EE 261 Helpful Formulas

Fourier series. If $f(t)$ is periodic with period T then its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

where

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt$$

Rayleigh (Parseval). If $f(t)$ is periodic of period T then

$$\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

The Fourier transform.

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i s x}\,dx$$

The inverse Fourier transform.

$$\mathcal{F}^{-1}f(x) = \int_{-\infty}^{\infty} f(s)e^{2\pi i s x}\,ds$$

Fourier transform theorems.

Linearity.

$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(s) + \beta G(s)$$

Stretch.

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F(\frac{s}{a})$$

Shift.

$$\mathcal{F}\{f(x-a)\} = e^{-i2\pi a s}F(s)$$

Shift and stretch.

$$\mathcal{F}\{f(ax-b)\} = \frac{1}{|a|}e^{-i2\pi s b/a}F\left(\frac{s}{a}\right)$$

Rayleigh (Parseval).

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} F(s)\overline{G(s)}\,ds$$

Modulation.

$$\mathcal{F}\{f(x)\cos(2\pi s_0 x)\} = \tfrac{1}{2}\big(F(s-s_0) + F(s+s_0)\big)$$

Derivative.

$$\mathcal{F}\{f'(x)\} = 2\pi i s F(s)$$

$$\mathcal{F}\{f^{(n)}(x)\} = (2\pi i s)^n F(s)$$

$$\mathcal{F}\{x^n f(x)\} = (\tfrac{i}{2\pi})^n F^{(n)}(s)$$

Moments.

$$\int_{-\infty}^{\infty} f(x)dx = F(0)$$

$$\int_{-\infty}^{\infty} x f(x)dx = \frac{i}{2\pi}F'(0)$$

$$\int_{-\infty}^{\infty} x^n f(x)dx = \left(\frac{i}{2\pi}\right)^n F^{(n)}(0)$$

Autocorrelation.

$$\mathcal{F}\{\overline{g}\star g\} = |G(s)|^2$$

Crosscorrelation.

$$\mathcal{F}\{\overline{g}\star f\} = \overline{G(s)}F(s)$$

Miscellaneous:

$$\mathcal{F}\left(\int_{-\infty}^{x} g(\xi)d\xi\right) = \tfrac{1}{2}G(0)\delta(s) + \frac{G(s)}{i2\pi s}$$

Symmetry and duality properties.

The *reversal* of $f(x)$ is $f^-(x) = f(-x)$.

$\mathcal{F}^{-1}f = \mathcal{F}f^-$

$\mathcal{F}f^- = (\mathcal{F}f)^-$

If f is even (odd) then $\mathcal{F}f$ is even (odd)

If f is real valued, then $\overline{\mathcal{F}f} = (\mathcal{F}f)^-$

Convolution.

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

$f*g = g*f$

$(f*g)*h = (f*g)*h$

$f*(g+h) = f*g + f*h$

Smoothing: If f is k-times continuously differentiable, then so is $f*g$ and

$$\frac{d^k}{dx^k}(f*g) = \left(\frac{d^k}{dx^k}f\right)*g$$

Convolution theorem.

$$\mathcal{F}(f*g)=(\mathcal{F}f)(\mathcal{F}g)$$
$$\mathcal{F}(fg)=\mathcal{F}f*\mathcal{F}g$$

Autocorrelation. If $f(x)$ has finite energy, i.e.,

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx<\infty\,,$$

then

$$(\overline{f}\star f)(x)=\int_{-\infty}^{\infty}f(y)\overline{f(y-x)}\,dy=f(x)*\overline{f(-x)}$$

Cross correlation. Let $g(x)$ and $h(x)$ be functions with finite energy. Then

$$\begin{aligned}(\overline{g}\star h)(x)&=\int_{-\infty}^{\infty}\overline{g(y)}h(y+x)\,dy\\&=\int_{-\infty}^{\infty}\overline{g(y-x)}h(y)\,dy\\&=\overline{(\overline{h}\star g)(-x)}\end{aligned}$$

Rectangle and triangle functions

$$\Pi(x)=\begin{cases}1 & |x|<\frac{1}{2}\\ 0 & |x|\geq\frac{1}{2}\end{cases}\qquad\Lambda(x)=\begin{cases}1-|x| & |x|\leq 1\\ 0 & |x|>1\end{cases}$$

$$\mathcal{F}\Pi(s)=\operatorname{sinc}s=\frac{\sin\pi s}{\pi s}\qquad\mathcal{F}\Lambda(s)=\operatorname{sinc}^2 s$$

Scaled rect function

$$\Pi_T(x)=\Pi\left(\frac{x}{T}\right)=\begin{cases}1 & |x|<\frac{T}{2}\\ 0 & |x|\geq\frac{T}{2}\end{cases}\qquad\mathcal{F}\Pi_T(s)=T\operatorname{sinc}Ts$$

The delta function $\boldsymbol{\delta(x)}$

Scaling.

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

Sifting.

$$\int_{-\infty}^{\infty}\delta(x-a)f(x)dx=f(a)$$
$$\int_{-\infty}^{\infty}\delta(x)f(x+a)dx=f(a)$$

Convolution.

$$\delta(x)*f(x)=f(x)$$
$$\delta(x-a)*f(x)=f(x-a)$$
$$\delta(x-a)*\delta(x-b)=\delta(x-(a+b))$$

Product.

$$f(x)\delta(x)=f(0)\delta(x)$$

Fourier transforms.

$$\mathcal{F}\delta=1$$
$$\mathcal{F}(\delta(x-a))=e^{-2\pi isa}$$

Derivatives.

$$\int_{-\infty}^{\infty}\delta^{(n)}(x)f(x)dx=(-1)^n f^{(n)}(0)$$
$$\delta'(x)*f(x)=f'(x)$$
$$x\delta(x)=0$$
$$x\delta'(x)=-\delta(x)$$

Fourier transform of cosine and sine

$$\mathcal{F}\cos 2\pi at=\frac{1}{2}(\delta(s-a)+\delta(s+a))$$
$$\mathcal{F}\sin 2\pi at=\frac{1}{2i}(\delta(s-a)-\delta(s+a))$$

Unit step and sgn

$$H(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}\qquad\mathcal{F}H(s)=\tfrac{1}{2}\left(\delta(s)+\frac{1}{\pi is}\right)$$

$$\operatorname{sgn}t=\begin{cases}-1 & t<0\\ +1 & t>0\end{cases}\qquad\mathcal{F}\operatorname{sgn}(s)=\frac{1}{\pi is}$$

EE 261 The Fourier Transform and its Applications

Being an Ancient Formula Sheet
Handed Down
To All EE 261 Students

Integration by parts:

$$\int_a^b u(t)v'(t)\,dt = \Big[u(t)v(t)\Big]_{t=a}^{t=b} - \int_a^b u'(t)v(t)\,dt$$

Even and odd parts of a function: Any function $f(x)$ can be written as

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{(even part)}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{(odd part)}}$$

Geometric series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

$$\sum_{n=M}^{N} r^n = \frac{r^M(1-r^{N-M+1})}{(1-r)}$$

Complex numbers: $z = x+iy$, $\bar{z} = x-iy$, $|z|^2 = z\bar{z} = x^2+y^2$

$$\frac{1}{i} = -i$$

$$x = \operatorname{Re} z = \frac{z+\bar{z}}{2}\,, \quad y = \operatorname{Im} z = \frac{z-\bar{z}}{2i}$$

Complex exponentials:

$$e^{2\pi it} = \cos 2\pi t + i\sin 2\pi t$$

$$\cos 2\pi t = \frac{e^{2\pi it}+e^{-2\pi it}}{2}\,, \quad \sin 2\pi t = \frac{e^{2\pi it}-e^{-2\pi it}}{2i}$$

Polar form:

$$z = x+iy \quad z = re^{i\theta}, \quad r = \sqrt{x^2+y^2}\,, \theta = \tan^{-1}(y/x)$$

Symmetric sum of complex exponentials (special case of geometric series):

$$\sum_{n=-N}^{N} e^{2\pi int} = \frac{\sin(2N+1)\pi t}{\sin \pi t}$$

Fourier series If $f(t)$ is periodic with period T its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$$

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\,dt$$

Orthogonality of the complex exponentials:

$$\int_0^T e^{2\pi int/T} e^{-2\pi imt/T}\,dt = \begin{cases} 0\,, & n \neq m \\ T\,, & n = m \end{cases}$$

The normalized exponentials $(1/\sqrt{T})e^{2\pi int/T}$, $n = 0, \pm 1, \pm 2, \ldots$ form an orthonormal basis for $L^2([0,T])$

Rayleigh (Parseval): If $f(t)$ is periodic of period T then

$$\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

The Fourier Transform:

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}dx$$

The Inverse Fourier Transform:

$$\mathcal{F}^{-1}f(x) = \int_{-\infty}^{\infty} f(s)e^{2\pi isx}ds$$

Symmetry & Duality Properties:

Let $f^-(x) = f(-x)$.

- $\mathcal{F}\mathcal{F}f = f^-$
- $\mathcal{F}^{-1}f = \mathcal{F}f^-$
- $\mathcal{F}f^- = (\mathcal{F}f)^-$
- If f is even (odd) then $\mathcal{F}f$ is even (odd)
- If f is real valued, then $\overline{\mathcal{F}f} = (\mathcal{F}f)^-$

Convolution:

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

- $f*g = g*f$
- $(f*g)*h = (f*g)*h$
- $f*(g+h) = f*g + f*h$

Smoothing: If f (or g) is p-times continuously differentiable, $p \geq 0$, then so is $f*g$ and

$$\frac{d^k}{dx^k}(f*g) = (\frac{d^k}{dx^k}f)*g$$

Convolution Theorem:

$$\mathcal{F}(f*g) = (\mathcal{F}f)(\mathcal{F}g)$$

$$\mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$$

Autocorrelation: Let $g(x)$ be a function satisfying $\int_{-\infty}^{\infty} |g(x)|^2\, dx < \infty$ (finite energy) then

$$\begin{aligned}(\overline{g} \star g)(x) &= \int_{-\infty}^{\infty} g(y)\overline{g(y-x)}\, dy \\ &= g(x) * \overline{g(-x)}\end{aligned}$$

Cross correlation: Let $g(x)$ and $h(x)$ be functions with finite energy. Then

$$\begin{aligned}(\overline{g} \star h)(x) &= \int_{-\infty}^{\infty} \overline{g(y)}h(y+x)\, dy \\ &= \int_{-\infty}^{\infty} \overline{g(y-x)}h(y)\, dy \\ &= \overline{(\overline{h} \star g)(-x)}\end{aligned}$$

Rectangle and triangle functions

$$\Pi(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| \geq \frac{1}{2} \end{cases} \quad \Lambda(x) = \begin{cases} 1-|x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$\mathcal{F}\Pi(s) = \operatorname{sinc} s = \frac{\sin \pi s}{\pi s}, \quad \mathcal{F}\Lambda(s) = \operatorname{sinc}^2 s$$

Scaled rect function

$$\Pi_p(x) = \Pi(x/p) = \begin{cases} 1, & |x| < \frac{p}{2} \\ 0, & |x| \geq \frac{p}{2} \end{cases}, \quad \mathcal{F}\Pi_p(s) = p \operatorname{sinc} ps$$

Gaussian

$$\mathcal{F}(e^{-\pi t^2}) = e^{-\pi s^2}$$

One-sided exponential decay

$$f(t) = \begin{cases} 0\,, & t < 0\,, \\ e^{-at}\,, & t \geq 0\,. \end{cases} \qquad \mathcal{F}f(s) = \frac{1}{a + 2\pi i s}$$

Two-sided exponential decay

$$\mathcal{F}(e^{-a|t|}) = \frac{2a}{a^2 + 4\pi^2 s^2}$$

Fourier Transform Theorems

Linearity: $\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(s) + \beta G(s)$

Stretch: $\mathcal{F}\{g(ax)\} = \frac{1}{|a|}G(\frac{s}{a})$

Shift: $\mathcal{F}\{g(x-a)\} = e^{-i2\pi a s}G(s)$

Shift & stretch: $\mathcal{F}\{g(ax-b)\} = \frac{1}{|a|}e^{-i2\pi s b/a}G(\frac{s}{a})$

Rayleigh (Parseval):

$$\begin{aligned}\int_{-\infty}^{\infty} |g(x)|^2\, dx &= \int_{-\infty}^{\infty} |G(s)|^2\, ds \\ \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx &= \int_{-\infty}^{\infty} F(s)\overline{G(s)}\, ds\end{aligned}$$

Modulation:

$$\mathcal{F}\{g(x)\cos(2\pi s_0 x)\} = \tfrac{1}{2}[G(s-s_0) + G(s+s_0)]$$

Autocorrelation: $\mathcal{F}\{\overline{g} \star g\} = |G(s)|^2$

Cross Correlation: $\mathcal{F}\{\overline{g} \star f\} = \overline{G(s)}F(s)$

Derivative:

- $\mathcal{F}\{g'(x)\} = 2\pi i s G(s)$
- $\mathcal{F}\{g^{(n)}(x)\} = (2\pi i s)^n G(s)$
- $\mathcal{F}\{x^n g(x)\} = (\frac{i}{2\pi})^n G^{(n)}(s)$

Moments:

$$\int_{-\infty}^{\infty} f(x)dx = F(0)$$

$$\int_{-\infty}^{\infty} x f(x)dx = \frac{i}{2\pi}F'(0)$$

$$\int_{-\infty}^{\infty} x^n f(x)dx = (\frac{i}{2\pi})^n F^{(n)}(0)$$

Miscellaneous:

$$\mathcal{F}\left\{\int_{-\infty}^{x} g(\xi)d\xi\right\} = \tfrac{1}{2}G(0)\delta(s) + \frac{G(s)}{i2\pi s}$$

The Delta Function: $\delta(x)$

Scaling: $\delta(ax) = \frac{1}{|a|}\delta(x)$

Sifting: $\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$

$\int_{-\infty}^{\infty} \delta(x)f(x+a)dx = f(a)$

Convolution: $\delta(x) * f(x) = f(x)$, $\delta(x-a) * f(x) = f(x-a)$

Product: $h(x)\delta(x) = h(0)\delta(x)$

$\delta(x-a) * \delta(x-b) = \delta(x-(a+b))$

Fourier Transform: $\mathcal{F}\delta = 1$

$\mathcal{F}(\delta(x-a)) = e^{-2\pi i s a}$

Derivatives:

- $\int_{-\infty}^{\infty} \delta^{(n)}(x)f(x)dx = (-1)^n f^{(n)}(0)$
- $\delta'(x) * f(x) = f'(x)$
- $x\delta(x) = 0$
- $x\delta'(x) = -\delta(x)$

Fourier transform of cosine and sine

$$\mathcal{F}\cos 2\pi at = \frac{1}{2}(\delta(s-a)+\delta(s+a))$$
$$\mathcal{F}\sin 2\pi at = \frac{1}{2i}(\delta(s-a)-\delta(s+a))$$

Unit step and sgn

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases} \quad \mathcal{F}u(s) = \frac{1}{2}\left(\delta(s)+\frac{1}{\pi i s}\right)$$

$$\operatorname{sgn} t = \begin{cases} -1, & t<0 \\ 1, & t>0 \end{cases} \quad \mathcal{F}\operatorname{sgn}(s) = \frac{1}{\pi i s}$$

The Shah Function: $\text{III}(x)$

$$\text{III}(x) = \sum_{n=-\infty}^{\infty}\delta(x-n)\,, \quad \text{III}_p(x) = \sum_{n=-\infty}^{\infty}\delta(x-np)$$

Sampling:	$\text{III}(x)g(x) = \sum_{n=-\infty}^{\infty} g(n)\delta(x-n)$
Periodization:	$\text{III}(x) * g(x) = \sum_{n=-\infty}^{\infty} g(x-n)$
Scaling:	$\text{III}(ax) = \frac{1}{a}\text{III}_{1/a}(x), \quad a>0$
Fourier Transform:	$\mathcal{F}\text{III} = \text{III}\,, \quad \mathcal{F}\text{III}_p = \frac{1}{p}\text{III}_{1/p}$

Sampling Theory For a bandlimited function $g(x)$ with $\mathcal{F}g(s) = 0$ for $|s| \geq p/2$

$$\mathcal{F}g = \Pi_p(\mathcal{F}g * \text{III}_p)$$

$$g(t) = \sum_{k=-\infty}^{\infty} g(t_k)\,\text{sinc}(p(x-t_k)) \quad t_k = k/p$$

Fourier Transforms for Periodic Functions
For a function $p(x)$ with period L, let $f(x) = p(x)\sqcap(\frac{x}{L})$. Then

$$\begin{aligned} p(x) &= f(x) * \sum_{n=-\infty}^{\infty}\delta(x-nL) \\ P(s) &= \frac{1}{L}\sum_{n=-\infty}^{\infty} F(\frac{n}{L})\delta(s-\frac{n}{L}) \end{aligned}$$

The complex Fourier series representation:

$$p(x) = \sum_{n=-\infty}^{\infty}\alpha_n e^{2\pi i \frac{n}{L}x}$$

where

$$\begin{aligned} \alpha_n &= \frac{1}{L}F(\frac{n}{L}) \\ &= \frac{1}{L}\int_{-L/2}^{L/2} p(x)e^{-2\pi i \frac{n}{L}x}dx \end{aligned}$$

Linear Systems Let L be a linear system, $w(t) = Lv(t)$, with impulse response $h(t,\tau) = L\delta(t-\tau)$.

Superposition integral:

$$w(t) = \int_{-\infty}^{\infty} v(\tau)h(t,\tau)d\tau$$

A system is time-invariant if:

$$w(t-\tau) = L[v(t-\tau)]$$

In this case $L(\delta(t-\tau) = h(t-\tau)$ and L acts by convolution:

$$\begin{aligned} w(t) &= Lv(t) = \int_{-\infty}^{\infty} v(\tau)h(t-\tau)d\tau \\ &= (v*h)(t) \end{aligned}$$

The transfer function is the Fourier transform of the impulse response, $H = \mathcal{F}h$ The eigenfunctions of any linear time-invariant system are $e^{2\pi i\nu t}$, with eigenvalue $H(\nu)$:

$$Le^{2\pi i\nu t} = H(\nu)e^{2\pi i\nu t}$$

The Discrete Fourier Transform

Nth root of unity:

Let $\omega = e^{2\pi i/N}$. Then $\omega^N = 1$ and the N powers $\underline{1} = \omega^0,\ \omega,\ \omega^2,\ \ldots\omega^{N-1}$ are distinct and evenly spaced along the unit circle.

Vector complex exponentials:

$$\underline{1} = (1,1,\ldots,1)$$
$$\underline{\omega} = (1,\omega,\omega^2,\ldots,\omega^{N-1})$$
$$\underline{\omega}^k = (1,\omega^k,\omega^{2k},\ldots,\omega^{(N-1)k})$$

Cyclic property

$$\underline{\omega}^N = \underline{1} \quad \text{and} \quad \underline{1},\underline{\omega},\underline{\omega}^2,\ldots\underline{\omega}^{N-1} \quad \text{are distinct}$$

The vector complex exponentials are orthogonal:

$$\underline{\omega}^k\cdot\underline{\omega}^\ell = \begin{cases} 0, & k\not\equiv\ell \mod N \\ N, & k\equiv\ell \mod N \end{cases}$$

The DFT of order N accepts an N-tuple as input and returns an N-tuple as output. Write an N-tuple as $\underline{f} = (\underline{f}[0],\underline{f}[1],\ldots,\underline{f}[N-1])$.

$$\underline{\mathcal{F}}\,\underline{f} = \sum_{k=0}^{N-1}\underline{f}[k]\underline{\omega}^{-k}$$

Inverse DFT:

$$\underline{\mathcal{F}}^{-1}\underline{f} = \frac{1}{N}\sum_{k=0}^{N-1} \underline{f}[k]\underline{\omega}^k$$

Periodicity of inputs and outputs: If $\underline{F} = \underline{\mathcal{F}}\,\underline{f}$ then both $\underline{f}$ and $\underline{F}$ are periodic of period N.

Convolution

$$(\underline{f} * \underline{g})[n] = \sum_{k=0}^{N-1} \underline{f}[k]\,\underline{g}[n-k]$$

Discrete δ:

$$\underline{\delta}_k[m] = \begin{cases} 1, & m \equiv k \mod N \\ 0, & m \not\equiv k \mod N \end{cases}$$

DFT of the discrete δ $\quad \underline{\mathcal{F}}\,\underline{\delta}_k = \underline{\omega}^{-k}$

DFT of vector complex exponential $\quad \underline{\mathcal{F}}\,\underline{\omega}^k = N\underline{\delta}_k$

Reversed signal: $\underline{f}^-[m] = \underline{f}[-m]$ $\quad \underline{\mathcal{F}}\,\underline{f}^- = (\underline{\mathcal{F}}\,\underline{f})^-$

DFT Theorems

Linearity: $\quad \underline{\mathcal{F}}\{\alpha\underline{f} + \beta\underline{g}) = \alpha\underline{\mathcal{F}}\,\underline{f} + \beta\underline{\mathcal{F}}\,\underline{g}$

Parseval: $\quad \underline{\mathcal{F}}\,\underline{f} \cdot \underline{\mathcal{F}}\,\underline{g} = N(\underline{f} \cdot \underline{g})$

Shift: Let $\tau_p\underline{f}[m] = \underline{f}[m-p]$. Then $\underline{\mathcal{F}}\,(\tau_p\underline{f}) = \underline{\omega}^{-p}\underline{\mathcal{F}}\,\underline{f}$

Modulation: $\quad \underline{\mathcal{F}}\,(\underline{\omega}^p\underline{f}) = \tau_p(\underline{\mathcal{F}}\,\underline{f})$

Convolution: $\quad \underline{\mathcal{F}}\,(\underline{f} * \underline{g}) = (\underline{\mathcal{F}}\,\underline{f})(\underline{\mathcal{F}}\,\underline{g})$

$$\underline{\mathcal{F}}\,(\underline{f}\,\underline{g}) = \tfrac{1}{N}(\underline{\mathcal{F}}\,\underline{f} * \underline{\mathcal{F}}\,\underline{g})$$

The Hilbert Transform The Hilbert Transform of $f(x)$:

$$\mathcal{H}f(x) = -\frac{1}{\pi x} * f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi$$

(Cauchy principal value)

Inverse Hilbert Transform $\quad \mathcal{H}^{-1}f = -\mathcal{H}f$

Impulse response: $\quad -\frac{1}{\pi x}$

Transfer function: $\quad i\,\mathrm{sgn}\,(s)$

Causal functions: $g(x)$ is causal if $g(x) = 0$ for $x < 0$. A casual signal Fourier Transform $G(s) = R(s) + iI(s)$, where $I(s) = \mathcal{H}\{R(s)\}$.

Analytic signals: The analytic signal representation of a real-valued function $v(t)$ is given by:

$$\begin{aligned} \mathcal{Z}(t) &= \mathcal{F}^{-1}\{2H(s)V(s)\} \\ &= v(t) - i\mathcal{H}v(t) \end{aligned}$$

Narrow Band Signals: $\quad g(t) = A(t)\cos[2\pi f_0 t + \phi(t)]$

Analytic approx: $\quad z(t) \approx A(t)e^{i[2\pi f_0 t + \phi(t)]}$

Envelope: $\quad |\,A(t)\,| = |\,z(t)\,|$

Phase: $\quad \arg[z(t)] = 2\pi f_0 t + \phi(t)$

Instantaneous freq: $\quad f_i = f_0 + \frac{1}{2\pi}\frac{d}{dt}\phi(t)$

Higher Dimensional Fourier Transform In n-dimensions:

$$\mathcal{F}f(\underline{\xi}) = \int_{\mathbf{R}^n} e^{-2\pi i \underline{x}\cdot\underline{\xi}} f(\underline{x})\,d\underline{x}$$

Inverse Fourier Transform:

$$\mathcal{F}^{-1}f(\underline{x}) = \int_{\mathbf{R}^n} e^{2\pi i \underline{x}\cdot\underline{\xi}} f(\underline{\xi})\,d\underline{\xi}$$

In 2-dimensions (in coordinates):

$$\mathcal{F}f(\xi_1, \xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(x_1\xi_1 + x_2\xi_2)} f(x_1, x_2)\,dx_1 dx_2$$

The Hankel Transform (zero order):

$$F(\rho) = 2\pi\int_0^{\infty} f(r)J_0(2\pi r\rho)r dr$$

The Inverse Hankel Transform (zero order):

$$f(r) = 2\pi\int_0^{\infty} F(\rho)J_0(2\pi r\rho)\rho d\rho$$

Separable functions: If $f(x_1, x_2) = f(x_1)f(x_2)$ then

$$\mathcal{F}f(\xi_1, \xi_2) = \mathcal{F}f(\xi_1)\mathcal{F}f(\xi_2)$$

Two-dimensional rect:

$$\Pi(x_1, x_2) = \Pi(x_1)\Pi(x_2), \quad \mathcal{F}\Pi(\xi_1, \xi_2) = \mathrm{sinc}\,\xi_1\,\mathrm{sinc}\,\xi_2$$

Two dimensional Gaussian:

$$g(x_1, x_2) = e^{-\pi(x_1^2 + x_2^2)}, \quad \mathcal{F}g = g$$

Fourier transform theorems

Shift: Let $(\tau_{\underline{b}}f)(\underline{x}) = f(\underline{x} - \underline{b})$. Then

$$\mathcal{F}(\tau_b f)(\underline{\xi}) = e^{-2\pi i\underline{\xi}\cdot\underline{b}}\mathcal{F}f(\underline{\xi})$$

Stretch theorem (special):

$$\mathcal{F}(f(a_1x_1, a_2, x_2)) = \frac{1}{|a_1||a_2|}\mathcal{F}f(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2})$$

Stretch theorem (general): If A is an $n \times n$ invertible matrix then

$$\mathcal{F}(f(A\underline{x})) = \frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\underline{\xi})$$

Stretch and shift:

$$\mathcal{F}(f(A\underline{x}+\underline{b})) = \exp(2\pi i\underline{b}\cdot A^{-\mathsf{T}}\underline{\xi})\frac{1}{|\det A|}\mathcal{F}f(A^{-\mathsf{T}}\underline{\xi})$$

***III*'s and lattices** *III* for integer lattice

$$\begin{aligned} III_{\mathbf{Z}^2}(\underline{x}) &= \sum_{\underline{n}\in\mathbf{Z}^2}\delta(\underline{x}-\underline{n}) \\ &= \sum_{n_1,n_2=-\infty}^{\infty}\delta(x_1-n_1,x_2-n_2) \\ \mathcal{F}III_{\mathbf{Z}^2} &= III_{\mathbf{Z}^2} \end{aligned}$$

A general lattice $\mathcal{L}$ can be obtained from the integer lattice by $\mathcal{L} = A(\mathbf{Z}^2)$ where A is an invertible matrix.

$$III_{\mathcal{L}}(\underline{x}) = \sum_{\underline{p}\in\mathcal{L}}\delta(\underline{x}-\underline{p}) = \frac{1}{|\det A|}III_{\mathbf{Z}^2}(A^{-1}\underline{x})$$

If $\mathcal{L} = A(\mathbf{Z}^2)$ then the reciprocal lattice is $\mathcal{L}^* = A^{-\mathsf{T}}\mathbf{Z}^2$
Fourier transform of $III_{\mathcal{L}}$:

$$\mathcal{F}III_{\mathcal{L}} = \frac{1}{|\det A|}III_{\mathcal{L}^*}$$

Radon transform and Projection-Slice Theorem: Let $\mu(x_1,x_2)$ be the density of a two-dimensional region. A line through the region is specified by the angle ϕ of its normal vector to the x_1-axis, and its directed distance ρ from the origin. The integral along a line through the region is given by the Radon transform of μ:

$$\mathcal{R}\mu(\rho,\phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x_1,x_2)\delta(\rho-x_1\cos\phi-x_2\sin\phi)\,dx_1dx_2$$

The one-dimensional Fourier transform of $\mathcal{R}\mu$ with respect to ρ is the two-dimensional Fourier transform of μ:

$$\mathcal{F}_\rho\mathcal{R}(\mu)(r,\phi) = \mathcal{F}\mu(\xi_1,\xi_2), \quad \xi_1 = r\cos\phi,\ \xi_2 = r\sin\phi$$

The list being compiled originally
by John Jackson
a person not known to me,
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Trigonometry Function Identities

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Indentities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\csc^2\theta - \cot^2\theta = 1$$

Even/Odd Indentities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\frac{\pi}{2} \text{ radians} = 90°$$

Sum/Difference Indentities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta - \phi) - \cos(\theta + \phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta - \phi) + \cos(\theta + \phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta + \phi) + \sin(\theta - \phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta + \phi) + \sin(\theta - \phi)]}{2}$$