

Example 2

t x	v(t) f(x)
1	20.1
2	22.5
3	26.6
4	28.9

$$\int_1^4 f(x) dx = ??$$

by Simpson's 3/8 Rule (because we have 4 points)

$$\int_1^4 f(x) dx = \frac{3(4)}{8} (f(1) + 3f(2) + 3f(3) + f(4))$$

or by trapezoidal

$$\int_1^4 f(x) dx = \frac{4}{2} (f(1) + f(4))$$

Example

Derive trapezoidal error or Rule.

we use $P(x)$ and $\int_{x_0}^{x_1} f(x) dx \approx \int_{x_0}^{x_1} P(x) dx$

$$= \int_{x_0}^{x_1} \left(\frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1 \right) dx \quad \begin{matrix} x = x_0 + ht \\ x_1 = x_0 + h \end{matrix} \quad dx = h dt$$

$$= \int_0^1 \left(\frac{h(t-1)}{-h} y_0 + \frac{ht}{h} y_1 \right) h dt = -y_0 h \int_0^1 (t-1) dt + h y_1 \int_0^1 t dt$$

$$= \frac{y_0 h}{2} + \frac{h y_1}{2} = \frac{h}{2} (y_0 + y_1)$$

$$\text{Error} = \int_{x_0}^{x_1} E(x) dx = \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c) dx$$

$$= \int_0^1 h(t) h(t-1) \frac{f^{(2)}(c)}{2} h dt = \frac{f^{(2)}(c)}{2} \int_0^1 h(t) h(t-1) h dt$$

$$= \frac{h^3 f^{(2)}(c)}{2} \int_0^1 (t^2 - t) dt = -\frac{h^3 f^{(2)}(c)}{12}$$

Def:-

The degree of precision or accuracy of a quadrature Formula is the largest positive integer k is such that the Formula is exact For x^k , $k=0,1,2,\dots$

Example:-

Find the degree of accuracy of Simpson's method:-

$$\frac{h}{3} (f_0 + 4f_1 + f_2)$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ x_0 & x_1 & x_2 \end{array}$$

<u>F(x)</u>	<u>Formula</u>	<u>Exact</u>	<u>Error</u>
$x^0 = 1$	$\frac{1}{3} (f_0 + 4f_1 + f_2)$ $\frac{1}{3} (1 + 4(1) + 1) = 2$	$\int_0^2 1 dx = 2$	0
$x^1 = x$	$\frac{1}{3} (0 + 4(1) + 2) = 2$	$\int_0^2 x dx = \frac{x^2}{2} \Big _0^2 = 2$	0
x^2	$\frac{1}{3} (0 + 4(1) + 4) = \frac{8}{3}$	$\int_0^2 x^2 dx = \frac{x^3}{3} \Big _0^2 = \frac{8}{3}$	0
x^3	$\frac{1}{3} (0 + 4(1) + 8) = 4$	$\int_0^2 x^3 dx = \frac{x^4}{4} \Big _0^2 = 4$	0
x^4	$\frac{1}{3} (0 + 4(1) + 16) = \frac{20}{3}$	$\int_0^2 x^4 dx = \frac{x^5}{5} \Big _0^2 = \frac{32}{5}$	$\frac{32}{5} - \frac{20}{3} \neq 0$

degree of accuracy of Simpson's is 3

Note:-

degree of accuracy of trapezoidal is 1

degree of accuracy of Simpson $\frac{1}{3}$ is 3

degree of accuracy of Simpson $\frac{3}{8}$ is 3

Theory:-

Error = $k f^{(n+1)}(\xi)$, k is the degree of accuracy

Example

For Simpson's $\frac{1}{3}$ method

$$\text{Error} = k f^{(4)}(\xi)$$

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

$$f'''(x) = 24x$$

$$f^{(4)}(x) = 24$$

$$\text{Error} = k f^{(4)}(\xi)$$

$$\frac{32}{5} - \frac{20}{3} = k(24)$$

$$\frac{96-100}{15} = 24k$$

$$k = \frac{-4}{15 \times 24} = -\frac{1}{90}$$

- if $f(x) = (x-x_0)^4$

$$\int_{x_0}^{x_2} f(x) dx$$

Error = Exact - Formula

$$\text{Exact} = \int_{x_0}^{x_2} f(x) dx = \int_{x_0}^{x_2} (x-x_0)^4 dx = \frac{(x-x_0)^5}{5} \Big|_{x_0}^{x_2} = \frac{32h^5}{5}$$

$$\text{Formula} = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] = \frac{20}{3} h^5$$

$$\text{Error} = -\frac{1}{90} h^5$$

7
6.1

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

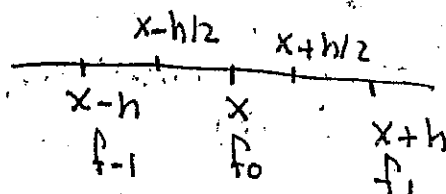
$$f_x(x, y) = \frac{f(x+h, y) - f(x-h, y)}{2h}$$

$$f_y(x, y) = \frac{f(x, y+h) - f(x, y-h)}{2h}$$

10
6.2

$$f'(x + \frac{h}{2}) = \frac{f_1 - f_0}{h}$$

$$f'(x - \frac{h}{2}) = \frac{f_0 - f_{-1}}{h}$$



$$f''(x) = (f'(x))' = \frac{f'(x+h/2) - f'(x-h/2)}{2(h/2)}$$

$$= \frac{\frac{f_1 - f_0}{h} - \frac{f_0 - f_{-1}}{h}}{h}$$

$$f''(x) = \frac{f_1 - 2f_0 + f_{-1}}{h^2}$$

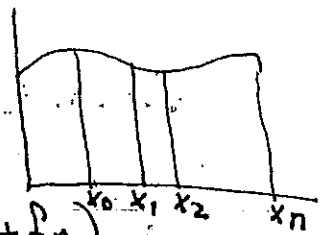
$$f'''(x) = (f'(x))'' = (f''(x))'$$

7.2. Composite Rules

1. Composite trapezoidal Rule

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx$$

$$= \frac{h_1}{2} (f_0 + f_1) + \frac{h_2}{2} (f_1 + f_2) + \dots + \frac{h_n}{2} (f_{n-1} + f_n)$$



$$h_k = h$$

$$= \frac{h}{2} (f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n)$$

$$= \frac{h}{2} \sum_{k=1}^n (f_{k-1} + f_k) = T(f, h)$$

EXAMPLE

t	v(t)
0	10
1	12
2	13
3	15

$$D(3) = \frac{1}{2} (f(0) + 2f(1) + 2f(2) + f(3))$$

EXAMPLE

t	v(t)
1	10
3	15
4	20
5	21

$$D(5) = \frac{2}{2} (f(1) + f(3)) + \frac{1}{2} (f(3) + f(4)) + \frac{1}{2} (f(4) + f(5))$$

• Error For Composite trapezoidal.

$$\text{Error} = -\frac{h^3 f^{(2)}(c_1)}{12} - \frac{h^3 f^{(2)}(c_2)}{12} + \dots - \frac{h^3 f^{(2)}(c_n)}{12}$$

$$= -\frac{h^3}{12} (f^{(2)}(c_1) + f^{(2)}(c_2) + \dots + f^{(2)}(c_n))$$

$$= -\frac{h^3}{12} (n f^{(2)}(c)) \quad h = \frac{b-a}{n}$$

$$= -\frac{h^3}{12} \left(\frac{b-a}{n} f^{(2)}(c) \right)$$

$$E_T(f, h) = -\frac{(b-a) f^{(2)}(c) h^2}{12} \approx O(h^2)$$

• EXAMPLE

Find the number m at step size h so that $|E_T(f, h)| \leq 5 \times 10^{-9}$

of the approximation $\int_2^7 \frac{dx}{x} = T(f, h)$

where m is the number of trapezoidal composite

$$m = n$$

$$|E_T(f, h)| \leq 5 \times 10^{-9}$$

$$\frac{(b-a) f^{(2)}(c) \left(\frac{b-a}{n}\right)^2}{12} \leq 5 \times 10^{-9}$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$\max |f''(x)| = \frac{2}{8} = \frac{1}{4}$$

$$2 \leq x \leq 7$$

فنا قسّمنا

فقط 2

$$\frac{(b-a) f^{(2)}(c) \left(\frac{b-a}{n}\right)^2}{12} \leq 5 \times 10^{-9}$$

$$\frac{5(0.25) \left(\frac{5}{n}\right)^2}{12} \leq 5 \times 10^{-9}$$

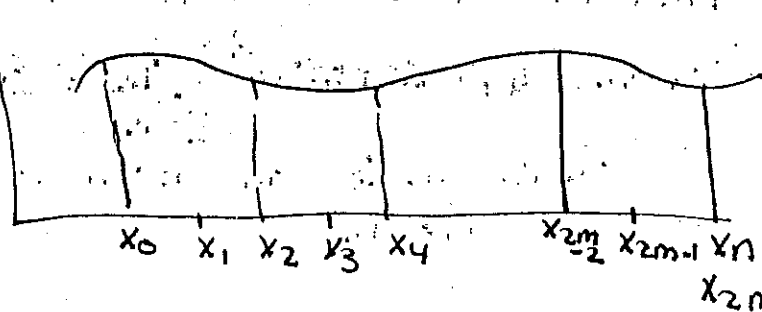
$$n \geq \sqrt{\frac{5 \times 0.25 \times 25}{12 \times 5 \times 10^{-9}}} = 22821.77$$

$$n = 22822$$

$$h = \frac{b-a}{n} = \frac{5}{22822} = 0.000219$$

2. Composite Simpson's $\frac{1}{3}$ Rule.

$$\int_{x_0}^{x_{2m}} f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \int_{x_4}^{x_6} f(x) dx + \dots + \int_{x_{2m-2}}^{x_{2m}} f(x) dx$$



$$\begin{aligned} & n = 2m \\ & m = \frac{n}{2} \\ & h_2 = h \\ & = \frac{h_1}{3} (f_0 + 4f_1 + f_2) + \frac{h_2}{3} (f_2 + 4f_3 + f_4) + \dots + \frac{h_n}{3} (f_{2m-2} + 4f_{2m-1} + f_{2m}) \\ & = \frac{h}{3} (f_0 + 4f_1 + f_2 + f_2 + 4f_3 + f_4 + \dots + f_{2m-2} + 4f_{2m-1} + f_{2m}) \\ & = \frac{h}{3} \sum_{k=1}^m (f_{2k-2} + 4f_{2k-1} + f_{2k}) = S(f, h) \end{aligned}$$

• Error For Composite Simpson

$$\begin{aligned} E_S(f, h) &= -\frac{h^5 f^{(4)}(c_1)}{90} - \frac{h^5 f^{(4)}(c_2)}{90} - \dots - \frac{h^5 f^{(4)}(c_m)}{90} \\ &= -\frac{h^5}{90} (f^{(4)}(c_1) + f^{(4)}(c_2) + \dots + f^{(4)}(c_m)) \\ &= -\frac{h^5}{90} (m f^{(4)}(c)) \end{aligned}$$

$$= -\frac{h^5}{90} \left(\frac{b-a}{2h} f^{(4)}(c) \right)$$

$$= -\frac{(b-a)h^4}{180} f^{(4)}(c)$$

$$\approx Ch^4$$

$$M = \frac{b-a}{2h}$$

EXAMPLE

Find the number $\underline{m}$ and step size h that $|E_S(f, h)| \leq 5 \times 10^{-9}$ of the approximation $\int_2^7 \frac{dx}{x} = S(f, h)$.

$$|E_S(f, h)| \leq 5 \times 10^{-9}$$

$$\left| \frac{(b-a) \left(\frac{b-a}{2m} \right)^4 f^{(4)}(c)}{180} \right| < 5 \times 10^{-9}$$

$$\frac{5 \cdot \left(\frac{5}{2m} \right)^4 \cdot 0.75}{180} < 5 \times 10^{-9}$$

$$m > \sqrt[4]{\frac{5 \cdot 5^4 \cdot 0.75}{2^4 \cdot 180 \cdot 5 \times 10^{-9}}} = 112.9$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}$$

$$f^{(4)}(x) = \frac{24}{x^5}$$

$$\max_{2 \leq x \leq 7} |f^{(4)}(x)| = \frac{24}{2^5}$$

$$= \frac{24}{32} = 0.75$$

7.5 Gauss - Legendre Formulas

2 points Formula

$$\int_a^b f(x) dx \approx w_1 f(x_1) + w_2 f(x_2)$$

عندنا 2 نقاط و 2 حوامل

we assume degree of precision 3

1. $E(1) = 0 \rightarrow \int_{-1}^1 1 dx = 2 \rightarrow \text{Formula} = w_1(1) + w_2(1) \rightarrow w_1 + w_2 = 2$
2. $E(x) = 0 \rightarrow \int_{-1}^1 x dx = 0 \rightarrow \text{Formula} = w_1 x_1 + w_2 x_2 \rightarrow w_1 x_1 + w_2 x_2 = 0$
3. $E(x^2) = 0 \rightarrow \int_{-1}^1 x^2 dx = \frac{2}{3} \rightarrow \text{Formula} = w_1 x_1^2 + w_2 x_2^2 \rightarrow w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}$
4. $E(x^3) = 0 \rightarrow \int_{-1}^1 x^3 dx = 0 \rightarrow \text{Formula} = w_1 x_1^3 + w_2 x_2^3 \rightarrow w_1 x_1^3 + w_2 x_2^3 = 0$

$$\text{Exact} = \int_{-1}^1 f(x) dx$$

$$\text{Formula} = w_1 f(x_1) + w_2 f(x_2)$$

$$\text{Exact} = \text{Formula}$$

حل المعادلات

$$w_1 x_1^3 = -w_2 x_2^3$$

$$w_1 x_1 = -w_2 x_2$$

$$x_1^2 = x_2^2$$

$$x_1 = x_2 \text{ or } \boxed{x_1 = -x_2}$$

$$w_1 x_1 + w_2 (-x_1) = 0$$

$$x_1 (w_1 - w_2) = 0$$

$$w_1 - w_2 = 0$$

$$\boxed{w_1 = w_2}$$

$$w_1 + w_2 = 2$$

$$2w_1 = 2$$

$$\boxed{w_1 = 1}, \boxed{w_2 = 1}$$

$$1(x_1)^2 + 1(x_1)^2 = \frac{2}{3}$$

$$2x_1^2 = \frac{2}{3}$$

$$x_1^2 = \frac{1}{3}$$

$$\rightarrow \boxed{x_1 = \pm \frac{1}{\sqrt{3}}}$$

$$\int_{-1}^1 f(x) dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= G_2(f)$$

Gauss - Legendre

2 points Formula

EXAMPLE

Estimate $\int_{-1}^1 \frac{1}{x+2} dx$ using

$$\frac{1}{(x+2)} = \frac{-1}{x+2}$$

$$1. T(f, 2) = \frac{2}{2} (f(-1) + f(1)) = 1.3333$$

$$2. S(f, 2) = \frac{1}{3} (f(-1) + 4f(0) + f(1)) = 1.1111$$

$$3. G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1.09091$$

$$\text{Exact} = 1.09861$$

$$E = \text{Exact} - \text{Formula} = \frac{24}{(x+2)^5} = 1.09091 - 1.09861$$

$$\text{The Error For } G_2(f) = \frac{f^{(4)}(c)}{135}$$

$$\text{Error} = \frac{1}{135} f^{(4)}(c)$$

$$f(x) = x^4$$

Gauss Legendre 3 points Formula should have 5 degree of accuracy.

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$G_3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$\text{Error} = \frac{f^{(6)}(c)}{15.750}$$

• Gauss - Legendre Formulas

... Gauss Legendre two pts Formula

$$G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \text{ with error} = \frac{1}{135} f^{(4)}(c)$$

has 3 - degree of accuracy

- Gauss legendre three pts Formula

$$G_3(F) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \text{ with error} = \frac{1}{15750} f^{(5)}$$

has 5 - degree of accuracy

$$G_n(F) = w_1 f(x_1) + w_2 f(x_2) + \dots + w_n f(x_n)$$

has $2n-1$ degree of accuracy

- $G_8(F) \rightarrow \text{error} = \frac{f^{(16)}(c)}{(16!)^3 17!} \quad \text{Very accurate Formula}$

- Theorem

if x_n 's are the pts of Gauss legendre Formula, and w_n 's are the weights in $[-1, 1]$ to apply the Formula on $[a, b]$ we use the transformation.

$$t = \frac{a+b}{2} + \frac{b-a}{2} x, \quad [-1, 1] \rightarrow [a, b]$$

$$dt = \frac{b-a}{2} dx$$

$$\int_a^b f(t) dt = \frac{b-a}{2} \sum_{k=1}^n w_k f\left(\frac{a+b}{2} + \frac{b-a}{2} x_k\right)$$

$$= \frac{b-a}{2} (w_1 f(\frac{a+b}{2} + \frac{b-a}{2} x_1) + w_2 f(\frac{a+b}{2} + \frac{b-a}{2} x_2) + \dots)$$

EXAMPLE

Use $G_3(f)$ to estimate $\int_1^5 \frac{1}{t} dt$

$$G_3(f) = 2 \left[\frac{5}{9} f\left(3 + 2\left(-\sqrt{\frac{3}{5}}\right)\right) + \frac{8}{9} f(3 + 2(0)) + \frac{5}{9} f\left(3 + 2\sqrt{\frac{3}{5}}\right) \right]$$
$$= 1.602694$$

- Gauss Legendre Formula are very accurate.