\dot{x} + kx = F(t) = [mv(0-) - m\dot{x}(0+)]\delta(t)$$

Taking the $\mathcal{L}_-$ transform of both sides of this last equation, we see that

$$(M + m)[s^2X(s) - sx(0-) - \dot{x}(0-)] + b[sX(s) - x(0-)] + kX(s)$$
$$= mv(0-) - m\dot{x}(0+)$$

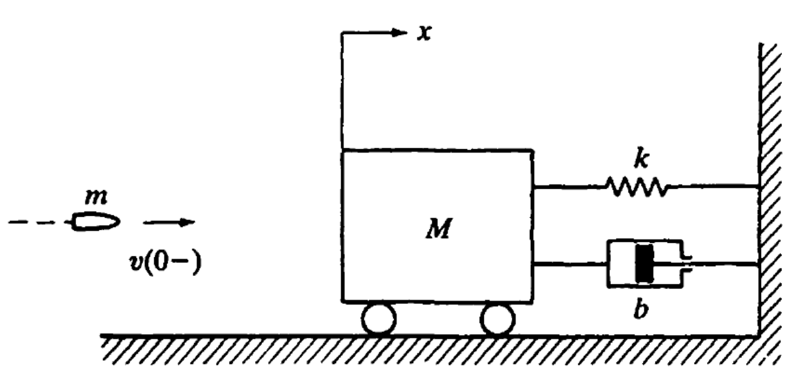


Figure 4–14 Mechanical system subjected to an impulse input.

Also, noting that $x(0-) = 0$ and $\dot{x}(0-) = 0$, we have

$$X(s) = \frac{mv(0-) - m\dot{x}(0+)}{(M+m)s^2 + bs + k} \tag{4–24}$$

To determine the value of $\dot{x}(0+)$, we apply the initial-value theorem:

$$\begin{aligned}\dot{x}(0+) &= \lim_{t\to 0+} \dot{x}(t) = \lim_{s\to\infty} s[sX(s)] \\ &= \lim_{s\to\infty} \frac{s^2[mv(0-) - m\dot{x}(0+)]}{(M+m)s^2 + bs + k} \\ &= \frac{mv(0-) - m\dot{x}(0+)}{M+m}\end{aligned}$$

from which we get

$$mv(0-) - m\dot{x}(0+) = (M+m)\dot{x}(0+)$$

or

$$\dot{x}(0+) = \frac{m}{M+2m} v(0-)$$

The magnitude of the impulse input is

$$mv(0-) - m\dot{x}(0+) = (M+m)\dot{x}(0+) = \frac{m(M+m)}{M+2m} v(0-)$$

To find the response of the system to $F(t)$ (which is an impulse input whose magnitude is not unity), we modify Equation (4–26) to the following form:

$$\frac{X(s)}{\mathcal{L}[\delta(t)]} = \frac{1}{(M+m)s^2 + bs + k} \frac{m(M+m)\,v(0-)}{M+2m}$$

```
% This code is to simulate the Impulse response of the system in the example
clear
clc
close


% Parameters:
M = 50;          % Mass of Block in kg
m = 0.01;        % Mass of Bullet in kg
b = 100;         % Damping Ratio in N-s/m
k = 2500;        % Spring Stiffness Coefficient in N/m
v0 = 800;        % Initial Velocity of the bullet in m/s


F = m*(M+m)*v0/(M+2*m);
n = F*[1];
d = [M+m b k];


sys = tf(n,d);


t = [0:0.01:6];
x = impulse(sys,t);


plot(t,x)
xlabel('Time (s)')
ylabel('x(t)')
title('Impulse Response')
```

- Response to an Arbitrary Input

Consider once again the system shown in Figure 4–7. (See **Example 4–3.**) Assume that $m = 10$ kg, $b = 20$ N-s/m, $k = 100$ N/m, and $u(t)$ is a unit-ramp input—that is, the displacement u increases linearly, or $u = \alpha t$, where $\alpha = 1$. We shall obtain the unit-ramp response using the command

lsim(sys,u,t)

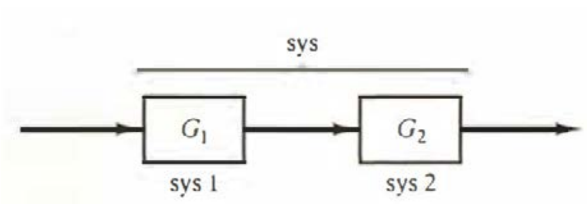
The transfer function of the system, derived in **Example 4–3**, is

$$\frac{Y(s)}{U(s)} = \frac{2s + 10}{s^2 + 2s + 10}$$

```
% This code is to simulate the step response of the system in the example
clear
clc
close


num = [2 10];
den = [1 2 10];


sys = tf(num,den);


t = [0:0.01:4];


u = t;


x = lsim(sys,u,t);


plot(t,x)
xlabel('Time (s)')
ylabel('x(t)')
title('Impulse Response')
```

- **Block Diagram Reduction**

1. **Series Connection:**



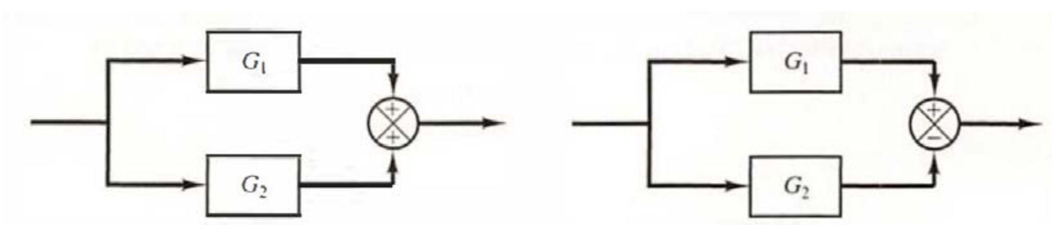
Consider the case where

$$G_1 = \frac{10}{s^2 + 2s + 10}, \qquad G_2 = \frac{5}{s + 5}$$

```
>> n1 = [10];
>> d1 = [1 2 10];

>> n2 = [5];
>> d2 = [1 5];
```

```
>> G1 = tf(n1,d1)

G1 =

        10
  --------------
  s^2 + 2 s + 10

Continuous-time transfer function.


>> G2 = tf(n2,d2)

G2 =

    5
  -----
  s + 5

Continuous-time transfer function.

>> Geq = series(G1,G2)

Geq =

            50
  -----------------------
  s^3 + 7 s^2 + 20 s + 50

Continuous-time transfer function.
```

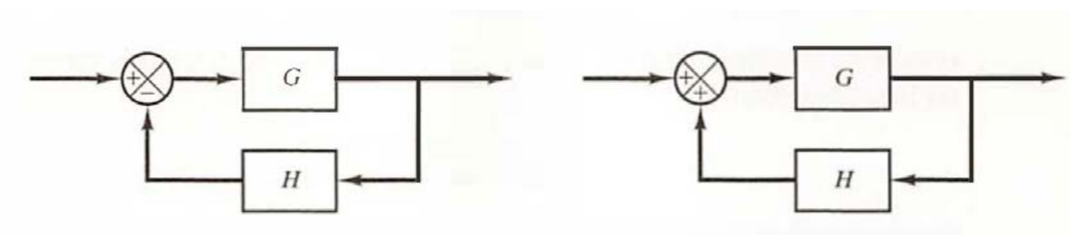
2. **Parallel Connection:**




```
>> n1 = [10];
>> d1 = [1 2 10];

>> n2 = [5];
>> d2 = [1 5];

>> G1 = tf(n1,d1)
```

```
G1 =

        10
  --------------
  s^2 + 2 s + 10

Continuous-time transfer function.


>> G2 = tf(n2,d2)

G2 =

    5
  -----
  s + 5

Continuous-time transfer function.

>> Geq = parallel(G1,G2)

Geq =

    5 s^2 + 20 s + 100
  -----------------------
  s^3 + 7 s^2 + 20 s + 50

Continuous-time transfer function.
```

3. **Feedback Connection:**



Consider the case where

$$G = \frac{5}{s^2 + 2s}, \qquad H = 0.1s + 1$$


```
>> ng1 = [5];
>> dg1 = [1 2 0];
>> G = tf(ng1,dg1)

G =

      5
  ---------
  s^2 + 2 s

Continuous-time transfer function.

>> nh1 = [0.1 1];
>> dh1 = [1];
>> H = tf(nh1,dh1)

H =

  0.1 s + 1

Continuous-time transfer function.

>> Geq = feedback(G,H)

Geq =

         5
  ---------------
  s^2 + 2.5 s + 5

Continuous-time transfer function.
```