Ch. 14 partial Derivatives:

14.1 Functions of Several Variables:

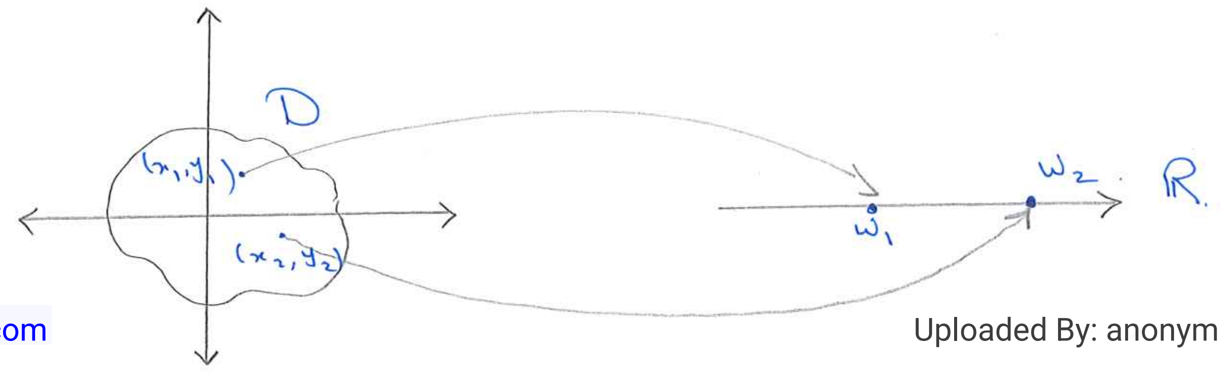
Def: Suppose D is a set of n-tuples of real numbers $(x_1, x_2, ..., x_n)$. A real-valued function $f: D \longrightarrow \mathbb{R}$ is a rule that assigns a Unique real number

$$w = f(x_1, ..., x_n)$$

to each element in D.

- D is the domain
- set of w-values is the range.

Example:



Independent Variable = Input $(x_1, ..., x_n)$ → $\boxed{F}$ → output = dependent Variable $w = f(x_1, ..., x_n)$

Example: Find $f(1,-2)$ if $f(x,y) = x^2+3xy$.

$$f(1,-2) = 1 + 3(1)(-2) = -5.$$

Example: Find $f(2,3,5)$ if $f(x,y,z) = \dfrac{2x}{1+3xy+z^2}$.

$$f(2,3,5) = \frac{2(2)}{1+3(2)(3)+(5)^2} = \frac{4}{44}$$

Example: Find domain & Range of the following:

1) $z = \sqrt{y-x^2}$, domain $y \geq x^2$, Range $[0,\infty)$

2) $z = \dfrac{1}{xy}$, Domain $xy \neq 0$, Range $(-\infty,0)\cup(0,\infty)$

3) $z = \sin xy$, Domain entire plane ~~space~~ , Range $[-1,1]$.

4) $w = \sqrt{x^2+y^2+z^2}$, Domain: Entire space $= \{(x,y,z) \in \mathbb{R}^3\}$, Range $[0,\infty)$.

5) $w = xy \ln z$, Domain - space $z > 0$, Range $(-\infty,\infty)$

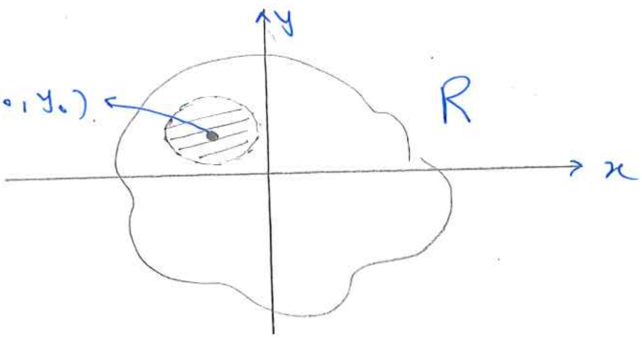
6) $w = \dfrac{1}{x^2+y^2+z^2}$, Domain $(x,y,z) \neq (0,0,0)$, Range $(0,\infty)$

Functions of two Variables:

Def: Interior point:

A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if it is the center of a disk of positive radius that lies entirely in R.

(x_0, y_0, z_0) in a space is Interior point of R if it is the center of a solid ball that Lies entirely in R



Def: Boundary point:

A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R.

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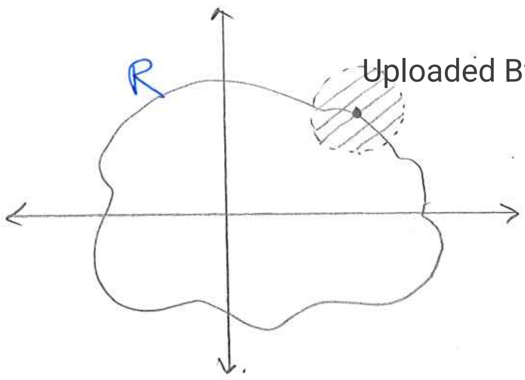
(Note $(x_0, y_0)\begin{cases} \in \\ \notin \end{cases} R$)



(x_0, y_0, z_0) is boundary.
if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside R as well as points lie inside R

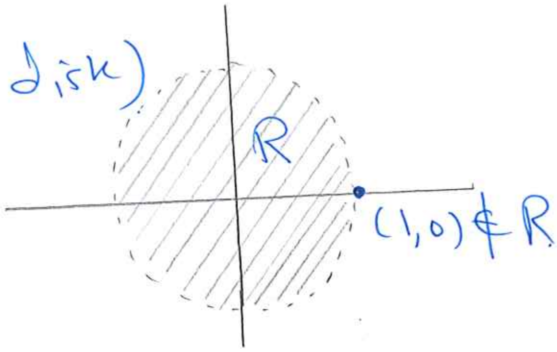
Def: (Interior of R) is the set of all interior points of R.

Def (Boundary of R) is the set of all boundary points of R

Def: The region R is open if it consists entirely of interior points. (i.e every point in R is interior).

Def: The region R is closed if it contains all its boundary pt.

Example: $x^2+y^2<1$ (open Unit disk)

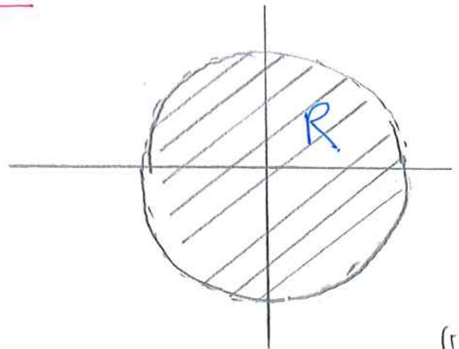


- Interior points of R
  $\{(x,y)\in\mathbb{R}: x^2+y^2<1$

- Boundary points of R:
  $\{(x,y)\in R : x^2+y^2=1\}$

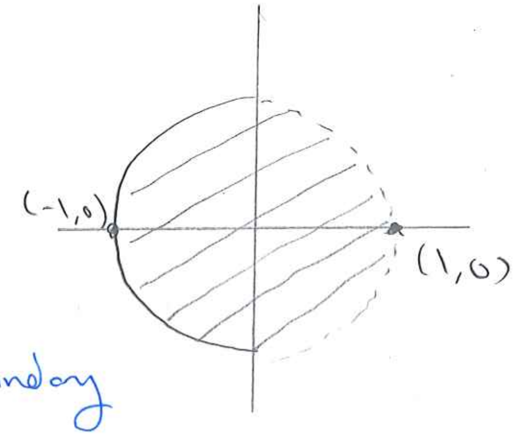
- R is open, since every point $(x,y)\in R$ is Interior

Example: $x^2+y^2\leq 1$



R is closed, since every it contains all its boundary pts.

Example: • R is not open, since $(-1,0) \in R$, but $(-1,0)$ is not Interior.

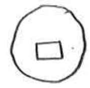


• R is not closed since $(1,0)$ is boundary but $(1,0) \notin R$

Def: (Bounded): A region in the plane is bounded if it lies inside a disk of fixed radius

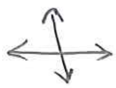
• If its not bounded, we say R is unbounded.

Example: Bounded sets:

line segments , triangles , rectangles 

disks 

Example: Unbounded sets:

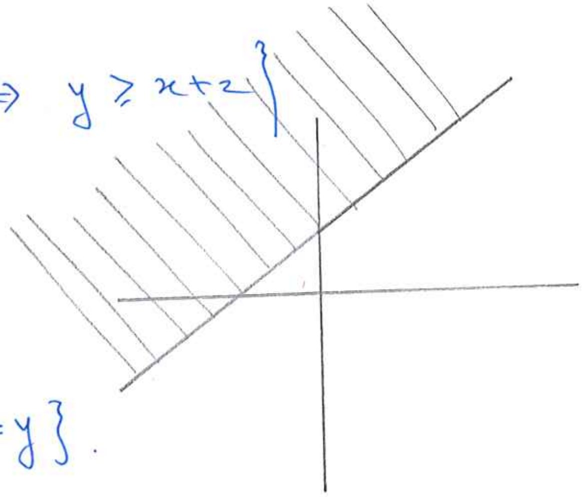
Lines , Coordinate axes , quadrants 

half planes , the plane 

Example: Find and sketch the domain of:

1) $f(x,y) = \sqrt{y-x-2}$.

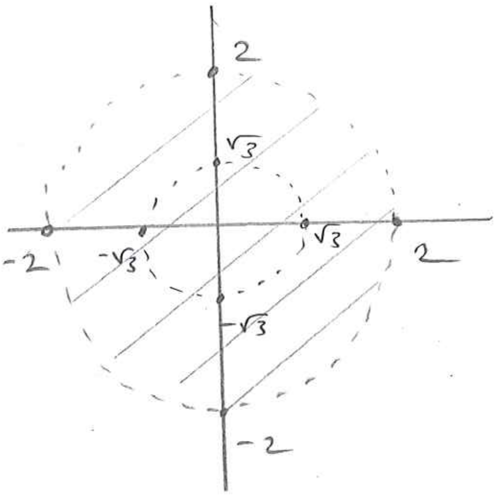
$D = \{ (x,y) \in \mathbb{R}^2 : y-x-2 \geq 0 \Leftrightarrow y \geq x+2 \}$



- R is unbounded
- closed since it contains all the boundary points $= \{ (x,y) \in \mathbb{R}^2 : x+2 = y \}$.
- Interior: $\{ (x,y) \in \mathbb{R}^2 : y > x+2 \}$.

2) $f(x,y) = \dfrac{1}{\ln(4-x^2-y^2)}$.

$D = \{ (x,y) \in \mathbb{R}^2 : 4-x^2-y^2 > 0 \text{ \& } 4-x^2-y^2 \neq 1 \}$



$4-x^2-y^2 > 0 \Rightarrow x^2+y^2 < 4$

$4-x^2-y^2 \neq 1 \Rightarrow x^2+y^2 \neq 3$

$(0, \sqrt{3}) \notin D$

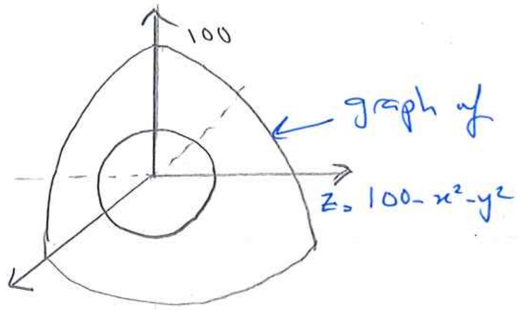
Def: The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = C$ is called a level curve of f.

Def: The set of all points $(x,y,f(x,y))$ in space, for (x,y) in the domain of f, is called the graph of f.

The graph of f is also called the surface $z = f(x,y)$.

Example: Find and sketch the level curves of

1) $f(x,y) = 100 - x^2 - y^2$ at $c = 75$



$75 = c = 100 - x^2 - y^2$

$\Rightarrow \quad x^2 + y^2 = 25$

Domain is xy-plane.

Range: $\{(x,y) \in \mathbb{R}^2 ; 100 > x^2 + y^2\}$.

2) $f(x,y) = \sqrt{y-x}$, $c = 1$

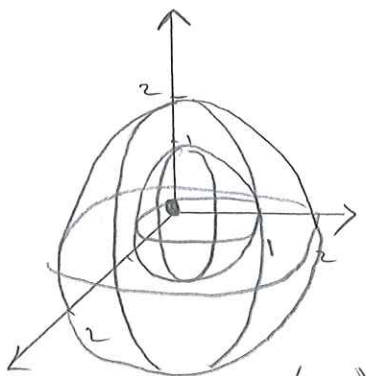
$1 = y - x \Rightarrow y = 1 + x$ (line)

3) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$, $c = 0, 1, 2$

$c = 0 \Rightarrow \quad 0 = x^2 + y^2 + z^2$. (point)

$c = 1 \Rightarrow \quad 1 = x^2 + y^2 + z^2$ (sphere)

$c = 2 \Rightarrow \quad 4 = x^2 + y^2 + z^2$ (sphere)



Example:

1) Interior of sphere is <u>bounded</u>.

2) The open half space $z > 0$ is <u>unbounded</u>

3) first octant $x > 0, y > 0, z > 0$ is <u>unbounded</u>

4) space it self is <u>unbounded</u>

5) Lines is <u>unbounded</u>. (closed)

6) closed half space $z \geq 0$ is <u>unbounded</u>. (closed)

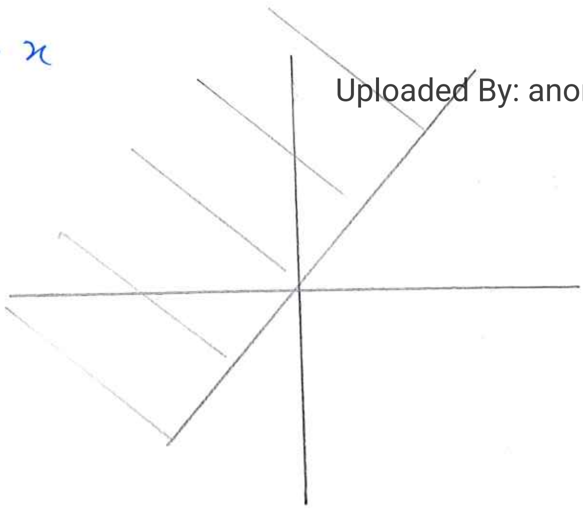
Example: $f(x,y) = \sqrt{y - x}$.

1) Domain $= \{(x,y) \in \mathbb{R}^2 : y \geq x\}$.

2) Range: $[0, \infty)$

3) Level Curves : $c = \sqrt{y - x} \Rightarrow c^2 = y - x \Rightarrow \boxed{y = x + d}$

4) Boundary of the domain $y = x$



5) D is closed.

6) D is unbounded

14.1 (17) $f(x,y) = y - x$

- Domain $= \mathbb{R}^2$
- Range $= \mathbb{R}$
- Level curve: $y - x = c \Rightarrow y = x + c$ (line) // $y = x$
- Boundary: No Boundary points
- Both open and closed
- unbounded.

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(22) $f(x,y) = \frac{y}{x^2}$.

- Domain $= \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,y)\}$.
- Range $= \mathbb{R}$
- level Curve $\frac{y}{x^2} = c \Rightarrow y = cx^2$ without the origin.
- Boundary : $x = 0$
- open
- unbounded.

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(23) $f(x,y) = \dfrac{1}{\sqrt{16-x^2-y^2}}$

- Domain $= \{(x,y) \in \mathbb{R}^2 : 16 > x^2+y^2\}$
- Range : $z \geq \frac{1}{4}$
- Level Curve : $\dfrac{1}{\sqrt{16-x^2-y^2}} = c$

$\Rightarrow \quad 16c^2 = c^2x^2 + c^2y^2 \Rightarrow 16 = x^2+y^2$

$\therefore$ circle with radius 4 centered at origin

- Boundary : $= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 16\}$
- Interior : $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 16\}$
- open Region
- bounded.

14.2 Limits and Continuity in Higher Dimensions:

Example: $\lim_{(x,y)\to(1,0)} \dfrac{e^y}{2x-1} = \dfrac{e^0}{2(1)-1} = \dfrac{1}{1} = 1$

2) $\lim_{(x,y,z)\to(0,1,2)} \dfrac{\cos x - \ln y + z^2}{x^2 + z^2} = \dfrac{\cos 0 - \ln 1 + 2^2}{(0)^2 + (2)^2} = \dfrac{5}{4}$.

3) $\lim_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x} = \lim_{x\to 0} \dfrac{e^0 \sin x}{x} = 1$

4) $\lim_{(x,y)\to(2,2)} \dfrac{x-y}{x^4-y^4} = \lim_{(x,y)\to(2,2)} \dfrac{x-y}{(x^2-y^2)(x^2+y^2)}$

$= \lim_{(x,y)\to(2,2)} \dfrac{x-y}{(x-y)(x+y)(x^2+y^2)} = \lim_{(x,y)\to(2,2)} \dfrac{1}{(x+y)(x^2+y^2)} = \dfrac{1}{32}$.

5) $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{(x^2+y^2)} = \lim_{u\to 0} \dfrac{\sin u}{u} = 1$

22

6) $\lim_{(x,y)\to(0,0)} \dfrac{1-\cos xy}{xy}$, Let $u = xy$
$(x,y)\to(0,0)$
$\Rightarrow u \to 0$

$\lim_{u\to 0} \dfrac{1-\cos u}{u} \overset{L.H}{=} \lim_{u\to 0} \dfrac{\sin u}{1} = 0$

Thm: properties of limits of functions of two variables

Assume $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ & $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$.

1. Sum Rule: Difference $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \pm g(x,y) \right) = L \pm M$

2. Constant Multiple root: $\lim_{(x,y)\to(x_0,y_0)} k\, f(x,y) = k\, L$.

3. Product Rule: $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \cdot g(x,y) \right) = L \cdot M$.

4. Quotient Rule: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$.

5. Power Rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n positive Integer,

6. Root Rule: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{\frac{1}{n}}$

where $n \in \mathbb{Z}^+$ & if n is even, $L > 0$.

Two - Path Test for Nonexistence of a limit.

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \longrightarrow (x_0, y_0)$ then the $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example:
Two paths

$$\left. \begin{aligned} y &= kx \\ y &= kx^2 \end{aligned} \right\} \text{ satisfy } (x_0, y_0).$$

Example(1): Show that the function $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$. has no limits as $(x,y) \longrightarrow (0,0)$.

$\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{2x^2y}{x^4+y^2}$, along the path $y = kx^2 \Rightarrow \lim\limits_{x\to 0} \dfrac{2x^2y}{x^4+y^2}$

$$= \lim_{x\to 0} \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \lim_{x\to 0} \frac{2k\,x^4}{x^4+k^2x^4} = \lim_{x\to 0} \frac{2k}{1+k^2}$$

$$= \frac{2k}{1+k^2} = \begin{cases} 0 & , \text{ if } k = 0 \\ 1 & , \text{ if } k = 1 \end{cases}$$

k Not constant.
k is parameter

So the limit Does not exist.

Remark: Having the same limit along all straight line ($y = kx$) does not meen the limit exists.

Explination of the Remark ↓

Example (2):

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^2y}{x^4+y^2} \quad \text{along path } y = kx$$

$$= \lim_{x\to 0} \frac{2x^2(kx)}{x^4+(kx)^2} = \lim_{x\to 0} \frac{2kx^3}{x^4+kx^2} = \lim_{x\to 0} \frac{2kx}{x^2+k^2} = 0$$

$\Rightarrow$ So this does not mean $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = 0$

since limit DNE.

Example: Find a level Curve for the function

$$f(x,y,z) = \frac{x - \sin y + e^z}{x^2+y^2} \quad \text{through } (1,0,0).$$

$$c = f(1,0,0) = \frac{1 - \sin 0 + e^0}{1^2+0^2} = 2.$$

$$\Rightarrow \quad 2 = \frac{x - \sin y + e^z}{x^2+y^2}.$$

Level Curve is Surface.

Example: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{|xy|}$ DNE.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = \lim_{x\to 0} \frac{x(kx)}{|x(kx)|} \text{ along the path } y=kx.$$

$$= \lim_{x\to 0} \frac{kx^2}{|kx^2|} = \lim_{x\to 0} \frac{kx^2}{|k|x^2} = \frac{k}{|k|} = \begin{cases} 1, & \text{if } k>0 \\ -1, & \text{if } k<0 \end{cases}$$

Example: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4}{x^4+y^4}$ DNE. $y=kx^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4} = \lim_{x\to 0} \frac{x^4}{x^4+(kx^2)^4} = \lim_{x\to 0} \frac{x^4}{x^4[1+k^4x^4]}$$

$$= \lim_{x\to 0} \frac{1}{1+k^4x^4} = 1, \text{ No Conclusion.}$$

Now: $$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^4} = \lim_{x\to 0} \frac{x^4}{x^4+(kx)^4} = \lim_{x\to 0} \frac{1}{1+k^4}$$

$$= \begin{cases} 1, & \text{if } k=0 \\ \frac{1}{2}, & \text{if } k=1 \end{cases}$$

$\Rightarrow$ The limit DNE.

Def: The function $f(x,y)$ is Continuous at (x_0,y_0)

if
1) f is defined at (x_0,y_0)

2) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

3) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

Def: f is continuous on Region R if it is Continuous at every point in R.

Examples:

$\boxed{1}$ $f(x,y) = \dfrac{x+y}{x-y}$

f is continuous at a point $(x,y)\in\mathbb{R}^2$ (s.t) $x \neq y$.

$\boxed{2}$ $f(x,y) = \dfrac{x+y}{2+\cos xy}$

f is Continuous at all point in plane $(\mathbb{R}^2)$

$\boxed{3}$ $f(x,y,z) = \ln(1-x^2+y^2-z)$

f is Continuous at all points $(x,y,z)\in\mathbb{R}^3$

(s.t) $1-x^2+y^2-z > 0$

Remark: 1) The extension of results for functions of two variables is apply for functions of several variables.

2) If $f(x,y)$ is continuous at (x_0,y_0) and $g(x)$ is continuous at $z = f(x_0,y_0)$ then the composition $g \circ f$ is continuous at (x_0,y_0).

$h = (g \circ f)(x_0,y_0) = g(f(x_0,y_0))$.

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55 Does knowing that:

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell any thing about $\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}$?

$$\lim_{(x,y)\to(0,0)} 1 - \frac{x^2y^2}{3} = 1 \quad \& \quad \lim_{(x,y)\to(0,0)} 1 = 1$$

$\Rightarrow$ by Sandwish thm:

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1.$$

14.2 50 Show that the limit does not exist

for $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2} = \frac{0}{0}$

Substitue $y=-1 \Rightarrow$

$$\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1} \frac{-x+1}{x^2-1} = \lim_{x\to 1} \frac{1-x}{(x-1)(x+1)}$$

$$= \boxed{\frac{-1}{2}}$$, No Conclusion.

Choose $y=-x^2 \Rightarrow$

$$\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1} \frac{-x^3+1}{x^2-x^4} = \lim_{x\to 1} \frac{1-x^3}{x^2(1-x^2)}$$

$$= \lim_{x\to 1} \frac{(1-\cancel{x})(x^2+x+1)}{x^2(1-\cancel{x})(1+x)} = \boxed{\frac{3}{2}}$$

So we have two different limits along two different paths $\Rightarrow$ limit DNE.

$\frac{2}{3}$ (47)

14.2 (63) $f(x,y) = \dfrac{y^2}{x^2+y^2}$ $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{0}{0}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{(kx^2)^2}{x^2+(kx^2)^2} = \lim_{x\to 0} \frac{k^2x^4}{x^2+k^2x^4} =$$

$$= \lim_{x\to 0} \frac{k\ x^2}{1+k^2x^2} = 0$$ (No, conclusion).

Using polar Coordinates:

Let $x = r\cos\theta$, $y = r\sin\theta$, then Using the path $y = r\sin\theta$ we have

$$\lim_{r\to 0} \frac{r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r^2\sin^2\theta}{r^2} = \sin^2\theta$$

Since we have $\sin^2\theta$ which depends on θ & has the value between 0 & 1.

OR : $y = kx$:

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim_{x\to 0} \frac{k^2x^2}{x^2+k^2x^2} = \lim_{x\to 0} \frac{k^2}{1+k^2}$$

$$= \begin{cases} 0 & , k=0 \\ \frac{1}{2} & , k=1 \end{cases}$$

$\left(\frac{3}{3}\right)(47)$

14.3 partial Derivatives:

$f(x) = 2x^3 - 5x + 1$

$f'(x) = 6x^2 - 5$ ordinary derivative.

Def: The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided that the limit exist. (Similarly $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$).

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Note: • The first partial derivative of $f(x,y)$ with respect to x is denoted by $f_x = \frac{\partial f}{\partial x}$

• The second partial derivatives of $f(x,y)$ w.r.t x is $f_{xx} = \frac{\partial^2 f}{\partial x^2}$

• The second mixed partial derivative of $f(x,y)$ is

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x}.$$

Example: Let $f(x,y) = \cos y + e^{x} + 2xy$, Find:

1) $f_x(\ln 2, \pi)$

$f_x = e^{x} + 2y \Rightarrow f_x(\ln 2, \pi) = 2 + 2\pi$.

2) $f_y(e, \pi)$

$f_y = -\sin y + 2x \Rightarrow f_y(e, \pi) = 0 + 2\pi = 2\pi$.

3) $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \boxed{2}$.

Example: $f(x,y,z) = xy + yz + xz$ find $f_z(2,-1,1)$.

$f_z = y + x \Longrightarrow f_z(2,-1,1) = 1$

Theorem: The Mixed derivative Theorem:

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open Region Containing a point (a,b) and all are continuous at (a,b), then:

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Example: Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$

$$\frac{\partial w}{\partial x} = y \quad \& \quad \frac{\partial^2 w}{\partial x \partial y} = 1 . = \frac{\partial^2 w}{\partial y \partial x}$$

Example: $g(x,y) = x^2 y + \cos y + y \sin x$.

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial y}\left(2xy + y \cos x \right) = 2x + \cos x .$$

Example: Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2 z + x^2 y$.

$$f_y = -2x(2y)z + x^2 = -4xyz + x^2$$

$$f_{yx} = -4yz \quad , \quad f_{yxy} = -4z \quad , \quad f_{yxyz} = \boxed{-4}$$

Example: Use partial derivative definition to find $\left.\frac{\partial f}{\partial x}\right|_{(1,2)}$ for $f(x,y) = 1 - x + y - 3x^2 y$.

$$\frac{\partial f(1,2)}{\partial x} = \lim_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h}$$

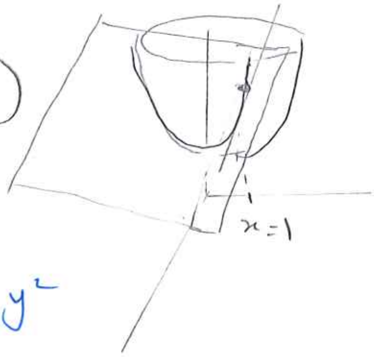
$$= \lim_{h \to 0} \frac{\left(1 - (1+h) + 2 - 3(1+h)^2(2)\right) - \left(1 - 1 + 2 - 3(1)(2)\right)}{h}$$

$$= \lim_{h \to 0} \frac{-h - 12h - 6h^2}{h} = \lim_{h \to 0} -13 - 6h = \boxed{-13}$$

Example: The plane $x=1$ intersects the paraboloid $Z = x^2+y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,2,3)$.

$$\left.\frac{\partial Z}{\partial y}\right|_{(1,2)} = 2y = 4.$$

(x constant)



we can assume the graph $Z = x^2+y^2 = (1)^2+y^2$

$\therefore Z = 1+y^2.$

$$\left.\frac{\partial Z}{\partial y}\right|_{y=2} = 2y = 4.$$

Differentiability:

Def: A function $Z = f(x,y)$ is differentiable at (x_0,y_0) if ① $f_x(x_0,y_0)$ and ② $f_y(x_0,y_0)$ exist and ΔZ satisfies:

$$\Delta Z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which $\epsilon_1, \epsilon_2 \longrightarrow 0$ as both $\Delta x, \Delta y \longrightarrow 0$

- we say f is differentiable if its differentiable at every point in its domain. (we say its graph is a smooth surface)

$\frac{1}{3}$

Differentiability: Error in Differential Approximation.

Recall: Let $f(x)$ be differentiable at $x = x_0$

and suppose that $dx = \Delta x$ is an increment of x.

so f changes as x changes from x_0 to $x_0 + \Delta x$

The true change: $\Delta f = f(x_0 + \Delta x) - f(x_0)$

The differential estimation: $df = f'(x_0)\Delta x$

Now: $\overset{\text{error}}{\Delta f - df} = f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x$

$$= \Bigg[\underbrace{\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0)}_{\text{call this part } \epsilon}\Bigg]\Delta x$$

$$= \epsilon \, \Delta x.$$

$$\Rightarrow \underbrace{\Delta f}_{\text{true change}} = df + \epsilon\,\Delta x = \underbrace{f'(x_0)\Delta x}_{\text{estimated change}} + \underbrace{\epsilon\,\Delta x}_{\text{error}}$$

in which $\epsilon \longrightarrow 0$ & $\Delta x \longrightarrow 0$

Now for functions of two variables:

Thm: The Increment Thm for functions of two Variables

Suppose that the first partial derivatives of $f(x,y)$ are defined throughout an open region R containing the point (x_0,y_0) & f_x & f_y are continuous at (x_0,y_0). Then:

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

should satisfy an equation of the form:

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

s.t $\quad \epsilon_1, \epsilon_2 \longrightarrow 0 \quad$ & $\quad \Delta x, \Delta y \longrightarrow 0$

Def: A function $z = f(x,y)$ is differentiable at (x_0,y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

(s.t) $\quad \epsilon_1, \epsilon_2 \longrightarrow 0 \quad$ and $\quad \Delta x, \Delta y \longrightarrow 0$

Def: f is differentiable if it is differentiable at every point in its domain, (we say that the graph is a smooth surface).

Thm: if the partial derivatives of f_x & f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Thm: If a function $f(x,y)$ is differentiable at (x_0,y_0) then f is continuous at (x_0,y_0).

Not cont $\Rightarrow$ Not diff

Example: $f(x,y)=\begin{cases} 0 & , \text{ if } xy \neq 0 \\ 1 & , \text{ if } xy = 0 \end{cases}$

1) prove that f is not continuous at origin.

2) show that f_x and f_y exist at origin. (i-e) Not differentiable

In 1 d. existence of derivative $\Rightarrow$ continuity. Here NO

sol:

1) $f(0,0)=1$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \neq 1.$$

That means that the tangent plane is above the surface
Not on it directly

2) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{1-1}{h} = 0$

$f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{1-1}{h} = 0$

Note: f is Not differentiable, Since

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0) = f(\Delta x,\Delta y) - f(0,0) = 0-1$$

$$= \boxed{-1} \neq \underbrace{f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y}_{=0}$$

14.3 (60) Use the limit def. of partial derivative to Compute the partial derivative of: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}\Big|_{(0,0)}$

$$f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{\sin h^3}{h^2} - 0}{h} = 1$$

---

(91) Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

show that $f_x(0,0)$ and $f_y(0,0)$ exist but f is not differentiable at $(0,0)$.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h \cdot 0^2}{h^2+0^4} - 0}{h} =$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y \to 0} \frac{(ky^2)y^2}{(ky^2)^2+y^4} = \frac{k}{k^2+1} \Rightarrow \text{Limit DNE}$$

$\Rightarrow f(x,y)$ is not Conti ~~on~~ at $(0,0)$, then by thm 4, $f(x,y)$ is not differentiable at $(0,0)$

$(\frac{1}{2})$

14.3 65 Find the value of $\partial z/\partial x$ at $(1,1,1)$ if the equation $xy + z^3 x - 2yz = 0$ defines z as a function of two Independent variables x and y and the partial derivative exists.

$$y + z^3 + x\left(3z^2 \frac{\partial z}{\partial x}\right) - 2y\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad y + z^3 + \left(3xz^2 - 2y\right)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-y - z^3}{(3xz^2 - 2y)}.$$

$$\Rightarrow \quad \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \frac{-2}{1} = \boxed{-2}$$

14.4 The Chain Rule:

$y = f(x)$, $x = x(t)$

$\frac{dy}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$

Thm: Chain Rule for functions of One Independent variable (t) and Two Intermediate Variables ($x(t)$, $y(t)$):

If $w = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the Composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

one Independent

or $$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Note: 1) $\frac{\partial z}{\partial x} = z_x$ while $\frac{dz}{dx} = z'(x)$

2) If $w = f(x,y,z)$, where $x = x(t)$, $y = y(t,s)$

$z = z(s)$, then:

- $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \underset{\partial f}{\frac{\partial w}{\partial y}} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot 0$

- $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot 0 + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{dz}{ds}$

Thm: If $w = f(x, y, z)$ such that

$x = g(r,s)$, $y = h(r,s)$, $z = k(r,s)$, then:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Example: Find $\partial w/\partial t$ if $w = xy + z$ for:

$x = \cos t$, $y = \sin t$, $z = t$ at $t = 0$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= -y\sin t + x\cos t + 1$$

$$= 1 + \cos 2t$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = 1 + \cos 0 = \boxed{2}$$

 $Z = \tan^{-1} x$, $x = e^{u} + \ln v$

Find $\frac{\partial z}{\partial u}$ at $(u, v) = (\ln 2, 1)$.

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\cdot\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)\cdot\left(e^{u}\right) = \frac{5e^{u}}{1+(e^{u}+\ln v)^2}$$

$$\left.\frac{\partial z}{\partial u}\right|_{(\ln 2, 1)} = \frac{5e^{\ln 2}}{1+\left(e^{\ln 2}+\ln 1\right)^2} = \boxed{2}.$$

Thm: Implicit differentiation:

suppose that $F(x,y)$ is differentiable and that the equation $F(x,y)=0$ defines y as a differentiable function of x, then at any ~~xxxx~~ point where $F_y \neq 0$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}.$$

proof: Let $w = F(x,y)$, then:

$$0 = \frac{dw}{dx} = F_x \cdot \frac{dx}{dx} + F_y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}.$$

Example: If $y^2 - x^2 - \sin xy = 0$, find dy/dx.

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(-2x - y\cos xy)}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

OR:

$$2yy' - 2x - (\cos xy)(xy' + y) = 0$$

$$2yy' - 2x - xy'\cos xy - y\cos xy = 0$$

$$y'(2y - x\cos xy) = 2x + y\cos xy$$

$$\therefore y' = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

Remark: Let $w = F(x,y,z) = 0$ (s.t) $z = f(x,y)$

$$\Rightarrow \quad 0 = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= F_x \cdot (1) + F_y \cdot (0) + F_z \cdot \frac{\partial z}{\partial x}.$$

$$\Rightarrow \quad F_x + F_z \cdot \frac{\partial z}{\partial x} = 0$$

Therefore $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$

& Similar Calculations if we differentiate w.r.t y.

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Example: If $x^3 + z^2 + y e^{xz} + z\cos y = 0$

Find $\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)}$.

$$\frac{\partial z}{\partial y} = -\frac{(e^{xy} - z\sin y)}{(2z + xy e^{xz} + \cos y)}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)} = \frac{-e^0 + 0}{0 + 0 + \cos 0} = -1$$

14.4

(43) If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$ and $w = z - x$, show that:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

14.4 (45) Laplace equations: show that if

$w = f(u,v)$ satisfies the Laplace equation:

$f_{uu} + f_{vv} = 0$ and if $u = \frac{(x^2 - y^2)}{2}$ & $v = xy$

then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$

Sol:

$$w_x = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial w}{\partial u}(x) + \frac{\partial w}{\partial v}(y)$$

$$w_{xx} = \frac{\partial w}{\partial u}(1) + x \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + y \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)$$

$$= \frac{\partial w}{\partial u} + x\left(\frac{\partial^2 w}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial u \partial v} \cdot \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial w}{\partial u} + x^2 \frac{\partial^2 w}{\partial u^2} + 2xy \frac{\partial^2 w}{\partial v \partial u} + y^2 \frac{\partial^2 w}{\partial v^2}.$$

Now:

$$w_y = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = -y \frac{\partial w}{\partial u} + x \frac{\partial w}{\partial v}$$

$$w_{yy} = -\frac{\partial w}{\partial u} + y^2 \frac{\partial^2 w}{\partial u^2} - 2xy \frac{\partial^2 w}{\partial v \partial u} + x^2 \frac{\partial^2 w}{\partial v^2}.$$

Thus $w_{xx} + w_{yy} = 0 = (x^2 + y^2)(w_{uu} + w_{vv})$.

Since $w_{uu} + w_{vv} = 0$

14.5 Directional Derivatives:

Recall: we find $f_x(x,y)$ & $f_y(x,y)$, where these derivatives represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively.

What if need to find the rate of change of f if we allow both x & y to change at the same time?

1) Note that one variable may be faster in changing than the other

2) one may be Increasing or decreasing & the other the opposite or the same.

 Therefore: We need to define what we called

Directional Derivatives:

14.5 Directional Derivatives and Gradient Vectors:

We know that if $f(x,y)$ is differentiable & $x = g(t)$

$y = h(t)$, then: [illegible]

Git

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \quad \cdots *$$

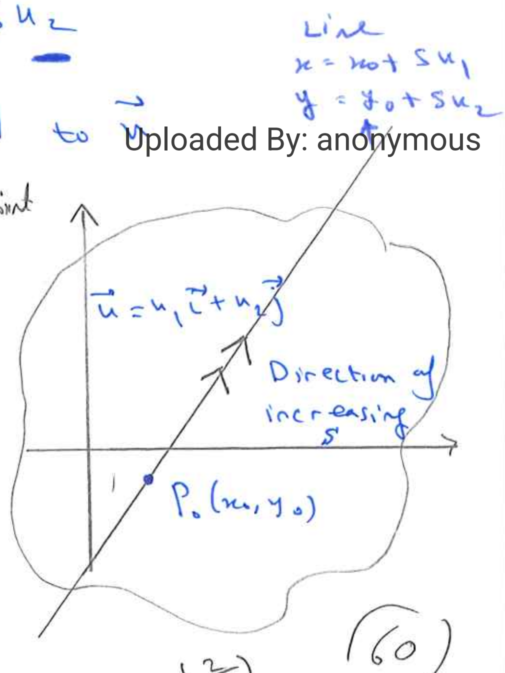
(No need)

At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation (*) gives the rate of change of f w.r.t increasing t and therefore depends on the direction of motion along the curve.

Suppose that $f(x,y)$ is defined throughout a region R in the xy-plane that $P_0(x_0, y_0) \in R$ and that $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ is a unit vector. Then:

$$x = x_0 + s u_1 \quad , \quad y = y_0 + s u_2$$

 parametrize the line through P_0 parallel to $\vec{u}$



If s measures arc length from P_0 (parameter, base point) in the direction of $\vec{u}$, we find the rate of change of f at P_0 in the direction of $\vec{u}$ by calculating

$$\left. \frac{df}{ds} \right|_{P_0}$$

Def: The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is:

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

- We call this Limit directional derivative and is denoted by: $(D_u f)_{P_0}$.

- $f_x(x_0, y_0)$ is the directional derivative of f at P_0 in the $\vec{i}$ direction. $f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$

- $f_y(x_0, y_0)$ is the directional derivative of f at P_0 in the $\vec{j}$ direction

Example: Find the derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of the unit vector

$$\vec{u} = \left(\frac{1}{\sqrt{2}}\right)\vec{i} + \left(\frac{1}{\sqrt{2}}\right)\vec{j}$$

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s\to 0} \frac{f\left(1+\frac{s}{\sqrt{2}}, 2+\frac{s}{\sqrt{2}}\right) - f(1,2)}{s}$$

The rate of change of f at P_0 in the direction of u is $\frac{5}{\sqrt{2}}$

$$= \lim_{s\to 0} \frac{\left(1+\frac{s}{\sqrt{2}}\right)^2 + \left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right) - (1+2)}{s}$$

$$= \lim_{s\to 0} \left(\frac{5}{\sqrt{2}} + s\right) = \boxed{\frac{5}{\sqrt{2}}}$$

Calculation and Gradients:

Consider the line $x = x_0 + s u_1$, $y = y_0 + s u_2$

through $P_0(x_0, y_0)$, parametrized with arc length parameter

Unit vector

s increasing in the direction of $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$

Using chain Rule we have:

$$\left(\frac{df}{ds}\right)_{u, P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0} \vec{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \vec{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1 \vec{i} + u_2 \vec{j}\right]}_{\text{Direction } \vec{u}}$$

Def: The gradient vector (gradient) of $f(x,y)$ at a point

$P_0(x_0, y_0)$ is the vector:

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

f_x at $P_0(x_0, y_0)$ & f_y at $P_0(x_0, y_0)$

Thm: The directional Derivative is a Dot Product:

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0, y_0)$

then
$$\left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

Example: Find the derivative of $f(x,y) = x e^{y} + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j}$.

$$\text{Direction of } \vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j} = \vec{u}$$

$$f_x(2,0) = \left(e^{y} - y \sin xy\right)\Big|_{(2,0)} = e^{0} - 0 = 1$$

$$f_y(2,0) = \left(x e^{y} - x \sin xy\right)\Big|_{(2,0)} = 2e^{0} - 2.0 = 2$$

$$\therefore \nabla f\Big|_{(2,0)} = \vec{i} + 2\vec{j}.$$

$$(D_u f)\Big|_{(2,0)} = \nabla f\Big|_{(2,0)} \cdot \vec{u}$$

$$= (\vec{i} + 2\vec{j}) \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)$$

$$= \frac{3}{5} - \frac{8}{5} = -\frac{3}{3} = -1$$

Note: $D_u f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta = |\nabla f| \cos\theta$

where θ is the angle between $\vec{u}$ & ∇f.

Properties of the Directional Derivative $D_u f$.

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\vec{u}$ is in the direction of ∇f.

& in this case: $D_u f = |\nabla f|$.

2. f decreases most rapidly in the direction of $-\nabla f$.

(i.e) $D_u f = |\nabla f| \cos\pi = -|\nabla f|$.

3. There will be NO change in f in ~~any~~ the direction of any vector $\vec{v}$ that is orthogonal to $\nabla f \neq 0$

(i.e) $D_u f = |\nabla f| . \cos\frac{\pi}{2} = 0$.

Example: Find the Directions in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

(a) Increases most rapidly at $(1,1)$.

(b) decreases most rapidly at $(1,1)$

(c) what are the directions of Zero change in f at $(1,1)$?

(a) The function increases most rapidly in the direction of ∇f at $(1,1)$.

$$\nabla f = x\vec{i} + y\vec{j} \Rightarrow \nabla f\Big|_{(1,1)} = \vec{i} + \vec{j}.$$

Its direction is

$$\vec{u} = \frac{\vec{i}+\vec{j}}{|\vec{i}+\vec{j}|} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1,1)$

$$\Rightarrow -\vec{u} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

(c) The directions of zero change at $(1,1)$ are the directions orthogonal to ∇f.

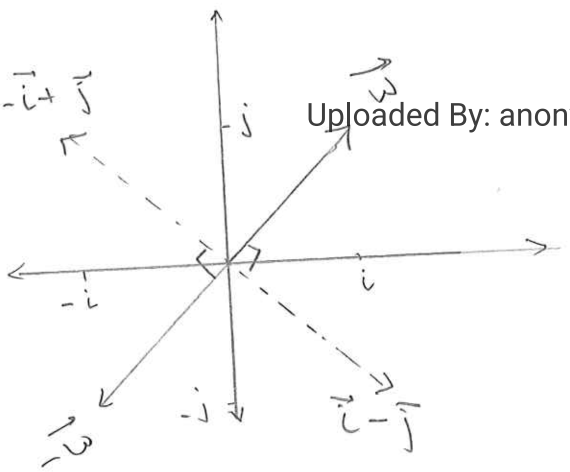
$$\vec{n} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} \quad \text{and} \quad -\vec{n} = \frac{-1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}.$$

Note: If $\vec{w} = \vec{i} + \vec{j}$

then $-\vec{w} = -\vec{i} - \vec{j}$

$\vec{n} = -\vec{i} + \vec{j}$

or $\vec{n} = \vec{i} - \vec{j}$



$\vec{w} \cdot \vec{n} = 0$

$1(n_1) + 1(n_2) = 0 \Rightarrow n_1 = 1$
$n_2 = -1$

Gradients and Tangents to level Curve:

Let $f(x,y)$ be differentiable and has a Constant value c (i.e) $f(x,y)=C$ along a smooth Curve $\vec{r}=g(t)\vec{i}+h(t)\vec{j}$.

then $$\frac{d}{dt} f(g(t),h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dh}{dt} = 0$$

$$\Longrightarrow \quad \Big(\underbrace{\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}}_{\nabla f}\Big)\cdot\Big(\underbrace{\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j}}_{\frac{d\vec{r}}{dt}}\Big) = 0$$

This says that

$\nabla f\big|_{(x_0,y_0)}$ is normal to the tangent vector $\frac{d\vec{r}}{dt}\big|_{(x_0,y_0)}$

which Implies that it's normal to the Curve. $\big|_{(x_0,y_0)}$

Question: How to find the equation of the tangent lines to level Curve?

We need $P_0(x_0,y_0)$

2. We need normal to the tangent, which is:

$$\vec{n} = \nabla f(x_0,y_0) = f_x(x_0,y_0)\vec{i} + f_y(x_0,y_0)\vec{j}$$

Equation of the tangent Line is:

$$f_x(x_0,y_0)\,x + f_y(x_0,y_0)\,y = f_x(x_0,y_0)\,x_0 + f_y(x_0,y_0)\,y_0$$

Equation of the Tangent Line:

$$y - y_0 = m(x - x_0) \quad , \quad m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$

$$\Rightarrow \quad y - y_0 = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} (x - x_0)$$

$$\Rightarrow \quad y - y_0 = \frac{-f_x(x_0,y_0)}{f_y(x_0,y_0)}(x - x_0)$$

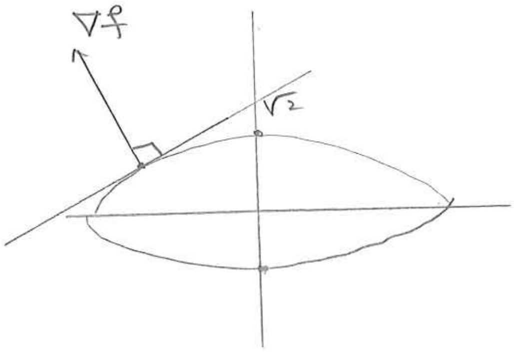
$$\Rightarrow \quad (y - y_0) f_y(x_0,y_0) = -f_x(x_0,y_0)(x - x_0)$$

$$\Rightarrow \quad f_x(x_0,y_0)\,x + f_y(x_0,y_0)\,y = f_x(x_0,y_0)\,x_0 + f_y(x_0,y_0)$$

Example: Find an equation for the tangent to the ellipse:

$$\frac{x^2}{4} + y^2 = 2 \quad \text{at } (-2,1)$$

$$\nabla f \Big|_{(-2,1)} = \frac{x}{2}\vec{i} + 2y\vec{j} \Big|_{(-2,1)} = -\vec{i} + 2\vec{j}$$



Tangent line:

$$(-1)(x) + (2)y = (-1)(-2) + 2(1)$$

$$-1(x+2) + 2(y-1) = 0$$

$$\therefore \quad x - 2y = -4$$

Algebra Rules for Gradients:

1. $\nabla(f+g) = \nabla f + \nabla g$.
2. $\nabla(f-g) = \nabla f - \nabla g$.
3. $\nabla(kf) = k\nabla f$.
4. $\nabla(fg) = f\nabla g + g\nabla f$

5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

Example: $f(x,y) = x+y$, $g(x,y) = xy$

$$\nabla(f+g) = \nabla(x+y+xy) = (1+y)\vec{i} + (1+x)\vec{j}$$

$$\nabla f + \nabla g = (\vec{i}+\vec{j}) + (y\vec{i} + x\vec{j}).$$

14.5 (29) Let $f(x,y) = x^2 - xy + y^2 - y$.

Find the directions $\vec{u}$ and the values of $D_u f(1,-1)$ for which:

a). $D_u f(1,-1)$ is Largest.

$$\nabla f \Big|_{(1,-1)} = (2x - y)\vec{i} + (2y - 1 - x)\vec{j} \Big|_{(1,-1)}$$

$$= 3\vec{i} + -4\vec{j}$$

$$\therefore \vec{u} = \frac{3i}{\sqrt{25}} - \frac{4}{\sqrt{25}}\vec{j} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$

b) $D_u f(1,-1)$ is smallest

$$-\frac{\nabla f \Big|_{(1,-1)}}{|\nabla f|} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

c) $D_u f(1,-1) = 0 \implies$ orthogonal

$$\vec{n} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$$

d) $D_u f(1,-1) = 4$

$$D_u f(1,-1) = \nabla f \Big|_{(1,-1)} \cdot \vec{u} = (3i - 4\vec{j}) \cdot (u_1\vec{i} + u_2\vec{j})$$

$$= 3u_1 - 4u_2 = 4$$

$\implies u_2 = \frac{3}{4}u_1 - 1$ & we know $u_1^2 + u_2^2 = 1$ (Unit)

we have $u_1 = 0$, or $u_1 = \frac{24}{25}$

If $u_1 = 0 \implies u_2 = -1$, if $u_1 = \frac{24}{25} \implies u_2 = -\frac{7}{25}$

14.5 35 The derivative of $f(x,y)$ at $P_0(1,2)$ in the direction of $\overset{\vec{v_1}}{\vec{i}+\vec{j}}$ is $2\sqrt{2}$ and in the direction of $\overset{\vec{v_2}}{-2j}$ is -3. What is the derivative of f in the direction of $\overset{\vec{v_3}}{-\vec{i}-2\vec{j}}$?

- $\nabla f = f_x(1,2)\vec{i} + f_y(1,2)\vec{j}$.

& $\vec{u_1} = \dfrac{\vec{i}+\vec{j}}{|\vec{i}+\vec{j}|} = \dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j}$.

$\therefore (D_{\vec{u_1}} f)\Big|_{(1,2)} = \nabla f\Big|_{(1,2)} \cdot \vec{u} = \dfrac{1}{\sqrt{2}} f_x(1,2) + \dfrac{1}{\sqrt{2}} f_y(1,2) = 2\sqrt{2}$

$\therefore \quad f_x(1,2) + f_y(1,2) = 4 \quad \dots (*)$

- $(D_{u_2}) f\Big|_{(1,2)} = f_x(1,2)(0) + f_y(1,2)(-2) = -2f_y(1,2) = -3$

$\Rightarrow \quad \boxed{f_y(1,2) = \frac{3}{2}}$ from $(*)$ & $\boxed{f_x(1,2) = \frac{5}{2}}$

Thus $\nabla f = \frac{3}{2}\vec{i} + \frac{5}{2}\vec{j}$.

Now $\vec{u_3} = \dfrac{-\vec{i}-2\vec{j}}{\sqrt{5}} = \dfrac{-1}{\sqrt{5}}\vec{i} - \dfrac{2}{\sqrt{5}}\vec{j}$

$\therefore (D_{u_3} f)\Big|_{(1,2)} = \nabla f \cdot u_3 = \frac{3}{2}\left(-\frac{1}{\sqrt{5}}\right) + \left(\frac{5}{2}\right)\left(\frac{-2}{\sqrt{5}}\right)$

$= \dfrac{-3}{2\sqrt{5}} - \dfrac{10}{2\sqrt{5}} = \boxed{\dfrac{-13}{2\sqrt{5}}}$

14.6 Tangent Planes and differentials:

Let $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ be a smooth Curve

on the level surface $f(x,y,z) = C$, then:

$f(g(t), h(t), k(t)) = C$ is differentiable.

$$\frac{d}{dt} f(g(t), h(t), k(t)) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)\cdot\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dk}{dt}\vec{k}\right) = 0$$

$$\Rightarrow \quad \nabla f \cdot \overset{\text{Tangent}}{\frac{d\vec{r}}{dt}} = 0$$

This Implies that ∇f is orthogonal to Curve's velocity vector.

Def: The tangent plane at $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = C$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$

Note: All the tangent lines are lie in the Tangent plane.

Tangent line: P_0 & normal to the tangent which is ∇f

Def: The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f\big|_{P_0}$.

Recall: • Tangent plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Level surface

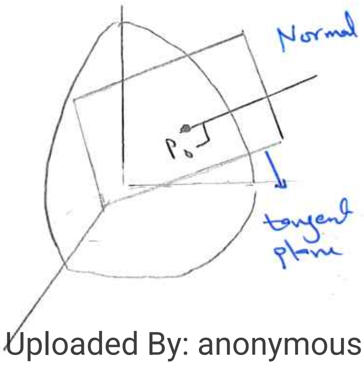
• Normal line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0) \,//\, \nabla f\big|_{P_0}$

$$x = x_0 + f_x(P_0)\,t \;,\quad y = y_0 + f_y(P_0)\,t \;,\; z = z_0 + f_z(P_0)\,t.$$

Example: Find the tangent plane and normal line of:

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0 \quad \text{at } P_0(1,2,4).$$

$$\nabla f\big|_{P_0} = (2x\vec{i} + 2y\vec{j} + \vec{k})\big|_{P_0} = 2\vec{i} + 4\vec{j} + \vec{k}$$



Tangent plane: $2(x-1) + 4(y-2) + (z-4) = 0$

$$\Leftrightarrow \; 2x + 4y + z = 14$$

Normal line : $x = 1 + 2t \quad , y = 2 + 4t \quad , z = 4 + t.$

$x_0 + f_x\big|_{(x_0,y_0,z_0)} t \qquad y = y_0 + f_y\, t \qquad z = z_0 + f_z\, t$

Plane Tangent to a surface $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$

Not $f(x,y,z) = c$

The plane tangent to the surface $z = f(x,y)$ at $P_0(x_0, y_0, z_0)$ $f(x_0,y_0)$

$z_0 = f(x_0,y_0)$

is
$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

$F(x,y,z) = f(x,y) - z = 0$ $\qquad f_z = -1$

Example: Find the plane tangent to the surface

$$z = x\cos y - y e^x \quad \text{at } (0,0,0).$$

$$f_x(0,0) = (\cos y - y e^x)\Big|_{(0,0)} = 1$$

$$f_y(0,0) = (-x\sin y - e^x)\Big|_{(0,0)} = -1$$

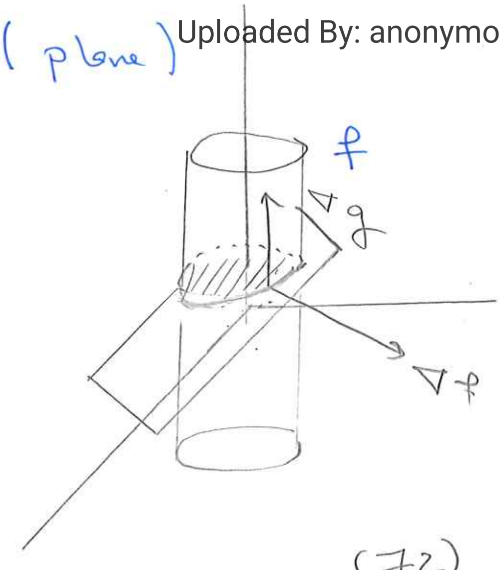
$\therefore$ Tangent plane: $1x - 1y - z = 0$.

Example: Find the parametric equation for the line tangent to the curve of intersection of the surfaces:

$f(x,y,z) = x^2 + y^2 - 2 = 0$ (cylinder)

$g(x,y,z) = x + z - 4 = 0$ (plane)

at $P_0(1,1,3)$



Solution The tangent line is orthogonal to both ∇f and ∇g.

Normal tang f ... normal tangent g

$$\nabla f \Big|_{(1,1,3)} = \left(2x\vec{i} + 2y\vec{j}\right)\Big|_{(1,1,3)} = 2\vec{i} + 2\vec{j}.$$

$$\nabla g \Big|_{(1,1,3)} = \left(\vec{i} + \vec{k}\right)\Big|_{(1,1,3)} = \vec{i} + \vec{k}.$$

Now: $\underbrace{\nabla f \times \nabla g}_{\perp \nabla f \,\&\, \nabla g}$ // to the Intersection at P_0.

$$\nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} - 2\vec{k}.$$

$\therefore$ The Tangent line is:

$\underbrace{x = 1 + 2t}_{x_0 + f_x(P_0)}$, $y = 1 - 2t$, $z = 3 - 2t$.

Estimating change in a specific Direction

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\vec{u}$ we use the formula:

$$df = \underbrace{\left(\nabla f\Big|_{P_0} \cdot \vec{u}\right)}_{\text{Directional derivative}} ds \longrightarrow \text{Distance Increment}$$

Example: Estimate how much the value of:

$$f(x,y,z) = y \sin x + 2yz.$$

will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,1,0)$ straight toward $P_1(2,2,-2)$.

First
$$\vec{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{2\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{4+1+4}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}.$$

$$\nabla f \Big|_{(0,1,0)} = y\cos x\,\vec{i} + (\sin x + 2z)\vec{j} + 2y\vec{k} \Big|_{(0,1,0)}$$

$$= \vec{i} + 2\vec{k}.$$

$$\Rightarrow \quad \nabla f\Big|_{P_0} \cdot \vec{u} = \left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}\right)\cdot\left(\vec{i} + 2\vec{k}\right)$$

$$= \left(\frac{2}{3} - \frac{4}{3}\right) = -\frac{2}{3}.$$

$$\therefore \quad df = \left(\nabla f\Big|_{P_0} \cdot \vec{u}\right)(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

Linearization:

Functions of two variables can be complicated, and we sometimes need to approximate them with simpler ones that give the accuracy required for specific applications without being so difficult to work with.

Recall If $f(x)$ is differentiable at $x = x_0$ then the linearization of f at x_0 is given by:

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

Def: The Linearization of $f(x,y)$ at (x_0, y_0) is:

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

$L(x,y)$ is called the standard Linear approximation of f at (x_0,y_0)

Note: The tangent plane to the surface $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$ is given by: No need

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0.$$

So we conclude that: $z = L(x,y)$ is a tangent plane to the surface $z = f(x,y)$ at (x_0, y_0). i.e. $(x_0, y_0, f(x_0,y_0))$

Example: Find the Linearization of:

$$f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at $(3,2)$.

$$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$$

$$= 8 + (2x-y)\Big|_{(3,2)}(x-3) + (-x+y)\Big|_{(3,2)}(y-2)$$

$$= 8 + 4(x-3) - 1(y-2) = 4x - y - 2$$

The error in the standard Linear approximation:

Let $|f_{xx}|, |f_{yy}|, |f_{xy}|$ have an Upper bound M.

Then $\underbrace{|E(x,y)|}_{= |f(x,y) - L(x,y)|} \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0| \right)^2$.

Example: Find an upper bound for the error in approximating $f(x,y)$ by $L(x,y)$ in the previous example over

$R: |x-3| \leq 0.1 \quad , \quad |y-2| \leq 0.1$.

$|f_{xx}| = |2| = 2$ & $|f_{xy}| = |-1| = 1$ & $|f_{yy}| = |1| = 1$

The largest is 2.

$$|E(x,y)| \leq \frac{1}{2}(2)\left(|x-3| + |y-2|\right)^2 = \left(|x-3| + (y-2)\right)^2$$

Now: Since $|x-3| \leq 0.1$ & $|y-2| \leq 0.1$ on R,

we have: $|E(x,y)| \leq (0.1 + 0.2)^2 = 0.04$

Upper bound as a %

As a Percentage of $f(3,2) = 8$, the error is No greater than:

$$\frac{0.04}{8} \times 100\% = 0.5\%$$

Differentials:

Estimating change in Direction

Recall: $\Delta f = f(x_0 + \Delta x) - f(x_0)$

The differential of f is $\quad df = f'(x_0)\,\Delta x$

Def: If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, then: the resulting change:

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

in the Linearization of f is called the total differential of f

when $dx = \Delta x = x_2 - x_1$ & $dy = \Delta y = y_2 - y_1$.

Example: A cylindrical Can is designed to have $r = 3$ cm with height 12 cm, but the radius and height are off by the amounts $dr = 0.08$ & $dh = -0.3$

Estimate the resulting absolute change in the Volume of the Can.

Sol: $P(3, 12)$, $V = r^2 \pi h$

$$dV = V_r(3,12)\,dr + V_h(3,12)\,dh$$

$$= 2r\pi h\Big|_{(3,12)} dr + r^2\pi\Big|_{(3,12)} dh$$

$$= 2(3)(\pi)(12)(0.08) + (3)^2\pi(-0.3) = 3.06\pi \cong 9.61 \text{ cm}^3$$

Example: A circular cylindrical tank with $h = 8$ m & $r = 2$ m. How sensitive are the tank's volumes to small variations in height and radius?

$$V = r^2 \pi h$$

$$dV = 2\pi r h \Big|_{(2,8)} dr + (r^2\pi)\Big|_{(2,8)} dh = (32\pi)dr + (4\pi)dh$$

- If one unit of r increased $\Rightarrow$ the Volume will Increases by (32π) units.
- If one unit of h increased $\Rightarrow$ the Volume will Increase by (4π) units

Hence the volume of the Con is 8 times more sensitive to the radius than to the height.

Example: Let $V = \pi r^2 h$. suppose that r is measured with error of no more than 2% and h is measured with error of no more than 0.5%. Estimate the resulting percentage error in Calculating V.

Given $\left|\frac{dr}{r}\right| \le 2\%$, $\left|\frac{dh}{h}\right| \le 0.5\%$

find $\frac{dv}{v}$.

$$dv = V_r\, dr + V_h\, dh$$

$$\frac{dv}{V} = \frac{2r\pi h\, dr + \pi r^2\, dh}{V}$$

$$\frac{dv}{V} = \frac{2\pi r h\, dr + \pi r^2\, dh}{\pi r^2 h}$$

$$\left|\frac{dv}{V}\right| = \left|\frac{2\,dr}{r} + \frac{dh}{h}\right| \le 2\left|\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$

$$\left|\frac{dv}{V}\right| \le 2\,(2\%) + 0.5\%$$

$$\left|\frac{dv}{V}\right| \le 4.5\%$$

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14.6 (14) parametric equations for the line tangent to the curve of intersection of:

$xyz = 1$ & $x^2 + 2y^2 + 3z^2 = 6$ at $(1,1,1)$

$$\nabla f = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}\,\Big|_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$$

$$\nabla g = 2x\,\vec{i} + 4y\,\vec{j} + 6z\,\vec{k}\,\Big|_{(1,1,1)} = 2\vec{i} + 4\vec{j} + 6\vec{k}.$$

$$v = \nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\vec{i} - 4\vec{j} + 2\vec{k}.$$

$\therefore$ Tangent line : $x = 1 + 2t$, $y = 1 - 4t$, $z = 1 + 2t$.

---

(19) by How much will $f(x,y,z) = \ln\sqrt{x^2 + y^2 + z^2}$ change if the Point $P(x,y,z)$ moves from $P_0(3,4,12)$ a distance of $ds = 0.1$ unit in the direction of $3\vec{i} + 6\vec{j} - 2\vec{k}$?

$$\nabla f\,\Big|_{(3,4,12)} = \frac{3}{169}\vec{i} + \frac{4}{169}\vec{j} + \frac{12}{169}\vec{k}$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{3}\vec{k}.$$

$$df = (\nabla f \cdot \vec{u})\,ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$$

14.6 (38) Find Linearization and upper bound for the error for

$$f(x,y) = \ln x + \ln y \quad \text{at } P_0(1,1)$$

$R: \ |x-1| \le 0.2 \ , \ |y-1| \le 0.2$

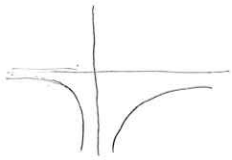
$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= 0 + 1(x-1) + 1(y-1)$$

$$= x + y - 2$$

$$|E(x,y)| \le \frac{1}{2} M \left(|x-1| + |y-1| \right)^2$$

$M = \max \{ |f_{xx}|, |f_{yy}|, |f_{xy}| \}$.



$f_{xx} = \dfrac{-1}{x^2}$ on $|x-1| \le 0.2$ (Increasing)

$\Longrightarrow \quad -0.2 \le x-1 \le 0.2 \Rightarrow 0.8 \le x \le 1.2$

$\therefore |f_{xx}(0.8)| \cong 1.5625$ & $|f_{xx}(1.2)| = 0.69$

$= \frac{1}{(0.8)^2}$

$\therefore |f_{xx}| \le 1.6$

$f_{yy} = -\dfrac{1}{y^2} \Longrightarrow \max |f_{yy}| = 1.6$

& $f_{xy} = 0$

therefore $M = 1.6$

$$\therefore |E(x,y)| \le \frac{1}{2}(1.6)\left(0.2 + 0.2\right)^2 \cong 0.128$$

(48) Find the Linearization at $P_0(0, 0, \frac{\pi}{4})$

for $f(x, y, z) = \sqrt{2}\cos x \sin(y+z)$. Then

find the upper bound of the error.

$|x| \le 0.01$, $|y| \le 0.01$, $|z - \frac{\pi}{4}| \le 0.01$.

$$L(x,y,z) = f(0,0,\tfrac{\pi}{4}) + f_x(0,0,\tfrac{\pi}{4})\,x + f_y(0,0,\tfrac{\pi}{4})\,y + f_z(0,0,\tfrac{\pi}{4})(z - \tfrac{\pi}{4})$$

$$= 1 + 0(x-0) + 1(y-0) + 1(z - \tfrac{\pi}{4})$$

$$= y + z - \tfrac{\pi}{4} + 1$$

Now $f_{xx} = -\sqrt{2}\cos x \sin(y+z)$

$f_{yy} = -\sqrt{2}\cos x \sin(y+z)$

$f_{zz} = -\sqrt{2}\cos x \sin(y+z)$

$f_{xy} = -\sqrt{2}\sin x \cos(y+z)$

$f_{xz} = -\sqrt{2}\sin x \cos(y+z)$

$f_{yz} = -\sqrt{2}\cos x \sin(y+z)$

Then $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|, |f_{yz}| \le \sqrt{2}$

Hence $|E(x,y,z)| \le \frac{1}{2}(\sqrt{2})(0.01 + 0.01 + 0.01)^2$

$$= 0.000636.$$

$(\frac{2}{2})$ (81)

14.7 Extreme Values and Saddle Points:

Def: Let $f(x,y)$ be defined on a Region R containing the point (a,b), then:

قمة الشكل في surface $z=f(x,y)$

1. $f(a,b)$ is a local maximum value of f if
   $f(a,b) \geq f(x,y)$, $\forall (x,y) \in D$ in an open disk centered at (a,b).

قاع الشكل الواد

2. $f(a,b)$ is a local minimum value of f if
   $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

Note: at the local maxima & local minima, the tangent planes, when they exist, are horizontal

Def: An Interior point of the domain of a function $f(x,y)$ where $f_x(a,b) = f_y(a,b) = 0$ or where one or both of $f_x(a,b)$ and $f_y(a,b)$ DNE is called a critical point.

If $\nexists$ (a,b) is max. then $f_x(a,b) = f_y(a,b) = 0$

المماس غير موجود

Def: A differential function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) 1 there are domain points (x,y) where $f(x,y) > f(a,b)$ 2 and domain points (x,y) where $f(x,y) < f(a,b)$.

The Corresponding point $(a,b,f(a,b))$ on the Surface $z = f(x,y)$ is called a saddle point of the Surface

saddle point.

Thm: First derivative Test for local extreme Values:

If $f(x,y)$ has a local maximum or minimum value at an Interior point (a,b) of its domain, and if the first derivatives exist, then $f_x(a,b) = 0$ & $f_y(a,b) = 0$

Relative Extreme

Thm: Second Derivative Test for local Extreme Values:

Suppose that $f(x,y)$ and its first and second partial derivatives are Continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$

critical points

then:

1) f has a local maximum at (a,b) if

$f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

2) f has a local minimum at (a,b) if

$f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

3). f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

4). The test is inconclusive at (a,b) if

$f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b)

Note: The expression $f_{xx} f_{yy} - f_{xy}^2$ is called the discriminant or Hessian of f.

we can write it as: $f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

Example: Find local maximum and local minimum & saddle point for:

$\boxed{1}$ $f(x,y) = x^2 - 4xy + y^2 + 6y + 2$.

$f_x = 2x - 4y = 0 \Longrightarrow x = 2y$

$f_y = -4x + 2y + 6 = 0 \Longrightarrow y = 2x - 3$

then $(x,y) = (2,1)$.

$f_{xx} = 2$, $f_{xx}(2,1) = 2 > 0$

$f_{yy} = 2$, $f_{yy}(2,1) = 2$

$f_{xy} = -4$, $f_{xy}(2,1) = -4$

$\therefore \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(2,1)} = (2)(2) - (-4)^2 = 4 - 16 = -12 < 0$

then f has a saddle point at $(2,1)$

$\boxed{2}$ $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

$f_x = 3x^2 + 6x = 0 \Longrightarrow 3x(x+2) = 0$

then $x = 0$ & $x = -2$.

$f_y = 3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0$

then $y = 0$ & $y = 2$.

Then critical points: $(0,0)$, $(0,2)$, $(-2,0)$, $(-2,2)$.

Case (1): $(0,0)$

$f_{xx} = 6x+6 \Rightarrow f_{xx}(0,0) = 6 > 0$

$f_{yy} = 6y-6 \Rightarrow f_{yy}(0,0) = -6$

$f_{xy} = 0$, then

$f_{xx} f_{yy} - f_{xy}^2 = (6)(-6) - 0 = -36 < 0$

Hence f has a saddle point at $(0,0)$

Case (2): $(0,2)$

$f_{xx}(0,2) = 6 > 0$

$f_{yy}(0,2) = 6 > 0$

$f_{xy}(0,2) = 0$

$\left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(0,2)} = (6)(6) - 0 = 36 > 0$

Hence f has a local minimum at $(0,2)$ which

is $f(0,2) = \boxed{-12}$

Case 3: $(-2,0)$

$f_{xx}(-2,0) = -6 < 0$, $f_{yy}(-2,0) = -6$, $f_{xy} = 0$

$$\left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(-2,0)} = (-6)(-6) - 0 = 36 > 0$$

then f has a local maximum at $(-2,0)$ which is

$$f(-2,0) = \boxed{-4}$$

Case 4: $(-2,2)$

$f_{xx}(-2,2) = -6 < 0$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$

$$\left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(-2,2)} = (-6)(6) - 0 = -36 < 0$$

Hence f has a saddle point at $(-2,2)$.

Absolute Maxima and Minima on closed Bounded Regions:

To find the absolute extreme of a continuous function

$f(x,y)$ on a closed and bounded region R

we have three steps:

1) Find all interior points of R where f has a local maxima and minima, then evaluate f at these points. (These are critical points).

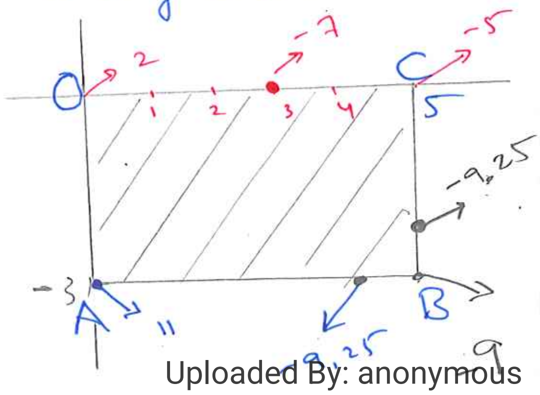
2) Find the Boundary points of R where f has local maxima and minima, then evaluate f at these points.

3.) Find Absolute max & Absolute min of (1) & (2).

Example: Find the absolute maxima and minima of

$$f(x,y) = x^2 + xy + y^2 - 6x + 2 \quad \text{on}$$

the rectangle : $0 \le x \le 5$ & $-3 \le y \le 0$.



Boundary points :

$\boxed{1}$ $OC \Rightarrow y = 0$

$f(x,0) = x^2 - 6x + 2$

$f_x = 2x - 6 = 0 \Rightarrow x = 3$

So we have critical point $(3,0)$

$f(3,0) = 9 - 18 + 2 = -7$

$f(0,0) = 2$

$f(5,0) = -3$

$\boxed{2}$ CB $\Rightarrow$ $x = 5$

$f(5, y) = 25 + 5y + y^2 - 30 + 2 = y^2 + 5y - 3$

$f_y = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$

$\Rightarrow (5, -\frac{5}{2})$

$f(5, -\frac{5}{2}) = -\frac{37}{4}$

$f(5, -3) = -9$

$\boxed{3}$ AB $\Rightarrow$ $y = -3$

$f(x, -3) = x^2 - 3x + 9 - 6x + 2 = x^2 - 9x + 11$

$f_x = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$

$\Rightarrow (\frac{9}{2}, -3)$

$f(\frac{9}{2}, -3) = -9.25$

$f(0, -3) = 0 - 0 + 11 = 11$

$\boxed{4}$ OA $\Rightarrow$ $x = 0$

$f(0, y) = y^2 + 2$

$f_y = 2y = 0 \Rightarrow y = 0$

$\Rightarrow (0, 0)$

5 Interior points :

$f_x = 2x + y - 6 = 0 \Rightarrow 2x + y = 6$
$f_y = x + 2y = 0 \Rightarrow 2x + 4y = 0$ } solve

so we have the point $(4,-2)$

$f(4,-2) = 16 - 8 + 4 - 24 + 2 = -10$.

Hence $f(x,y)$ has Absolute maxima at $(0,-3)$

which is $f(0,-3) = 11$

& $f(x,y)$ has Absolute minima at $(4,-2)$ which is -10

Example 53: Find three numbers whose ~~three~~ sum is 9 and whose sum of squares is a minimum.

Sol: $x + y + z = 9 \Rightarrow z = 9 - x - y$.

Let $s(x,y,z) = x^2 + y^2 + z^2 = x^2 + y^2 + (9-x-y)^2$.

$S_x = 2x + 2(9-x-y)(-1) = 0$
$S_y = 2y + 2(9-x-y)(-1) = 0$ } solve these two equations

$\Rightarrow x = 3,\ y = 3 \ \& \ z = 3$.

$S_{xx}(3,3,3) = 4 > 0$, $S_{yy}(3,3,3) = 4$

& $S_{xy}(3,3,3) = 2 \Rightarrow S_{xx}S_{yy} - S_{xy}^2 = 12 > 0$

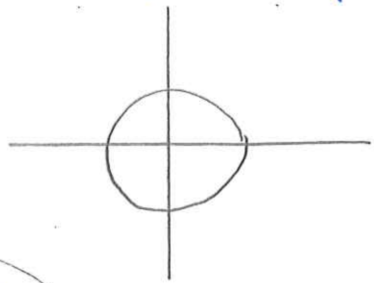
Hence S has a local minimum at $(3,3,3)$.

14.7

(41) A flat circular plate has the shape of the region $x^2+y^2 \leq 1$. The plate, including the boundary where $x^2+y^2=1$ is heated so that the temprature at (x,y) is

$$T(x,y) = x^2 + 2y^2 - x.$$

Find the temprature at the hottest and Coldest points on plate.



Interior:

$T_x \Big|_{(x,y)} = 2x - 1 = 0$, $T_y = 4y = 0$

$\therefore$ $x = \frac{1}{2}$, $y = 0$ with $T(\frac{1}{2},0) = -\frac{1}{4}$

Boundary: $x^2+y^2=1 \Rightarrow y^2 = 1 - x^2$, substitue in T

Then $T(x) = -x^2 - x + 2$ on $-1 \leq x \leq 1$

$T'(x) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$ & $y = \pm \frac{\sqrt{3}}{2}$

$T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{9}{4}$, $T(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{9}{4}$

$T(-1, 0) = 2$, $T(1,0) = 0$

The hottest is $\frac{9}{4}$, The clodest is $-\frac{1}{4}$

(45) show that $(0,0)$ is a critical point of $f(x,y) = x^2 + kxy + y^2$ no matter what value the Constant k has (Hint, $k=0$ & $k \neq 0$)

- If $k=0$, then $f(x,y) = x^2 + y^2$

$f_x = 2x = 0$ & $f_y = 2y = 0 \Longrightarrow (0,0)$ is the only critical point

- If $k \neq 0$, then:

$f_x = 2x + ky = 0 \Longrightarrow \boxed{y = -\frac{2}{k} x}$ (*)

$f_y = kx + 2y = 0 \Longrightarrow kx + 2\left(\frac{-2}{k}x\right) = 0$

$\therefore \quad kx - \frac{4x}{k} = 0 \Longrightarrow \left(k - \frac{4}{k}\right) x = 0$

$\Longrightarrow \quad x = 0$ or $k = \pm 2$

k=±2 from (*)

$\Longrightarrow \quad y = \left(\frac{-2}{k}\right)(0) = 0$, or $y = \pm x \Longrightarrow (0,0)$ is critical point

(56) Find the minimum distance from $z = \sqrt{x^2+y^2}$ to $(-6,4,0)$.

(x,y,z) to $(-6,4,1)$

$d = \sqrt{(x+6)^2 + (y-4)^2 + (z-0)^2}$

$D(x,y,z) = d^2 = (x+6)^2 + (y-4)^2 + z^2.$

$= (x+6)^2 + (y-4)^2 + \left(\sqrt{x^2+y^2}\right)^2$

$D = 2x^2 + 2y^2 + 12x - 8y + 52$

$D_x = 4x + 12 = 0$, $D_y = 4y - 8 = 0 \Longrightarrow$ Critical $(-3,2)$.

$z = \sqrt{13}$.

$d(-3,2,\sqrt{13}) = \sqrt{26}$ is the minimum distance

$D_{xx}\Big|_{(-3,2)} = 4$, $D_{yy}\Big|_{(-3,2)} = 4$, $D_{xy}\Big|_{(-3,2)} = 0$

$\Longrightarrow D_{xx} D_{yy} = 16 > 0$ & $D_{xx} > 0 \Longrightarrow$ Local minimum $\left(\frac{3}{2}\right)(91)$

14.8 Lagrange Multipliers:

The method of Lagrange Multipliers: (One Constraint)

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z)=0$.

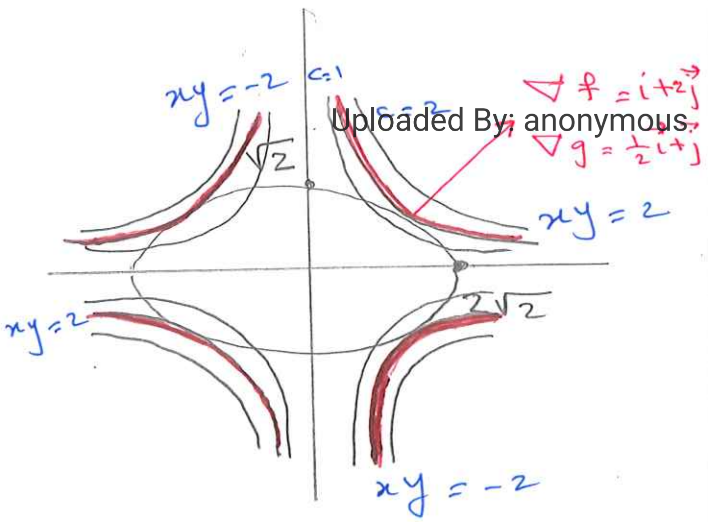
To find the local maximum and minimum values of f subject to the Constraint $g(x,y,z)=0$, we find the points (x,y,z) and lagrange multiplier λ such that simultaneously satisfy:

$$\nabla f = \lambda \nabla g \quad \& \quad g(x,y,z)=0$$

Example: Find the greatest and smallest values that the function $f(x,y)=xy$ takes on the ellipse:

$$\frac{x^2}{8}+\frac{y^2}{2}=1$$

Level Curves for $f(x,y)$.



$xy = C \Rightarrow y = \frac{c}{x}$

$c=1 \Rightarrow y = \frac{1}{x}$

$c=2 \Rightarrow y = \frac{2}{x}$

$c=-2 \Rightarrow y = \frac{-2}{x}$

- $f(x,y) = xy \Rightarrow \nabla f = f_x \vec{i} + f_y \vec{j}$

$$= y\vec{i} + x\vec{j}.$$

- $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 \Longrightarrow \nabla g = \frac{x}{4}\vec{i} + y\vec{j}.$

$$\nabla f = \lambda \nabla g \quad \& \quad g(x,y) = 0$$

$$\Rightarrow \quad y\vec{i} + x\vec{j} = \lambda\left(\frac{x}{4}\vec{i} + y\vec{j}\right) \quad \& \quad \frac{x^2}{8} + \frac{y^2}{2} = 1$$

$$\Rightarrow \begin{cases} y = \dfrac{\lambda x}{4} & \ldots (1) \\ x = \lambda y & \ldots (2) \\ \dfrac{x^2}{8} + \dfrac{y^2}{2} = 1 & \ldots (3) \end{cases}$$

substitute (2) in (1), $y = \frac{\lambda}{4}(\lambda y) \Leftrightarrow 4y = \lambda^2 y.$

$\Longrightarrow \quad y(4 - \lambda^2) = 0$, so we have two cases:

$\boxed{1}$ If $y = 0 \Longrightarrow x = y = 0$, but $(0,0) \notin$ ellipse $\Rightarrow y \neq 0$

$\boxed{2}$ if $y \neq 0 \Longrightarrow \boxed{\lambda = \pm 2}$, substitue in (2):

$x = \pm 2y \Rightarrow x^2 = 4y^2$, substitue in (3):

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \frac{y^2}{2} + \frac{y^2}{2} = 1 \Rightarrow y^2 = 1$$

$$\therefore \boxed{y = \pm 1} \quad \& \quad \boxed{x = \pm 2}$$

Therefore $f(x,y) = xy$ takes on its extreme values on the ellipse at $(\pm 2, 1), (\pm 2, -1) \Leftrightarrow \begin{cases} xy = 2. & \text{Abs Max} \\ xy = -2. & \text{Abs Min} \end{cases}$

The Geometry of the Solution:

The Level Curves of $f(x,y) = xy$ are the hyperbolas

$xy = C$.

. The further the hyperbolas Lie from the origin, the Larger the absolute value of f.

We want the extreme of $f(x,y)$, given that the point (x,y) also Lies on the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$.

Note: At these points any vector normal to $xy = C$ is normal also to the ellipse. $(\nabla f = \lambda \nabla g)$

$(2,1)$
Ex. $\nabla f = \vec{i} + 2\vec{j}$, $\nabla g = \frac{1}{2}\vec{i} + \vec{j}$ & $\nabla f = 2\nabla g$.

Thm: Orthogonal Gradient Theorem:

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth Curve:

$$C : \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + \ell(t)\vec{k}$$

If P_0 is a point on C where f has a local maximum or minimum, then: $\nabla f \perp C$ at P_0

Moreover, $\nabla f \cdot \vec{v} = 0$, where $\vec{v} = \frac{d\vec{r}}{dt}$

Proof: The Values of f on C are:

$$f(x,y,z) = f\big(g(t), h(t), l(t)\big)$$

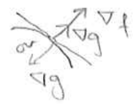
$$\frac{df}{dt} = f_x \frac{dg}{dt} + f_y \frac{dh}{dt} + f_z \frac{dl}{dt}$$

$$= \left(f_x \vec{i} + f_y \vec{j} + f_z \vec{k}\right) \cdot \left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dl}{dt}\vec{k}\right)$$

$$= \nabla f \cdot \vec{V}.$$

If f has a local max or local min at P_0, then

$$\frac{df}{dt}(P_0) = 0 \Longrightarrow \nabla f \cdot \vec{V} = 0 \Longrightarrow \nabla f \perp \vec{V} \quad . \frac{dr}{dt}$$



Example: Find the maximum and minimum Values of $f(x,y) = 3x + 4y$ on the Circle $x^2 + y^2 = 1$

- $f(x,y) = 3x + 4y$, $\nabla f = 3\vec{i} + 4\vec{j}$
- $g(x,y) = x^2 + y^2 - 1$, $\nabla g = 2x\vec{i} + 2y\vec{j}$.

$$\nabla f = \lambda \nabla g \quad \& \quad g(x,y) = 0$$

$$3\vec{i} + 4\vec{j} = \lambda\left(2x\vec{i} + 2y\vec{j}\right) \quad \& \quad x^2 + y^2 - 1 = 0$$

$$\begin{cases} 3 = 2\lambda x & \ldots (1) \\ 4 = 2\lambda y & \ldots (2) \\ x^2 + y^2 = 1 & \ldots (3) \end{cases}$$

from (1), $\lambda = \frac{3}{2x}$.

substitute in (2) $\Longrightarrow$ $4 = 2y\left(\frac{3}{2x}\right)$, which Implies:

$4x = 3y \iff x = \frac{3}{4}y$, substitute in (3)

$\frac{9}{16}y^2 + y^2 = 1 \iff \frac{25}{16}y^2 = 1$,

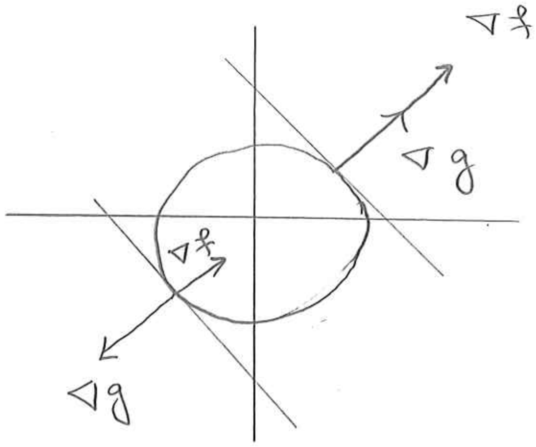
Therefore $y = \pm\frac{4}{5}$ & $x = \pm\frac{3}{5}$

$\left(\frac{3}{5}, \frac{4}{5}\right)$: $f\left(\frac{3}{5}, \frac{4}{5}\right) = 5$ $\Longrightarrow$ Abs. Max.

$\left(-\frac{3}{5}, -\frac{4}{5}\right)$: $f\left(-\frac{3}{5}, -\frac{4}{5}\right) = -5$ $\Longrightarrow$ Abs. Min.

$\left(\frac{3}{5}, -\frac{4}{5}\right)$: $f\left(\frac{3}{5}, -\frac{4}{5}\right) = -\frac{7}{5}$

$\left(-\frac{3}{5}, \frac{4}{5}\right)$: $f\left(-\frac{3}{5}, \frac{4}{5}\right) = \frac{7}{5}$



Level Curve $f(x,y) = c$

$3x + 4y = c$

$3x + 4y = 5$

$3x + 4y = -5$

$\nabla f\left(\frac{3}{5}, \frac{4}{5}\right) = 3\vec{i} + 4\vec{j}$

$\nabla g\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{6}{5}\vec{i} + \frac{8}{5}\vec{j}$

Exemple: Use the method of Lagrange multipliers to find the extreme values of the function:

$f(x,y,z) = xy + z^2$ on the sphere $x^2+y^2+z^2=1$

sol: $\vec{\nabla} f = \lambda \nabla \vec{g}$

$\Rightarrow y\vec{i} + x\vec{j} + 2z\vec{k} = \lambda(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$

$\Rightarrow y = 2\lambda x$ --- (1)

$x = 2\lambda y$ --- (2)

$2z = 2\lambda z$ --- (3)

$x^2+y^2+z^2 = 1$ ---(4)

substitute (1) in (2) $\Rightarrow x = 2\lambda(2\lambda)x = 4\lambda^2 x$

$\Rightarrow x(1-4\lambda^2) = 0$, so we have: either/or

1) $x = 0 \Rightarrow y = 0 \Rightarrow z = \pm 1$ & from (3) $\Rightarrow \lambda = 1$

$\Rightarrow (x,y,z,\lambda) = (0,0,\pm 1, 1)$

or 2) $x \neq 0 \Rightarrow \lambda = \frac{\pm 1}{2}$

(i) If $\lambda = \frac{1}{2}$, then $\boxed{y = x}$ & from (3) $\boxed{z = 0}$

substitute in (4), we have

$x = y = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow (x,y,z,\lambda) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, \frac{1}{2}\right)$

$\left(\frac{2}{3}\right)$ (96)

(ii) If $\lambda = -\frac{1}{2}$

$\Rightarrow$ $x = -y$ & from (3) $\Rightarrow$ $\boxed{z=0}$

Then substitute in (4), we have

$$x = \pm\frac{1}{\sqrt{2}} \Rightarrow y = \mp\frac{1}{\sqrt{2}}$$

$$\Rightarrow (x, y, z, \lambda) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, -\frac{1}{2}\right).$$

Now :

$$f(0, 0, \pm 1) = 1 \Rightarrow \underline{(\text{max})}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{2} \quad \underline{(\text{min})}$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) = \frac{1}{2}$$

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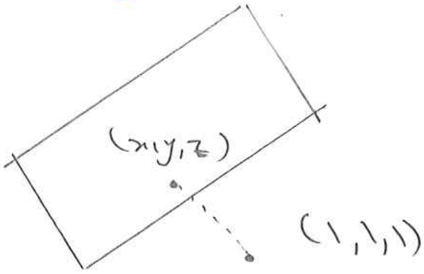
Example: Find the point on the plane $\underbrace{x+2y+3z=13}_{}$ closest to the point $(1,1,1)$.

$g(x,y,z)=$

We need to find the minimum value of d:

$$d = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$$



$$f(x,y,z) = d^2 = (x-1)^2+(y-1)^2+(z-1)^2$$

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$$

$$= 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k}.$$

$$\nabla g = \vec{i} + 2\vec{j} + 3\vec{k}.$$

$$\nabla f = \lambda \nabla g \qquad \& \qquad g(x,y,z) = 0.$$

$$\begin{cases} 2(x-1)\vec{i}+2(y-1)\vec{j}+2(z-1)\vec{k} = \lambda(\vec{i}+2\vec{j}+3\vec{k}) \\ x+2y+3z = 13. \end{cases}$$

$$\Rightarrow \begin{cases} 2(x-1) = \lambda & \dots(1) \\ 2(y-1) = 2\lambda & \dots(2) \\ 2(z-1) = 3\lambda & \dots(3) \\ x+2y+3z = 13 & \dots(4) \end{cases}$$

• substitute (1) in (2):

$$2(y-1) = 2(2(x-1)) \Longrightarrow y-1 = 2(x-1).$$

$$\therefore \quad y+1 = 2x \Longrightarrow x = \frac{y+1}{2}$$

• Substitute (2) in (3) $\quad y-1=\lambda$

$$2(z-1) = 3(y-1) \Longrightarrow 2z = 3y-1 \Longrightarrow z = \frac{3y-1}{2}$$

but (4) : $x+2y+3z = 13$

$$\Longrightarrow \quad \frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) = 12$$

$$\therefore \quad 7y = 14$$

$$y = 2 \quad , \quad x = \frac{3}{2} \quad , \quad z = \frac{5}{2}$$

So the point is $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$..

14.8 (17) Find the point on $x+2y+3z=13$ closest to $(1,1,1)$.

$$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$$

$$g(x,y,z) = x+2y+3z$$

$$\nabla f = \lambda \nabla g \quad \& \quad x+2y+3z-13=0$$

$$\nabla f = 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k}.$$

$$\nabla g = 1\vec{i} + 2\vec{j} + 3\vec{k}.$$

$\therefore \quad 2(x-1) = \lambda \quad , \quad 2(y-1) = 2\lambda \quad , \quad 2(z-1) = 3\lambda$

$\therefore \quad 2(x-1) = (y-1) \quad \& \quad 3(x-1) = (z-1)$

$\therefore \quad y = 2x-1 \quad \& \quad z = 3x-2$

Using: $g(x,y,z) = 13 \Rightarrow$

$$13 = x + 2(2x-1) + 3(3x-2) \Longrightarrow x = \frac{3}{2}, y = 2, z = \frac{5}{2}$$

(23) Find the maximum and minimum values of

$f(x,y,z) = x-2y+5z$ on sphere $x^2+y^2+z^2=30$.

$$\nabla f = \vec{i} - 2\vec{j} + 5\vec{k} \quad , \quad \nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

 x & $-2 = 2\lambda y$ & $5 = 2\lambda z$

& $x^2+y^2+z^2 = 30$ ---(*)

$$\Rightarrow \nabla f = \lambda \nabla g \Rightarrow i - 2j + 5k = \lambda(2x)i + \lambda(2y)j + \lambda(2z)\vec{k}$$

$$\Rightarrow 1 = 2x\lambda, \quad -2 = 2y\lambda \quad \& \quad 5 = 2z\lambda$$

$$\Rightarrow x = \frac{1}{2\lambda}, \quad y = \frac{-1}{\lambda} = -2x, \quad z = \frac{5}{2\lambda} = 5x \quad \text{-- subst. in } (*)$$

$$\Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow \boxed{x = \pm 1}$$

$\Rightarrow f(1,-2,5) = 30$ is maximum & $f(-1,2,-5) = -30$ is minimum