Exercises :

Q11 : let X_1 and X_2 have the joint p.d.f . $f(x_1, x_2) = \begin{cases} x_1 + x_2 & , 0<x_1<1, 0<x_2<1 \\ 0 & , \text{elsewhere} \end{cases}$.

Find the conditional mean and variance of X_2 given $X_1 = x_1$, $0 < x_1 < 1$.

$$E(X_2 | x_1) = \int_{-\infty}^{\infty} x_2 \, f_{2|1}(x_2 | x_1) \, dx_2$$

$$\circ \; f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2 = \int_0^1 (x_1 + x_2) \, dx_2 = x_1 x_2 + \frac{1}{2} x_2^2 \Big|_0^1$$

$$= x_1 + \frac{1}{2} \quad , 0 < x_1 < 1 .$$

$$\rightarrow f_1(x_1) = \begin{cases} x_1 + \frac{1}{2} & , 0<x_1<1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$\circ \; f_{2|1}(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} = \begin{cases} \frac{x_1 + x_2}{x_1 + \frac{1}{2}} & , 0<x_2<1 \\ 0 & , \text{elsewhere} \end{cases}$$

so $$E(X_2 | x_1) = \int_0^1 x_2 \left(\frac{x_1 + x_2}{x_1 + \frac{1}{2}} \right) dx_2 = \frac{1}{x_1 + \frac{1}{2}} \int_0^1 (x_2 x_1 + x_2^2) \, dx_2$$

$$= \frac{1}{x_1 + \frac{1}{2}} \left[\frac{x_1 x_2^2}{2} + \frac{x_2^3}{3} \Big|_0^1 \right]$$

$$= \frac{1}{x_1 + \frac{1}{2}} \left[\frac{3 \times x_1}{3 \times 2} + \frac{1 \times 2}{3 \times 2} \right]$$

$$= \frac{1}{x_1 + \frac{1}{2}} \left[\frac{3x_1 + 2}{6} \right]$$

$$= \frac{3x_1 + 2}{6x_1 + 3}$$

$\rightarrow$

cont. Q11 : $Var(X_2 | X_1) = E\left[(X_2 - E(X_2 | X_1))^2 \mid X_1 \right]$

$$= \int_{-\infty}^{\infty} \left(X_2 - \frac{3X_1+2}{6X_1+3} \right)^2 \cdot f_{2|1}(X_2|X_1) \; dX_2 \qquad 0<X_2<1$$

$$= \int_0^1 \left[X_2^2 - 2X_2\left(\frac{3X_1+2}{6X_1+3}\right) + \left(\frac{3X_1+2}{6X_1+3}\right)^2 \right] \left(\frac{X_1+X_2}{X_1+\frac{1}{2}}\right) dX_2$$

$$= \int_0^1 X_2^2\left(\frac{X_1+X_2}{X_1+\frac{1}{2}}\right) - 2X_2\left(\frac{3X_1+2}{6X_1+3}\right)\left(\frac{X_1+X_2}{X_1+\frac{1}{2}}\right) + \frac{(3X_1+2)^2}{(6X_1+3)^2}\left(\frac{X_1+X_2}{X_1+\frac{1}{2}}\right) dX_2$$

$$= \frac{1}{X_1+\frac{1}{2}} \int_0^1 X_1X_2^2 + X_2^3 - \left(\frac{3X_1+2}{6X_1+3}\right)(2X_2X_1 + 2X_2^2) + \left(\frac{3X_1+2}{6X_1+3}\right)^2 (X_1+X_2) \; dX_2 .$$

$$= \frac{1}{X_1+\frac{1}{2}} \left[\frac{X_1X_2^3}{3} + \frac{X_2^4}{4} - \left(\frac{3X_1+2}{6X_1+3}\right)\left(\frac{2X_1X_2^2}{2} + \frac{2X_2^3}{3}\right) + \left(\frac{3X_1+2}{6X_1+3}\right)^2\left(X_1X_2 + \frac{X_2^2}{2}\right) \Big|_0^1 \right]$$

$$= \frac{1}{X_1+\frac{1}{2}} \left[\frac{X_1}{3} + \frac{1}{4} - \left(\frac{3X_1+2}{6X_1+3}\right)\left(X_1 + \frac{2}{3}\right) + \left(\frac{3X_1+2}{6X_1+3}\right)^2\left(X_1 + \frac{1}{2}\right) \right]$$

$$= \frac{1}{X_1+\frac{1}{2}} \left[\frac{X_1}{3} + \frac{1}{4} - \left(\frac{3X_1^2 + 2X_1 + 2X_1 + \frac{4}{3}}{6X_1+3}\right) + \left(\frac{3X_1+2}{6X_1+3}\right)^2\left(X_1+\frac{1}{2}\right) \right]$$

$$= \frac{1}{X_1+\frac{1}{2}} \left[\frac{X_1}{3} + \frac{1}{4} + \frac{-3X_1^2 - 4X_1 - \frac{4}{3}}{6X_1+3} \right.$$

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Q12: let $f_{1|2}(x_1 | x_2) = \begin{cases} \frac{c_1 x_1}{x_2^2} , & 0 < x_1 < x_2 , \quad 0 < x_2 < 1 \\ 0 , & \text{elsewhere} . \end{cases}$

and $f_2(x_2) = \begin{cases} c_2 x_2^4 , & 0 < x_2 < 1 \\ 0 , & \text{elsewhere} . \end{cases}$

denote, respectively, the conditional p.d.f of X_1 given X_2 and the marginal p.d.f of X

a. the constant c_1 and c_2.

since $f_{1|2}(x_1|x_2)$ is p.d.f so:

$$\int_0^{x_2} \frac{c_1 x_1}{x_2^2} \, dx_1 = 1 \rightarrow \frac{c_1}{x_2^2}\left(\frac{x_1^2}{2}\Big|_0^{x_2}\right) = 1 \rightarrow \frac{c_1}{\cancel{x_2^2}}\left(\frac{\cancel{x_2^2}}{2}\right) = 1$$

$$\rightarrow \frac{c_1}{2} = 1$$

$$\rightarrow \boxed{c_1 = 2}$$

and since $f_2(x_2)$ is a p.d.f then:

$$\int_0^1 c_2 x_2^4 \, dx_2 = 1 \rightarrow \frac{c_2 \, x_2^5}{5}\Big|_0^1 = 1 \rightarrow \frac{c_2}{5} = 1 \rightarrow \boxed{c_2 = 5}$$

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b. the joint p.d.f of X_1 and X_2: $f_{1|2}(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$

$$f(x_1, x_2) = f_{1|2}(x_1 | x_2) \cdot f_2(x_2)$$

$$= \frac{2 x_1}{x_2^2} (5 x_2^4)$$

$$= 10 \, x_1 \, x_2^2 \quad \text{for} \quad 0 < x_1 < x_2 < 1$$

c. $Pr\left(\frac{1}{4} < X_1 < \frac{1}{2} \mid X_2 = \frac{5}{8}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_{1|2}\left(x_1 \mid \frac{5}{8}\right) \, dx_1$

$f_{1|2}\left(x_1 \mid \frac{5}{8}\right) = \frac{2x_1}{\left(\frac{5}{8}\right)^2} = 2x_1 \times \frac{64}{25} = \frac{128}{25} x_1$

$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{128}{25} x_1 \, dx_1 = \frac{128}{25} \frac{x_1^2}{2} \Big|_{\frac{1}{4}}^{\frac{1}{2}} = \frac{128}{50}\left(\frac{1}{4} - \frac{1}{16}\right)$

$= \frac{\overset{8}{\cancel{128}}}{50}\left(\frac{3}{\cancel{16}}\right)$

$= \frac{24 \div 2}{50 \div 2} = \frac{12}{25}$

$\Rightarrow Pr\left(\frac{1}{4} < X_1 < \frac{1}{2} \mid X_2 = \frac{5}{8}\right) = \frac{12}{25}$ ✓

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d. $Pr\left(\frac{1}{4} < X_1 < \frac{1}{2}\right) =$

$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2 = \int_{x_1}^{1} 10 x_1 x_2^2 \, dx_2 = \frac{10 x_1 x_2^3}{3} \Big|_{x_1}^{1} = \frac{10 x_1}{3} - \frac{10 x_1^4}{3} = \frac{10}{3}(x_1 - x_1^4)$

So $Pr\left(\frac{1}{4} < X_1 < \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_1(x_1) \, dx_1 = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{10}{3}(x_1 - x_1^4) \, dx_1 = \frac{10}{3}\left[\frac{x_1^2}{2} - \frac{x_1^5}{5} \Big|_{\frac{1}{4}}^{\frac{1}{2}}\right]$

$= \frac{10}{3}\left[\left(\frac{1}{8} - \frac{1}{160}\right) - \left(\frac{1}{32} - \frac{1}{5120}\right)\right]$

$= \frac{10}{3}\left(\frac{449}{5120}\right)$

$= \frac{449}{1536}$ ✓

Q13: let $f(x_1, x_2) = \begin{cases} 21 x_1^2 x_2^3 & , 0 < x_1 < x_2 < 1 \\ 0 & , \text{elsewhere} . \end{cases}$ be the joint p.d.f of X_1 and X_2.

a. Find the conditional mean and variance of X_1 given $X_2 = x_2$, $0 < x_2 < 1$.

$$E(X_1 \mid x_2) = \int_{-\infty}^{\infty} x_1 f_{1|2}(x_1 \mid x_2)\, dx_1 .$$

$$\circ\ f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 = \int_0^{x_2} 21 x_1^2 x_2^3\, dx_1 = \frac{21}{3} x_1^3 x_2^3 \Big|_0^{x_2} = 7 x_2^6 .$$

$$\circ\ f_{1|2}(x_1 \mid x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = \begin{cases} \frac{21 x_1^2 x_2^3}{7 x_2^6} = \frac{3 x_1^2}{x_2^3} & , 0 < x_1 < x_2 \\ 0 & , \text{elsewhere} \end{cases}$$

$$\Rightarrow E(X_1 \mid x_2) = \int_0^{x_2} x_1 \left(\frac{3 x_1^2}{x_2^3} \right) dx_1 = \int_0^{x_2} \frac{3 x_1^3}{x_2^3}\, dx_1 = \frac{3}{4} \frac{x_1^4}{x_2^3} \Big|_0^{x_2} = \frac{3}{4} x_2$$

$$\Rightarrow E(X_1 \mid x_2) = \begin{cases} \frac{3}{4} x_2 & , 0 < x_2 < 1 \\ 0 & , \text{elsewhere} . \end{cases}$$

$$\Rightarrow Var(X_1 \mid x_2) = E(X_1^2 \mid x_2) - \left(E(X_1 \mid x_2) \right)^2$$

$$= \int_0^{x_2} x_1^2 \left(\frac{3 x_1^2}{x_2^3} \right) dx_1 - \left(\frac{3}{4} x_2 \right)^2$$

$$\int \frac{3 x_1^4}{x_2^3}\, dx_1 = \frac{3}{5} \frac{x_1^5}{x_2^3} \Big|_0^{x_2} = \frac{3}{5} x_2^2$$

$$= \frac{16 \times 3}{16 \times 5} x_2^2 - \frac{5 \times 9}{5 \times 16} x_2^2$$

$$Var(X_1 \mid x_2) = \frac{3 x_2^2}{80}$$

C?

b. Find the distribution of $y = E(X_1)X_2$.

$$y = \frac{3}{4} X_2$$

$F(y) = Pr(Y \leq y) = Pr(\frac{3}{4} X_2 \leq y) = Pr(X_2 \leq \frac{4}{3} y)$

$$= \int_0^{\frac{4}{3}y} f_2(x_2)\, dx_2$$

$$= \int_0^{\frac{4}{3}y} 7x_2^6\, dx_2 = x_2^7 \Big|_0^{\frac{4}{3}y} = \left(\frac{4}{3} y\right)^7 .$$

So the p.d.f $f(y) = F'(y)$.

$$f(y) = 7\left(\frac{4}{3}\right)^7 y^6 .$$

C. Determine $E(y)$ and $Var(y)$:

By Remark : $E(y) = \int_0^{\frac{3}{4}} \qquad dy$

Q14: If X_1 and X_2 are Random variables of discrete type having p.d.f

$$f(x_1, x_2) = \begin{cases} \frac{x_1 + 2x_2}{18} , & (x_1, x_2) = (1,1), (1,2), (2,1), (2,2). \\ 0 , & \text{elsewhere.} \end{cases}$$

determine the conditional mean and variance of X_2 given X_1, for $X_1 = 1$ or 2.
Also compute $E(3X_1 - 2X_2)$.

$X_1 = 1$.

$E(X_2 | X_1 = 1) = \sum x_2 \; f_{2|1}(x_2 | x_1 = 1)$

$E(X_2 | X_1 = 2) = \sum x_2 \; f_{2|1}(x_2 | x_1 = 2)$

$$f_1(x_1) = \sum f(x_1, x_2) = \begin{cases} \sum f(1, x_2) = \frac{3}{18} + \frac{5}{18} = \frac{8}{18} , & x_1 = 1 \\ \sum f(2, x_2) = \frac{4}{18} + \frac{6}{18} = \frac{10}{18} , & x_1 = 2 \end{cases}$$

$$\Rightarrow f_{2|1}(x_2 | x_1 = 1) = \frac{f(1, x_2)}{f_1(1)} = \begin{cases} \frac{3/18}{8/18} = \frac{3}{8} , & x_2 = 1 \\ \frac{5/18}{8/18} = \frac{5}{8} , & x_2 = 2 \\ 0 , & \text{elsewhere} \end{cases}$$

$$\Rightarrow E(X_2 | X_1 = 1) = 1\left(\frac{3}{8}\right) + 2\left(\frac{5}{8}\right) = \frac{3 + 10}{8} = \frac{13}{8}$$

Now,

$$f_{2|1}(x_2 | x_1 = 2) = \frac{f(2, x_2)}{f_1(2)} = \begin{cases} \frac{4/18}{10/18} = \frac{4}{10} , & x_2 = 1 \\ \frac{6/18}{10/18} = \frac{6}{10} , & x_2 = 2 \\ 0 , & \text{elsewhere} \end{cases}$$

$\rightarrow$

. $E(X_2 | X_1 = 2) = 1\left(\frac{4}{10}\right) + 2\left(\frac{6}{10}\right) = \frac{4+12}{10} = \frac{16}{10}$.

$\rightarrow var(X_2 | X_1) = E(X_2^2 | X_1) - (E(X_2|X_1))^2$

$$\left.\begin{aligned} E(X_2^2 | X_1) &= \sum_{X=1}^{2} X_2^2 \, f_{2|1}(X_2|X_1) \\ &= 1\, f(X_2|1) + 4\, f(X_2|1) \\ &= \frac{3}{8} + 4\left(\frac{5}{8}\right) \\ &= \frac{23}{8} \end{aligned}\right\} \rightarrow X_1 = 1$$

$$\left.\begin{aligned} E(X_2^2|X_1) &= \sum X_2^2 \, f_{2|1}(X_2|X_1) \\ &= 1.\frac{4}{10} + 4\left(\frac{6}{10}\right) \\ &= \frac{28}{10} \end{aligned}\right\} \rightarrow X_1 = 2$$

$$\Rightarrow var(X_2|X_1) = \begin{cases} \frac{23}{8} - \left(\frac{13}{8}\right)^2 = \frac{15}{64}, & X_1 = 1 \\ \frac{28}{10} - \left(\frac{16}{10}\right)^2 = \frac{6}{25}, & X_1 = 2. \end{cases}$$

$\Rightarrow E(3X_1 - 2X_2) = 3E(X_1) - 2E(X_2)$.

. $E(X_1) = \sum_{1}^{2} X_1 f_1(X_1) = 1.\frac{8}{18} + 1.\frac{8}{18} + 2.\frac{10}{8} + 2.\frac{10}{8} =$