5 Applications of Differentiation

Definition A function f defined on an interval I is called **(strictly) increasing** on I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

and is called **(strictly) decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

First-Derivative Test for Monotonicity Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

EXAMPLE 1 Determine where the function

$$\boxed{f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3, \quad x \in \mathbf{R}}$$

is increasing and where it is decreasing.

f is cont. on $\mathbb{R}$.

$f'(x) = 3x^2 - \frac{3}{\cancel{2}}(\cancel{2}x) - 6$

$= 3x^2 - 3x - 6 = 3(x^2 - x - 2)$

$= 3(x-2)(x+1) = 0$

$\boxed{x=2}$ or $\boxed{x=-1}$

f': $+ + + + \;\; (-1) \;\; - - - - - - - - \;\; (2) \;\; + + + + +$

$f'(x) = 3(x-2)(x+1)$

$f'(4) = 3(2)(5) > 0$

$f'(0) = 3(-2)(1) < 0$

$f'(-2) = 3(-4)(-1) > 0$

f is increasing on $(-\infty, -1] \cup [2, \infty)$.

f is decreasing on $[-1, 2]$.

(18) $y = \ln x$, $x > 0$

$y' = \frac{1}{x} > 0$

y': $\cancel{\quad} \;\; (0) \;\; + + + + + +$

$\therefore f$ is increasing on $(0, \infty)$.

Definition A differentiable function $f(x)$ is **concave up** on an interval I if the first derivative $f'(x)$ is an increasing function on I. $f(x)$ is **concave down** on an interval I if the first derivative $f'(x)$ is a decreasing function on I.

Second-Derivative Test for Concavity Suppose that f is twice differentiable on an open interval I.

(a) If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

(b) If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

EXAMPLE 3 Determine where the function

$$f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3, \quad x \in \mathbf{R}$$

is concave up and where it is concave down.

Sol: f is cont. and diff'ble on $\mathbb{R}$.

$$f'(x) = 3x^2 - 3x - 6$$

$$\boxed{f''(x) = 6x - 3} = 0 \Rightarrow x = \frac{3}{6} = \frac{1}{2}$$

$f''(x)$: $- - - - - - - - - \quad \frac{1}{2} \quad + + + + + +$

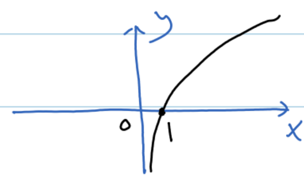
$f''(0) = -3 < 0$ $\qquad$ $f''(1) = 6 - 3 > 0$

f is concave up on $(\frac{1}{2}, \infty)$

" down on $(-\infty, \frac{1}{2})$.

(18) $y = \ln x$, $x > 0$.

$y' = \frac{1}{x} > 0$; $y'' = \frac{-1}{x^2} < 0$



y': $\times \quad 0 \quad + + + + +$

y'': $\times \quad 0 \quad - - - - - - - -$

$y = \ln x$ is increasing on $(0, \infty)$ and concave down on $(0, \infty)$

A continuous function has a local minimum at c if the function is decreasing to the left of c and increasing to the right of c. A continuous function has a local maximum at c if the function is increasing to the left of c and decreasing to the right of c.

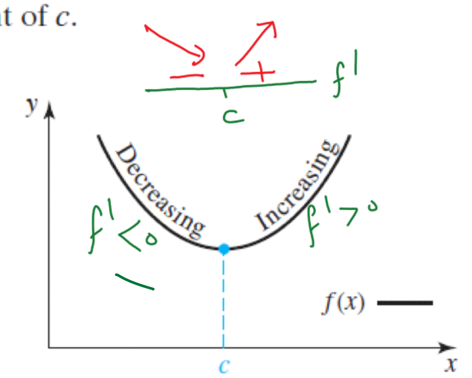


Figure 5.36 The function $y = f(x)$ has a local minimum at $x = c$.

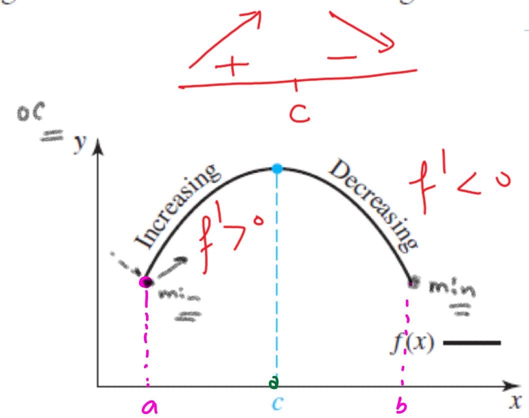


Figure 5.37 The function $y = f(x)$ has a local maximum at $x = c$.

1. Find all numbers c where $f'(c) = 0$.
2. Find all numbers c where $f'(c)$ does not exist.
3. Find the endpoints of the domain of f.

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Critical points

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$c \in Dom(f)$.

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EXAMPLE 2 Find all local and global extrema of

$D_f = (-\infty, \infty)$.

$$f(x) = x(1-x)^{2/3}, \quad x \in \mathbf{R}$$

$$f'(x) = (x)\,\tfrac{2}{3}(1-x)^{-\frac{1}{3}}(-1) + (1-x)^{\frac{2}{3}}(1).$$

$$= \frac{-2x}{3(1-x)^{\frac{1}{3}}} + \frac{(1-x)^{\frac{2}{3}}}{1}$$

$$= \frac{(-2x)(1) + (1-x)^{\frac{2}{3}}\left(3(1-x)^{\frac{1}{3}}\right)}{3(1-x)^{\frac{1}{3}}(1)}$$

$$= \frac{-2x + 3(1-x)}{3(1-x)^{\frac{1}{3}}} = \frac{-5x+3}{3(1-x)^{\frac{1}{3}}}$$

$$\boxed{f'(x) = \frac{-5x+3}{3(1-x)^{\frac{1}{3}}}}$$

$f'(x) = 0 \Rightarrow -5x+3=0 \Rightarrow \boxed{x = \frac{3}{5}} \in D_f$

$f'(x)$ DNE $\Rightarrow \frac{3}{3}(1-x)^{\frac{1}{3}} = \frac{0}{3}$

$\left[(1-x)^{\frac{1}{3}}\right]^3 = (0)^3$

$1-x=0 \Rightarrow \boxed{x=1} \in D_f$

f': $+ + + + + + \quad \frac{3}{5} \quad - - - - - - - - \quad 1 \quad + + + + + +$

$\left(\frac{3}{5}, f\left(\frac{3}{5}\right)\right)$ local max.

$(1, f(1)) = (1, 0)$ local min.

Inflection points are points where the concavity of a function changes—that is, where the function changes from concave up to concave down or from concave down to concave up.

EXAMPLE 3 Show that the function

$$f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x + 1, \quad x \in \mathbf{R}$$

has an inflection point at $x = 1$.

Sol: f is cont. + diff'ble on $\mathbb{R}$.

$f'(x) = \frac{3}{2}x^2 - \frac{6}{2}x + 2$

$= \frac{3}{2}x^2 - 3x + 2$

$f''(x) = 3x - 3 = 0 \Rightarrow \boxed{x=1}$

f'': $- - - - - - \quad 1 \quad + + + +$

$(1, f(1)) = (1, 2)$ is the inflection point.

f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

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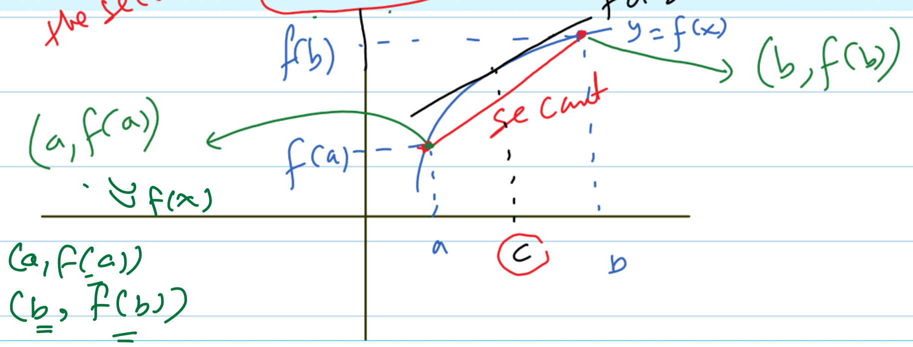
The Mean-Value Theorem (MVT) If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

conclusion

slope of the secant

slope of the tangent



35. Suppose $f(x) = x^2, x \in [0, 2]$.

(a) Find the slope of the secant line connecting the points $(0, 0)$ and $(2, 4)$.

$x_1 \; y_1$ $x_2 \; y_2$

(b) Find a number $c \in (0, 2)$ such that $f'(c)$ is equal to the slope of the secant line you computed in (a), and explain why such a number must exist in $(0, 2)$.

Sol. (a) Slope $= \frac{4-0}{2-0} = 2$

(b) By MVT, there is $c \in (0,2)$:

$f(x) = x^2$

$\boxed{f'(x) = 2x}$

$f'(c) = \frac{f(2) - f(0)}{2-0}$

$2c = \frac{4-0}{2-0} = 2$

$\boxed{c = 1} \in (0,2)$.

Rolle's Theorem If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

L'Hospital's Rule Suppose that f and g are differentiable functions and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$$

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Example:

[1] $\lim_{x\to 2}\frac{x^6-64}{x^2-4} \Rightarrow \lim_{x\to 2}\frac{2^6-64}{2^2-4} = \frac{0}{0}$

$\Rightarrow \lim_{x\to 2}\frac{6x^5}{2x} = \lim_{x\to 2} 3x^4 = 3.2^4 = 3.16 = 48$

[2] $\lim_{x\to\infty}\frac{\ln x}{x} \Rightarrow \lim_{x\to\infty}\frac{\ln\infty}{\infty} = \frac{\infty}{\infty}$

$\Rightarrow \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$

[3] $\lim_{x\to\infty}\frac{e^x}{x} \Rightarrow \frac{e^\infty}{\infty} = \frac{\infty}{\infty}$

$\Rightarrow \lim_{x\to\infty}\frac{e^x}{1} = \infty$ (DNE)

[4] $\lim_{x\to 0^+} x\ln x \Rightarrow 0.\ln 0^+ = 0.(-\infty)$

$\Rightarrow \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$

[5] $\lim_{x\to(\frac{\pi}{2})^-}(\tan x-\sec x) = \lim_{x\to\frac{\pi}{2}^-}\tan(\frac{\pi}{2})^- - \sec(\frac{\pi}{2})^-$

$= \infty - \infty$

But $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$

$$\Rightarrow \tan x - \sec x = \frac{\sin x}{\cos x} - \frac{1}{\cos x}$$

$$= \frac{\sin x - 1}{\cos x}$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} \tan x - \sec x = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x - 1}{\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{-\sin x} = -\lim_{x \to \frac{\pi}{2}^-} \cot x$$

$$= 0$$

$\boxed{6}$ $\lim\limits_{x \to 0^+} x^x = 0^0$

$$\Rightarrow x^x = e^{\ln(x^x)} = e^{x * \ln x}$$

Now, $\lim\limits_{x \to 0^+} x \cdot \ln x = 0$ by example 4 then.

$e^0 = \underline{\underline{1}}$, so $\lim\limits_{x \to 0^+} x^x = 1$