3.2 : Poisson distribution.

Def: a Random variable X is called a poisson Random variable with parameter m if it has the following p.d.f

$$f(x) = \begin{cases} \frac{m^x e^{-m}}{x!} & , x = 0, 1, \ldots \\ 0 & , \text{elsewhere} \end{cases}$$

where $m > 0$.

Remarks:

1. $A = \{0, 1, 2, \ldots\}$

   X discrete random variable.

2. $f(x)$ is a p.d.f.

   i. $f(x) > 0 \quad \forall x \in A$

   $f(x) = 0 \quad \forall x \in A^*$

   ii. $\sum_x f(x) = 1$.

Recall : " Taylor series "

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Proof ii:

$$e^{M} = \sum_{K=0}^{\infty} \frac{M^{K}}{K!} \Rightarrow e^{M} = \sum_{X=0}^{\infty} \frac{M^{X}}{X!}$$

$$e^{-m} \cdot e^{M} = e^{-m} \sum_{X=0}^{\infty} \frac{M^{X}}{X!}$$

$$1 = \sum_{X=0}^{\infty} \frac{m^{X}}{X!} e^{-m}$$

$$\Rightarrow \sum_{X=0}^{\infty} f(x) = 1$$

Question: Find the M.g.f, expected value, variance and standard dev. of X.

$X \sim \text{Poisson}(m)$ " X has poisson dist with parameter m "

→ M.g.f :

$$E(e^{tX}) = \sum_{X} e^{tX} f(x) = \sum_{X=0}^{\infty} e^{tX} \frac{m^{X} e^{-m}}{X!}$$

$$= e^{-m} \sum \frac{m^{X} e^{tX}}{X!}$$

$$= e^{-m} \sum \frac{(m\ \ )^{X}}{X!}$$

$$= e^{-M} e^{me^{t}}$$

$$= e^{m(e^{t}-1)}$$

$$\Rightarrow M(t) = e^{M(e^{t}-1)}, \quad t \in \mathbb{R}.$$

$$\Rightarrow M(t) = \exp\left(M(e^{t}-1)\right), \quad t \in \mathbb{R}.$$

→ the expected value $E(e^{tx})$ converges for all $t \in \mathbb{R}$, the expected value $E(e^{tx})$ is defined a round zero.

Hence, the m.g.f exists.

→ Expected value.

$E(x) = \mu = \bar{M}(0)$.

$\rightarrow \bar{M}(t) = M(t)\,[me^{t}]$

$\bar{M}(0) = M(0)\,[me^{0}]$

$= m$

$\Rightarrow E(x) = \mu = m$.

→ Variance :

$E(x^2) = \bar{\bar{M}}(0)$

$\bar{\bar{M}}(t) = M(t)\,(Me^{t}) + (me^{t})\,\bar{M}(t)$

$\bar{\bar{M}}(0) = M(0)\,(Me^{0}) + (me^{0})\,\bar{M}(0)$

$\bar{\bar{M}}(0) = m + m.m$

$\bar{\bar{M}}(0) = m + m^2$

so $Var(x) = \sigma^2 = E(x^2) - (E(x))^2 = m + m^2 - m^2$

$= m$.

$\Rightarrow Var(x) = m$.

→ standard deviation :

$\sigma = \sqrt{var(x)} = \sqrt{m}$

Summary: $X \sim \text{Poisson}(\mu)$. $\mu = m$

1. $f(x) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!} & , x = 0, 1, 2, \dots \\ 0 & , \text{elsewhere} \end{cases}$

where $\mu > 0$.

2. $M(t) = e^{\mu(e^t - 1)}$, $t \in \mathbb{R}$.

3. $E(X) = \mu$

4. $Var(X) = \sigma^2 = \mu$

5. $\sigma = \sqrt{\mu}$

check exp 1, 2, 3.