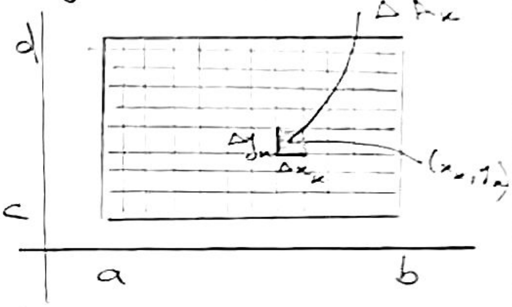
15.1 Double and Iterated Integrals over Rectangles

We defined the definite Integral of a cont. $f(x)$ over $[a,b]$ as a limit of Riemann Sum, Now:

<u>How to construct double Integral?</u>

Assume $f(x,y)$ is defined on a rectangle Region R

$R: a \leq x \leq b, \; c \leq y \leq d$



- We subdivide R into n small rectangles with ~~base~~ width Δx & height Δy.
- Each small rectangle has area:

  $\Delta A = \Delta x \Delta y$
- These small n rectangles form a partition P
- The number n gets large as Δx & Δy get smaller.
- If we order the areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, and in each ΔA_k we choose a point (x_k, y_k) and evaluate $f(x_k, y_k)$, then:

The Reimann Sum over R is:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

- As $\Delta x \longrightarrow 0$ & $\Delta y \longrightarrow 0$, the norm of the Partition $\|P\| = \max\{\Delta x, \Delta y\} \longrightarrow 0$ for any rectangle.

Hence $n \longrightarrow \infty$

Therefore:

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

$$= \lim_{\|P\|} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

$$= \iint_R f(x,y)\, dA$$

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

يعني الحسبة ب xy بطلعلي
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Remark: If $f(x,y)$ is positive over a rectangle Region R then the double Integral is the Volume of the 3-dimensional solid over the xy-plane. bounded below by R & above by the surface $z = f(x,y)$.

$$\text{Volume} = \iint_R f(x,y)\, dA \longleftarrow$$

Iterated
or repeated Integral (101)

Example: Find the volume under the plane $Z = 4 - x - y$ over the rectangle region $R: 0 \leq x \leq 2, \; 0 \leq y \leq 1$

$$V = \int_0^1 \int_0^2 (4 - x - y)\,dx\,dy = \int_0^1 \left[4x - \frac{x^2}{2} - yx \Big|_0^2 \right] dy$$

$$= \int_0^1 [8 - 2 - 2y]\,dy = 6y - y^2 \Big|_0^1 = 5$$

or:

$$V = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx = \int_0^2 \left[4y - xy - \frac{y^2}{2} \right] \Big|_0^1 dx$$

$$= \int_0^2 \left[4 - x - \frac{1}{2} \right] dx = 3.5x - \frac{x^2}{2} \Big|_0^2 = 7 - 2 = 5$$

Thm: Fubini's Theorem (First form)

If $f(x,y)$ is continuous throughout the rectangle $R: a \leq x \leq b, \; c \leq y \leq d$, then:

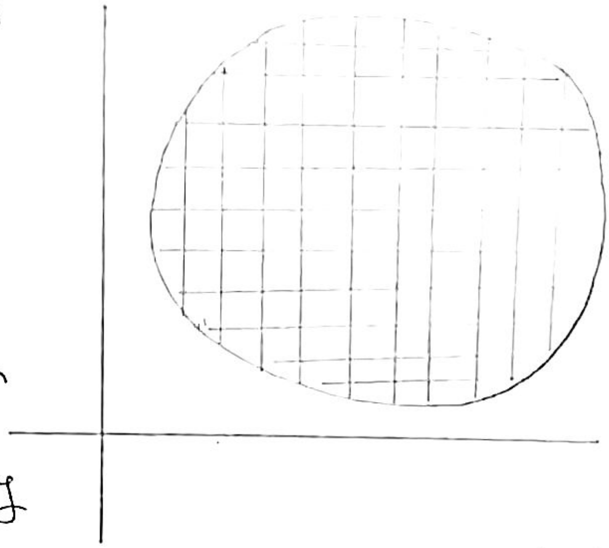
$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example: Find the Volume bounded above by $Z = 2\sin x \cos y$ & below by $0 \leq x \leq \frac{\pi}{2}, \; 0 \leq y \leq \frac{\pi}{4}$

$$V = \int_0^{\pi/4} \int_0^{\pi/2} 2\sin x \cos y\,dx\,dy = \sqrt{2}$$

15.2 Double Integrals over General Regions:

We define a double Integral of a function $f(x,y)$ over bounded general region R similarly as we did in the previous section but we approximately cover R by a grid of small rectangles that are completely inside R.



That is
$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k .$$

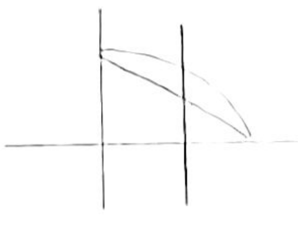
- If $z = f(x,y)$ is <u>positive</u> and continuous, then
$$\text{Volume} = \iint_R f(x,y)\,dA .$$

Thm: (Fubini's Theorem : Stronger form)

Let $f(x,y)$ be continuous on a region R :

1) If $R: a \le x \le b$, $g_1(x) \le y \le g_2(x)$ (s.t) g_1 & g_2 are continuous on $[a,b]$, then
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

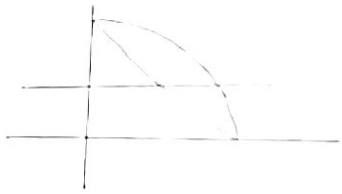
(vertical cross section)



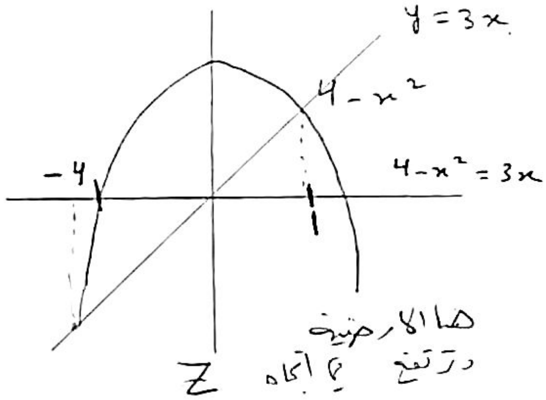
2) If $R: c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 Continuous on $[c,d]$, then:

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

(Horizontal cross section)



Example: Find the Volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$ while the top of the solid is bounded by the plane $Z = x + 4$.



Vertical:

$4 - x^2 = 3x$

$x^2 + 3x - 4 = 0 \Rightarrow x = -4, 1.$

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

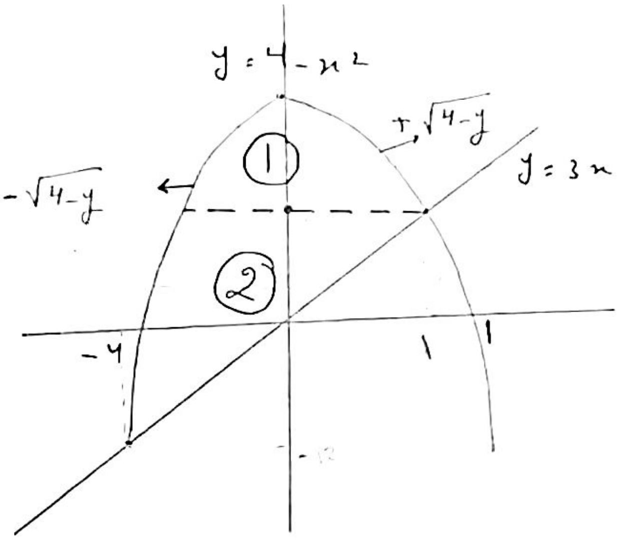
$$= \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\, dy\, dx = \int_{-4}^{1} \Big[(x+4)y\Big]_{3x}^{4-x^2} dx$$

$$= \int_{-4}^{1} \Big[(x+4)(4-x^2) - (x+4)(3x)\Big] dx = \dots = \frac{625}{12}$$

If we want to change the order of $dy\,dx$

$\Rightarrow \quad x = \frac{y}{3}$

$x = \pm\sqrt{4-y}$



we have two regions:

Then:

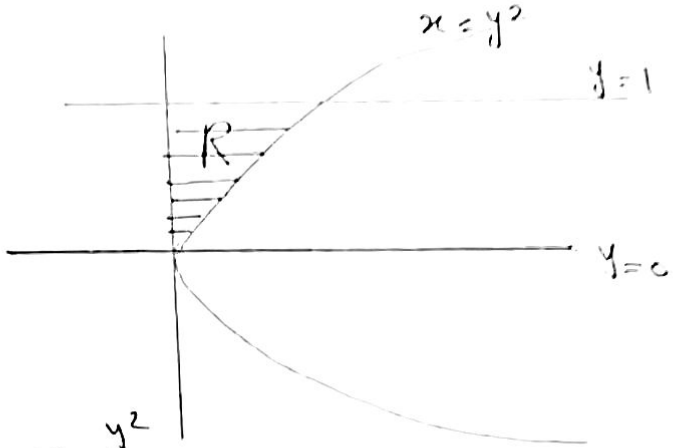
$$\int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\,dy\,dx = \int_{3}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} (x+4)\,dx\,dy + \int_{-12}^{3}\int_{-\sqrt{4-y}}^{y/3} (x+4)\,dx\,dy$$

$$= \int_{3}^{4} \left(8\sqrt{4-y}\right) dy + \int_{-12}^{3} \frac{y^2 + 33y + 72\sqrt{4-y} - 36}{18}\,dy$$

$$= \frac{16}{3} + \frac{187}{4} = \frac{64+561}{12} = \frac{625}{12}.$$

$\left(\frac{2}{2}\right)$ (104)

Example: Sketch the Region of Integration and evaluate the integral:



$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dx \, dy$$

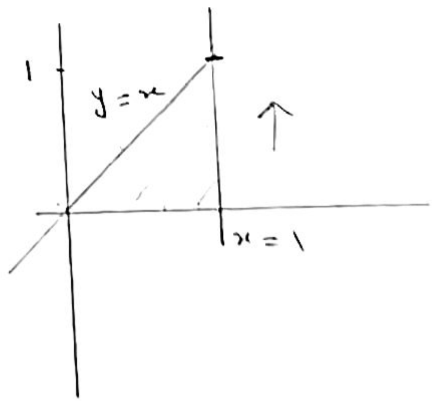
$$= \int_0^1 \frac{3y^3}{y} e^{xy} \Big|_0^{y^2} dy = \int_0^1 3y^2 e^{xy} \Big|_0^{y^2} dy$$

$$= \int_0^1 3y^2 \left[e^{y^3} - 1 \right] dy = \int_0^1 3y^2 e^{y^3} dy - \int_0^1 3y^2 \, dy$$

$$= e^{y^3} \Big|_0^1 - y^3 \Big|_0^1 = e - 1 - 1 = e - 2$$

Example: Evaluate $\iint_R \frac{\sin x}{x} \, dA$ where R is triangle in the xy plane bounded by x-axis & $y = x$ & $x = 1$

① $$\int_0^1 \int_0^x \frac{\sin x}{x} \underset{\text{vertical}}{dy} \, dx = \int_0^1 \frac{\sin x}{x} y \Big|_0^x dx$$



$$= \int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = 1 - \cos 1$$

② $$\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy = 1 - \cos 1$$

Note: It's hard here to Integrate $\frac{\sin x}{x}$ (dx)

Properties of Double Integrals:

If $f(x,y)$ & $g(x,y)$ are Continuous on bounded region R,

then:

1) $\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$

2) $\iint_R \left(f(x,y) \pm g(x,y) \right) dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$

3) ⓐ $\iint_R f(x,y)\, dA \geq 0$ if $f(x,y) \geq 0$ on R

ⓑ $\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$ if $f(x,y) \geq g(x,y)$ on R

4) If $R = R_1 \cup R_2$, then: (R_1 & R_2 nonoverlapping)

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

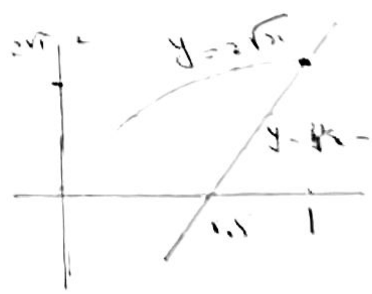
Example: Find the Volume of the solid the lies beneath

$z = 16 - x^2 - y^2$ & above R bounded by $y = 2\sqrt{x}$

& $y = 4x - 2 \rightarrow = 0 \Rightarrow x = 0.5$

$4x - 2 = 2\sqrt{x} \xrightarrow{x=1} \Rightarrow x \in [0.5, 1]$

horizontal cross. $dx\, dy$



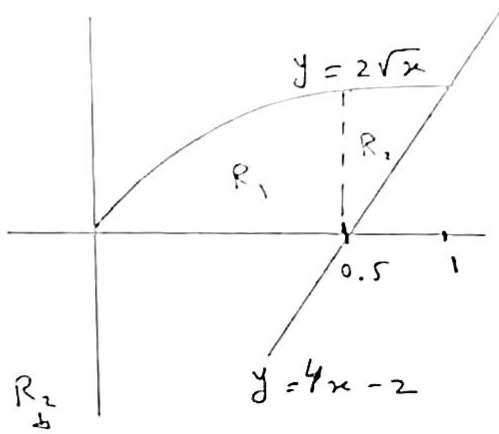
$$V = \int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy = \cdots = \frac{20803}{1680}$$

≈ 12.3827

(½) (106)

OR:

Volume $= \iint_R (16-x^2-y^2)\,dA$



$R = R_1 \cup R_2$

$R_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_2$

$$= \int_0^{0.5}\int_0^{2\sqrt{x}} (16-x^2-y^2)\,dy\,dx + \int_{0.5}^{1}\int_{4x-2}^{2\sqrt{x}} (16-x^2-y^2)\,dy\,dx$$

vertical

$$= \int_0^{0.5} -\frac{\sqrt{x}\,(6x^2+8x-96)}{3}\,dx + \int_{0.5}^{1} -\frac{\left(-76x^3+\sqrt{x}\,(6x^2+8x-96)+102x^2+144x-88\right)}{3}$$

$$= \frac{23991642}{3284995} + \frac{52916316}{10417961} \approx 12.3827.$$

Example: Find $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(x^2+1)(y^2+1)} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1\,dx}{x^2+1}\right]\frac{dy}{y^2+1}$ (*)

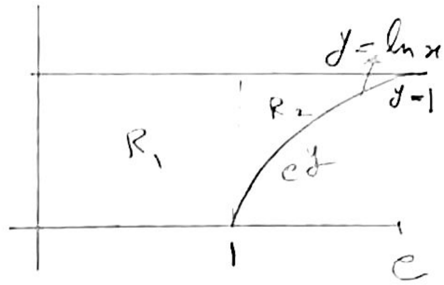
$$(*)\ \int_0^{\infty}\frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \tan^{-1}\Big|_0^b = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} 2\left(\frac{\pi}{2}\right)\frac{dy}{y^2+1} = \pi\int_{-\infty}^{\infty}\frac{dy}{y^2+1} = \pi\,(2)\int_0^{\infty}\frac{dy}{(y^2+1)} = \boxed{\pi^2}$$

$\left(\frac{2}{2}\right)(106)$

(16) Using horizontal & vertical cross section

Write $\iint_A dA$



$y = 0$, $x = 0$, $y = 1$, $y = \ln x$

(a) Vertical cross section

$$\int_0^1 \int_0^1 dy\, dx + \int_1^e \int_{\ln x}^1 dy\, dx$$

(b) horizontal cross section: $\displaystyle\int_0^1 \int_0^{e^y} dx\, dy$

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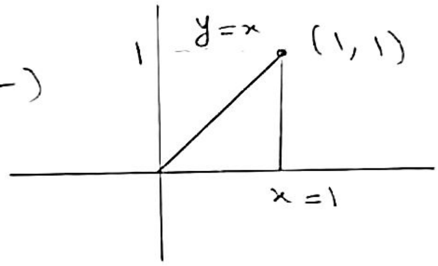
(49) $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$ (Reverse order)

NO

$y = x$



$$= \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx = \frac{e-2}{2}$$

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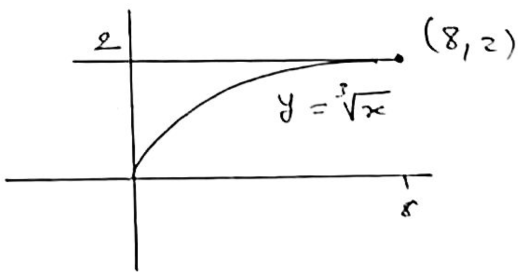
(54) $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\, dy\, dx$ (Reverse order)

NO



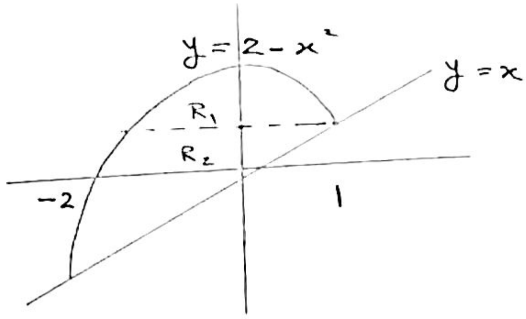
$$= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy = \frac{\ln 17}{4}$$

$\left(\frac{1}{3}\right)(107)$

(58) Find the Volume of the solid that is bounded above by the Cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ & $y = x$ in xy plane.

$2 - x^2 = x \Rightarrow x^2 + x - 2 = 0$

$\Rightarrow x = -2$ & 1



$$V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2 \, dy \, dx = \frac{63}{20}$$

$$\approx 3.15$$

OR: $y = 2 - x^2 \Rightarrow x = \pm\sqrt{2-y}$

$$V = \int_{1}^{2}\int_{-\sqrt{2-y}}^{\sqrt{2-y}} x^2 \, dx \, dy + \int_{-2}^{1}\int_{-\sqrt{2-y}}^{y} x^2 \, dx \, dy$$

$$= \int_{1}^{2} -\frac{2\sqrt{2-y}\,(y-2)}{3} \, dy + \int_{-2}^{1} \frac{y^3}{3} - \frac{\sqrt{2-y}\,(y-2)}{3} \, dy$$

$$= \frac{4}{15} + \frac{173}{60} \approx 3.15$$

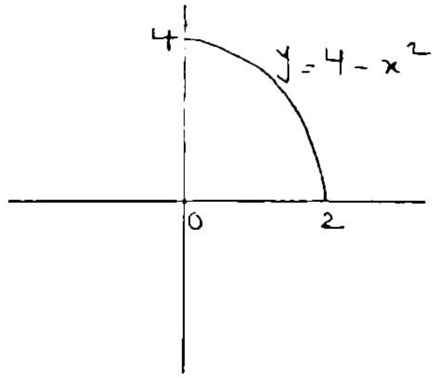
$\left(\frac{2}{3}\right)$ (107)

15.2 (62) Find the Volume of the solid cut from the first octant by the surface $Z = 4 - x^2 - y$.

In the xy plane, we assume $Z = 0$

$\Longrightarrow \quad 4 - x^2 - y = 0$

$\Longrightarrow \quad y = 4 - x^2$.



$$V = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\, dy\, dx = \frac{128}{15}$$

OR :

$$V = \int_0^4 \int_0^{\sqrt{4-y}} (4 - x^2 - y)\, dx\, dy = \frac{128}{15}$$

$\left(\frac{3}{3}\right)$ (107)

15.3 Area by Double Integration:

Recall: The Riemann sum in the definition of double Integral is:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k .$$

• If we let $f(x,y) = 1$, then the Riemann Sum becomes,

$$S_n = \sum_{k=1}^{n} \Delta A_k$$

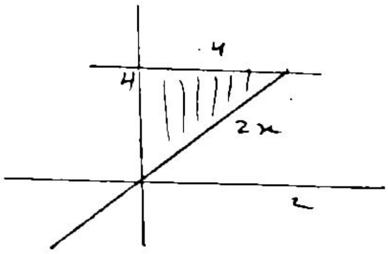
• Def: The area of a closed, bounded plane region R is

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k = \iint_R dA .$$

Example: Find the area of the Region R bounded by:

1) $x = 0$, $y = 2x$, $y = 4$

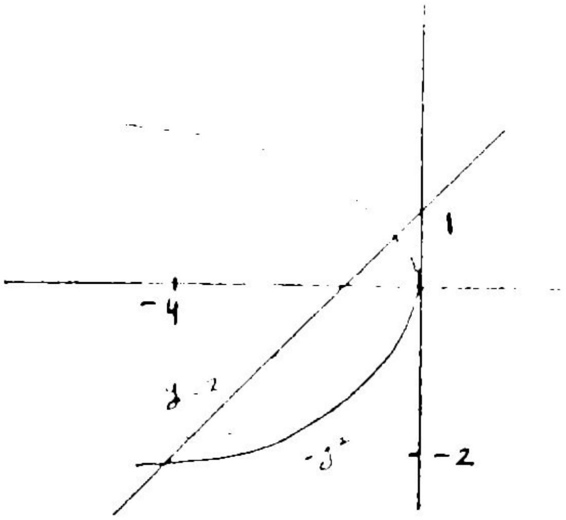
$$A = \iint_R dA = \int_0^2 \int_{y=2x}^{y=4} dy\, dA$$



$$= \int_0^4 \int_{x=0}^{x=\frac{y}{2}} dx\, dy = 4$$

(or 1 check)

2) $x = -y^2$, $y = x + 2$.



$y = -y^2 + 2$

$y^2 + y - 2 = 0$

$\therefore \quad y = -2 \ \& \ y = 1$

$$A = \iint_A dA = \int_{-2}^{1} \int_{y-2}^{-y^2} dx\, dy = \frac{9}{2}.$$

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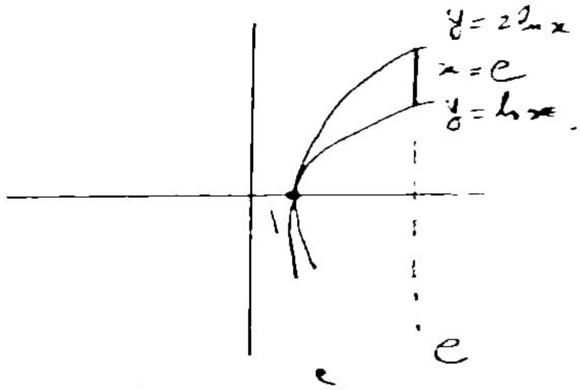
3) $y = \ln x$, $y = 2 \ln x$, $x = e$



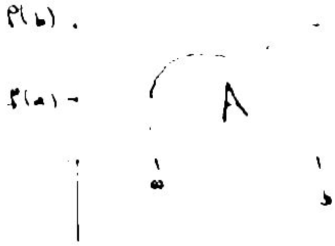
$$A = \int_{1}^{e} \int_{\ln x}^{2\ln x} dy\, dx$$

$$= \int_{1}^{e} y \Big|_{\ln x}^{2\ln x} dx = \int_{1}^{e} \overset{\text{بالاجزاء}}{\ln x}\, dx = x\ln x - x \Big|_{1}^{e} = \boxed{1}$$

Average Value : On bounded Region R

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\, dA$$

In 1D : $\text{av}(f) = \dfrac{1}{(b-a)} \displaystyle\int_a^b f(x)\, dx$



Example: Find the average value of $f(x,y) = \frac{1}{xy}$

over the square $\ln 2 \le x \le 2\ln 2$ & $\ln 2 \le y \le 2\ln 2$

$$A = \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} dx\, dy = \ln 2 \ln 2 = (\ln 2)^2$$

$$av(f) = \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} f(x,y)\, dx\, dy$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy}\, dx\, dy$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{y} \ln x \Big|_{\ln 2}^{2\ln 2} dy$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{y} \Big(\ln(\ln 4) - \ln(\ln 2) \Big) dy$$

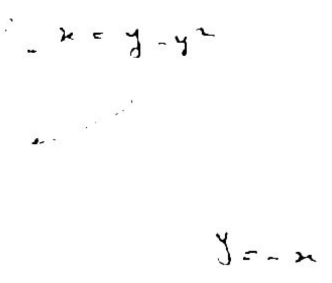
$$\ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$$

$$= \frac{1}{\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{y}\, dy$$

$$= \frac{1}{\ln 2} \Big(\ln y \Big) \Big|_{\ln 2}^{2\ln 2} = \frac{1}{\ln 2} \cdot \ln 2 = \boxed{1}$$

(4) sketch the region bounded by given lines and curves then express the region's area as a double Integral.

$x = y - y^2$ & $y = -x$



$$A = \int_0^2 \int_{-y}^{y-y^2} dx\, dy$$

$$= \int_0^2 (2y - y^2)\, dy = \frac{4}{3}$$

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(21) Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, 0 \le y \le 2$.

$$av(f) = \frac{1}{area} \int_0^2 \int_0^2 (x^2 + y^2)\, dy\, dx$$

$$area = \int_0^2 \int_0^2 dy\, dx = \int_0^2 y \Big|_0^2 dx = \int_0^2 2\, dx = 2x \Big|_0^2 = 4$$

$$\therefore av(f) = \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2)\, dy\, dx = \frac{1}{4} \int_0^2 \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^2 dx$$

$$= \frac{1}{4} \int_0^2 \left[2x^2 + \frac{8}{3}\right] dx = \frac{1}{4} \left[\frac{2x^3}{3} + \frac{8}{3}x\right]_0^2$$

$$= \frac{1}{4} \left[\frac{16}{3} + \frac{16}{3}\right] = \frac{1}{4}\left[\frac{32}{3}\right] = \boxed{\frac{8}{3}}$$

15.4 Double Integrals in Polar form:

Recall:

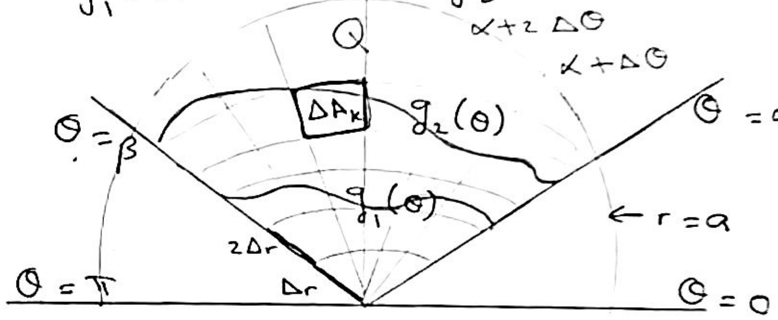
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\frac{y}{x} = \tan\theta.$$

Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ & $\theta = \beta$ and by continuous curves $r = g_1(\theta)$ & $r = g_2(\theta)$.



Suppose also $0 \le g_1(\theta) \le g_2(\theta) \le \overset{r=a}{a}$, $\forall \theta \in [\alpha, \beta]$

Then R lies in a fanshaped region Q

$$0 \le r \le a \quad \& \quad \alpha \le \theta \le \beta.$$

We cover Q by a grid of circular arcs and rays

" polar rectangles "

arcs have radius: $\Delta r, 2\Delta r, \ldots,$ $\boxed{\Delta r = a/m}$

rays are given by: $\theta = \alpha$; $\theta = \alpha + \Delta\theta, \ldots, \alpha + \Delta\theta m' = \beta$

$$\boxed{\Delta\theta = (\beta - \alpha) / m'}$$

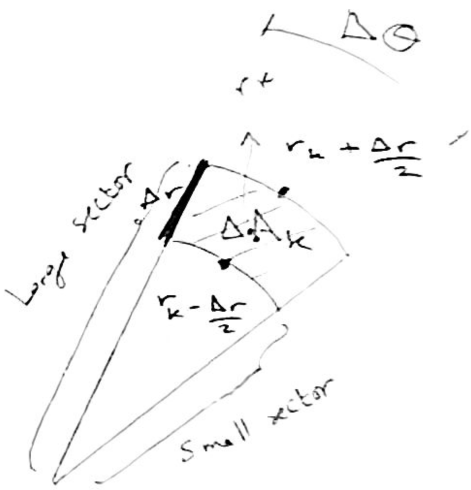
• We number the polar rectangles that lie inside R

ΔA_1 , ΔA_2 , ... , ΔA_n , Let $(r_k, \theta_k) \in$ polar rectangle k

Then : $S_n = \sum_{k=1}^{n} f(r_k, \theta_k) \, \Delta A_k$

Now Let Δr & $\Delta\theta \longrightarrow 0$, then :

$$\lim_{n \to \infty} S_n = \iint_R f(r, \theta) \, dA .$$



<u>How to find ΔA_k ??</u>

The area of a sector of a circle is

$$A = \frac{1}{2} \theta \cdot r^2$$

$\Rightarrow$ Area of small sector :

$$A_{small} = \frac{1}{2} \left(r_k - \frac{\Delta r}{2} \right)^2 \Delta\theta$$

$$A_{large} = \frac{1}{2} \left(r_k + \frac{\Delta r}{2} \right)^2 \Delta\theta .$$

$\Rightarrow$ ΔA_k = area of Large sector − area of small sector.

$$= \frac{\Delta\theta}{2} \left(\left(r_k + \frac{\Delta r}{2} \right)^2 - \left(r_k - \frac{\Delta r}{2} \right)^2 \right)$$

$$= \frac{\Delta\theta}{2} \left(2 r_k \Delta r \right) = r_k \, \Delta r \, \Delta\theta$$

Then $S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k \, \Delta r \, \Delta \theta$.

as $n \longrightarrow \infty$ & $\Delta r \longrightarrow 0$ & $\Delta \theta \longrightarrow 0$, then

$$\lim_{n \to \infty} S_n = \iint_R f(r,\theta)\, \underline{r}\, dr\, d\theta .$$

A version of Fubini's Theorem:

$$\iint_R f(r,\theta)\, dA = \int_{\alpha}^{\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, \underline{\underline{r}}\, dr\, d\theta$$

Remark: 1) If f is positive and Continuous, then:

$$\iint_R f(r,\theta)\, dA = \text{Volume in polar Coordinates.}$$

2) If $f(r,\theta) = 1$, then $\iint_R f(r,\theta)\, dA$ = area in Polar Coordinates.

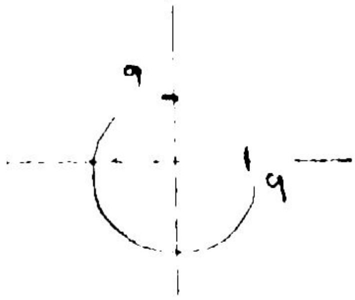
$\Longleftrightarrow$ $A = \iint_R r\, dr\, d\theta$, where R closed & bounded.

Example: (1) Describe the given region in polar Coordinates:
(outline)

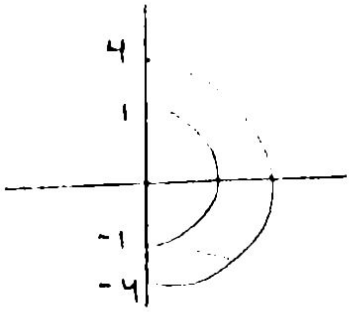
1) $x^2 + y^2 = 9^2$

$r^2 = 81$

$\Rightarrow \quad r = 9 \quad \& \quad \frac{\pi}{2} \leq \theta \leq 2\pi$



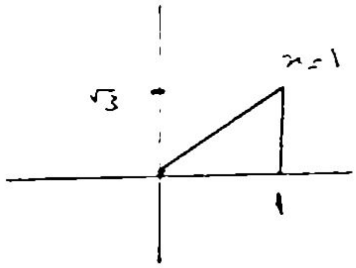
2) $1 \leq r \leq 4 \quad \& \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



3) $x = 1 \Rightarrow r\cos\theta = 1$

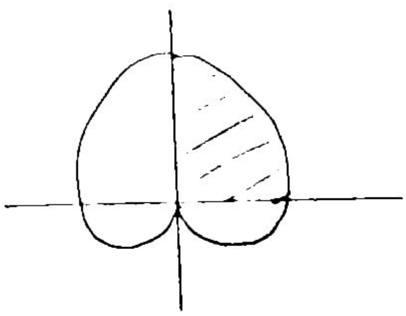
then $r = \sec\theta \quad \Rightarrow \quad 0 \leq r \leq \sec\theta$

$\& \quad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \quad \Rightarrow \quad 0 \leq \theta \leq \frac{\pi}{3}$



Example (31): Find the area of the region cut from the first quadrant by the Cardioid $r = 1 + \sin\theta$.

$$A = \int_0^{\frac{\pi}{2}} \int_0^{1+\sin\theta} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} \Big|_0^{1+\sin\theta} \right] d\theta$$



$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1+\sin\theta)^2\,d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(1 + 2\sin\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} 1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\left[\theta + -2\cos\theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{4} + 2\right] = \frac{3\pi}{8} + 1$$

Remark: Changing Cartesian integrals into Polar Integrals:

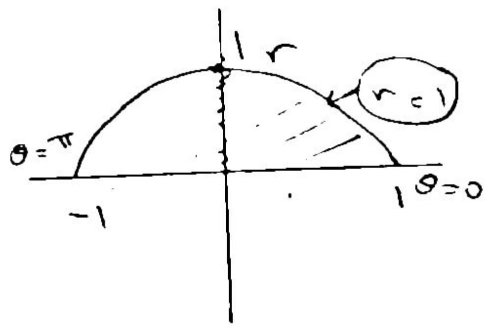
$$\iint_R f(x,y)\,dx\,dy = \iint_Q f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Why Polar Integrals are important?

Example: Find $\iint_R e^{x^2+y^2}\,dy\,dx$ where R is the semicircular region R bounded by the x-axis & $y=\sqrt{1-x^2}$.

$x^2+y^2=1$

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta$$

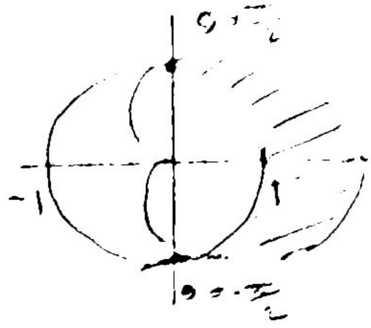


$$= \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\Big|_0^1\right] d\theta = \int_0^{\pi} \frac{1}{2}(e-1)\,d\theta$$

$$= \frac{1}{2}(e-1)\,\theta\Big|_0^{\pi} = \frac{\pi}{2}(e-1).$$

Example: Find the limits of Integration for integrating $f(r, \theta)$ over the region R that lies inside the Cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta$$



in the above example ↑

(39) If the Top of R is the plane $Z = x$ [$f(x,y)$]. Find the Volume.

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} \underline{r \cos\theta}\; r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

$$= 2 \int_{0}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

$$= \frac{2}{3} \int_{0}^{\frac{\pi}{2}} \left(3\cos^2\theta + 3\cos^3\theta + \cos^4\theta \right) d\theta$$

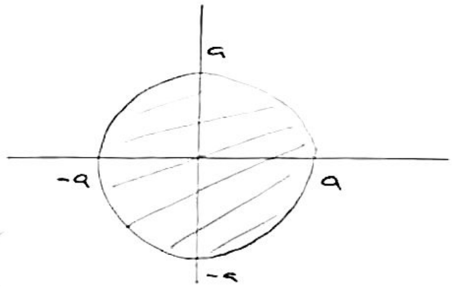
$$= \frac{2}{3} \left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32} \right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{4}{3} + \frac{5\pi}{8}.$$

The average Value of f over R is:

$$av(f) = \frac{1}{area(R)} \iint_R f(r, \theta)\, r\, dr\, d\theta.$$

Example: 33 Find the average height of the hemispherical sphere surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy-plane



$$\text{Area}(R) = \pi r^2 = a^2\pi.$$

$$\cancel{av(f) = \frac{1}{a^2\pi} \int_{-\pi/2}^{\pi/2}\int_{-a}^{a} \sqrt{a^2 - r^2}}\; r\, dr\, d\theta$$

الأرباع $\leftarrow$

$$= \frac{4}{a^2\pi} \int_0^{\pi/2}\int_0^{a} \sqrt{a^2 - r^2}\; r\, dr\, d\theta$$

$u = a^2 - r^2$

$du = -2r\,dr$

when $r = 0 \Rightarrow u = a^2$

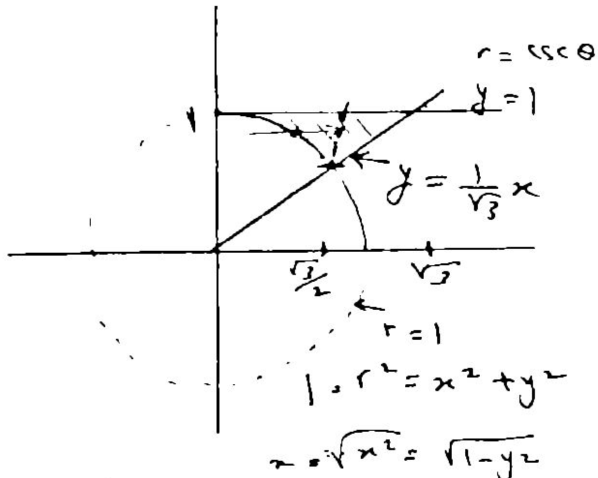
$r = a \Rightarrow u = 0$

تعويض

$$= \frac{4}{3\pi a^2} \int_0^{\pi/2} a^3\, d\theta = \frac{2a}{3}$$

15.4 (24) Convert from polar to Cartesian:

$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc\theta} r^2 \cos\theta \, dr \, d\theta$$



$r = 1$

$r = \csc\theta \Rightarrow r\sin\theta = 1 = y.$

$\theta = \frac{\pi}{6} \Rightarrow \tan\frac{\pi}{6} = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}} x$

$\theta = \frac{\pi}{2} \Rightarrow \tan\frac{\pi}{2} = \frac{y}{x}$ (No need)

1) Horizontal Cross section:

$$\int_{1/2}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{3}y} x \, dx \, dy$$

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$$\left[\begin{array}{l} y = \frac{1}{\sqrt{3}} x \\ x^2 + y^2 = 1 \end{array} \right]$$

$\Rightarrow x = \frac{\sqrt{3}}{2} \Rightarrow y = \frac{1}{2}$

or

2) Vertical Cross Section:

$$\int_{0}^{\sqrt{3}/2} \int_{\sqrt{1-x^2}}^{1} x \, dy \, dx + \int_{\sqrt{3}/2}^{\sqrt{3}} \int_{\frac{1}{\sqrt{3}}x}^{1} x \, dy \, dx$$

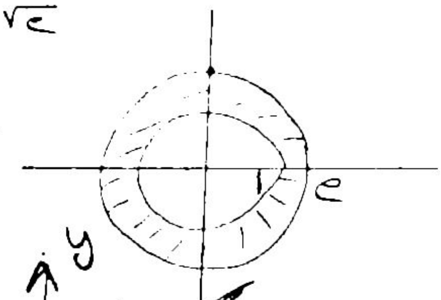
(37) Convert to a polar Integral:

$$f(x,y) = \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}, \quad 1 \le x^2+y^2 \le e$$

$$1 \le r^2 \le e$$

$$1 \le r \le \sqrt{e}$$

$$\int_0^{2\pi}\int_1^{\sqrt{e}} \frac{\ln r^2}{r}\, r\, dr\, d\theta = 2\pi(2-\sqrt{e}).$$



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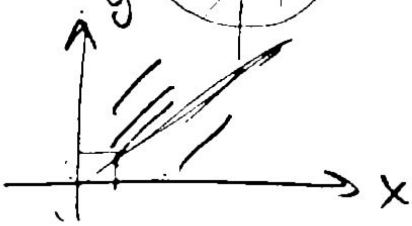
(41) Find $I = \int_0^{\infty} e^{-x^2}\, dx$



$$\underset{\text{positive}}{I^2} = \left(\int_0^{\infty} e^{-x^2}\, dx\right)\left(\int_0^{\infty} e^{-y^2}\, dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$0 < x < \infty$
$0 \le y < \infty$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\lim_{b\to\infty}\int_0^{b} r e^{-r^2}\, dr\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{2}\lim_{b\to\infty}\left(e^{-b^2}-1\right) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}.$$

---

(42) Evaluate $\int_0^{\infty}\int_0^{\infty} \frac{1}{(1+x^2+y^2)^2}\, dx\, dy$.

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\lim_{b\to\infty}\int_0^{b} \frac{r}{(1+r^2)^2}\, dr\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\lim_{b\to\infty} \frac{-1}{2(1+r^2)}\Bigg|_0^{b}\right] d\theta = \ldots \quad \text{check!}$$

Example: Find the volume under the plane $Z = 4 - x - y$ over the rectangle region $R: 0 \le x \le 2 \ , \ 0 \le y \le 1$.

$$V = \int_0^1 \int_0^2 (4 - x - y)\,dx\,dy = \int_0^1 \left[4x - \frac{x^2}{2} - yx \Big|_0^2 \right] dy$$

$$= \int_0^1 [8 - 2 - 2y]\,dy = 6y - y^2 \Big|_0^1 = 5$$

or:

$$V = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx = \int_0^2 \left[4y - xy - \frac{y^2}{2}\right] \Big|_0^1 dx$$

$$= \int_0^2 \left[4 - x - \tfrac{1}{2}\right] dx = 3.5x - \frac{x^2}{2} \Big|_0^2 = 7 - 2 = 5$$

Thm: Fubini's Theorem (First form)

If $f(x,y)$ is continuous throughout the rectangle $R: a \le x \le b, \ c \le y \le d$, then:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example: Find the Volume bounded above by $Z = 2\sin x \cos y$ & below by $0 \le x \le \frac{\pi}{2} \ , \ 0 \le y \le \frac{\pi}{4}$

$$V = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} 2\sin x \cos y\,dx\,dy = \sqrt{2}$$