Exercises 3.1 :

Q.1 Prove by induction that for each natural number n, each of the following is true :-

d) $5+10+15+\cdots+5n = \frac{5n(n+1)}{2}$

• Proof : using first principle of Mathematical induction :

a) The Statment is true for $n=1$ since $5 = \frac{5(1+1)}{2}$ ✓

b) Suppose that the Statment is true for $n=k$ that is

$$5+10+15+\cdots+5k = \frac{5k(k+1)}{2} \quad \cdots (1)$$

⇒ and need to prove that for $n=k+1$ is true, that is

$$5+10+15+\cdots+5k+5(k+1) = \frac{5(k+1)(k+2)}{2} \quad \cdots (2)$$

• Now by adding $5(k+1)$ to both sides of (1) we get

$$5+10+15+\cdots+5k+5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$$

$$= \frac{5k(k+1) + 2\cdot 5(k+1)}{2}$$

$$= \frac{5(k+1)(k+2)}{2}$$ which is (2) above ✓

f) $5+7+9+ \cdots + (2n+3) = n(n+4)$

• Proof: using first principle of Mathematical Induction :-

a) The statment is true for $n=1$ since $5 = 1(1+4)$ ✓

b) Suppose that the statment is true for $n=k$, that is
$$5+7+9+ \cdots + (2k+3) = k(k+4) \quad \cdots (1)$$

and need to prove that it is true for $n=k+1$, that is
$$5+7+9+ \cdots + (2k+3) + (2(k+1)+3) = (k+1)(k+5) \cdots (2)$$

• Now : adding $(2k+5)$ to both sides of (1), we get:

$$5+7+9+ \cdots +(2k+3) + (2k+5) = k(k+4) + (2k+5)$$
$$= k^2 + 4k + 2k + 5$$
$$= k(k+6) + 5$$
$$= k((k+5)+1) + 5$$
$$= k(k+5) + (k+5)$$
$$= (k+5)(k+1) \quad \text{which is (2) above}$$

g) $1(1+1) + 2(2+1) + 3(3+1) + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

• Proof : using first principle of Mathematical Induction :-

a) The statment is true for $n=1$ since $1(1+1) = \frac{1(1+1)(1+2)}{3}$

b) Suppose that the statment is true for $n=k$, that is

$$1(1+1) + 2(2+1) + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \text{--- (1)}$$

⇒ and need to prove that for $n=k+1$ is true, that is

$$1(1+1) + 2(2+1) + \dots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3} \quad \text{--- (2)}$$

• Now : adding $(k+1)(k+2)$ to both sides of (1), we get

$$1(1+1) + 2(2+1) + \dots k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$ which is (2) above

j) $7^n - 2^n$ is divisible by 5 :→ (5 divids $7^n - 2^n$)

• Proof : using first principle of Mathematical Induction:

a) The statment is true for $n=1$ since ~~...~~ $5 | 5$ ✓

b) Suppose that the statment is true for $n=k$, that is suppose $5 | 7^k - 2^k$ and need to prove that it is true for $n = k+1$, that is $5 | 7^{k+1} - 2^{k+1}$:

• Now : $5 | 7^k - 2^k \Rightarrow 7^k - 2^k = 5t \; ; \; t \in \mathbb{Z}$

$$7^{k+1} - 2^{k+1} = 7^k(7) - 2^k(2)$$
$$= 7^k(5+2) - 2^k(2)$$
$$= 7^k(5) + \underline{7^k(2) - 2^k(2)}$$
$$= 7^k(5) + 2(7^k - 2^k)$$
$$= 7^k(5) + 2(5t)$$
$$= 5(7^k + 2t)$$
$$= \boxed{5S} \; , \; S = 7^k + 2t \in \mathbb{Z}$$

true always

9) $\left(1+\frac{1}{2}\right)^n \geq 1+\frac{n}{2}$

• Proof : using first principle of Mathematical Induction :-

a) The statment is true for $n=1$ since $\left(1+\frac{1}{2}\right)^1 \geq 1+\frac{1}{2}$ ✓

b) Suppose that the statment is true for $n=k$, that is

$$\left(1+\frac{1}{2}\right)^k \geq 1+\frac{k}{2} \quad \text{--- } (1)$$

and need to prove that it is true for $n=k+1$, that is

$$\left(1+\frac{1}{2}\right)^{k+1} \geq 1+\frac{k+1}{2} \quad \text{--- } (2)$$

• Now : multiply both sides of (1) by $\left(1+\frac{1}{2}\right)$, we get

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)^k \geq \left(1+\frac{1}{2}\right)\left(1+\frac{k}{2}\right)$$

$$\left(1+\frac{1}{2}\right)^{k+1} \geq 1+\frac{k}{2}+\frac{1}{2}+\frac{k}{4}$$

$$\geq 1+\frac{k+1}{2}+\frac{k}{4}$$

$$\left(1+\frac{1}{2}\right)^{k+1} \geq 1+\frac{k+1}{2} \quad ; \frac{k}{4} > 0$$

which is (2) above

Q.5 Prove that the product of any three Consecutive natural numbers is divisible by 6 :⊃

- Assume three Consecutive natural numbers is $n, n+1, n+2$

$\Rightarrow$ Show that $(n).(n+1).(n+2)$ is divisible by 6 ??

- proof : using first principle of Mathematical of Induction :-

a) The Statment is true for $n=1$ since $(1)(2)(3) = 6$ divisible by 6. ✓

b) Suppose that the statment is true for $n=k$, that is $(k)(k+1)(k+2)$ divisible by 6 $\Rightarrow 6 \mid k(k+1)(k+2)$

and need to prove that it is true for $n=k+1$, that is $6 \mid (k+1)(k+2)(k+3)$.

- Now : givin $6 \mid k(k+1)(k+2) \Rightarrow k(k+1)(k+2) = 6t, t \in \mathbb{Z}$

need to prove that

$(k+1)(k+2)(k+3) = 6s$??

$$\Rightarrow (k+1)(k+2)(k+3) = k(k+1)(k+2) + 3(k+1)(k+2)$$
$$= 6t + 3.2c$$
$$= 6t + 6c$$
$$= 6(t+c) = 6s, \quad s = t+c$$

Q.16 Prove that for each even natural number n,

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{(-1)^n}{n}\right)=\frac{1}{2}$$

• Proof: using first extended principle of Mathematical Induction

a) The Statment is true for $n=2$ since $1-\frac{1}{2}=\frac{1}{2}$ ✓

b) Suppose that the Statment is true for $n=k$, that is

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{(-1)^k}{k}\right)=\frac{1}{2} \quad \text{---} (1)$$

k is even $\Rightarrow \left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{2}$

and need to prove that it is true for $n=k+2$, that is

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right)=\frac{1}{2}$$

• Now:

$$\left(\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)\right)\cdot\left(\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right)\right)= \;??$$

$$\frac{1}{2}\cdot\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right)=$$

see (1)

$$\frac{1}{2}\left(\frac{\cancel{k+2}}{\cancel{k+1}}\right)\left(\frac{\cancel{k+1}}{\cancel{k+2}}\right)=\frac{1}{2}$$ ✓

Q.18 Prove that for each natural number $n \geq 2$,

$$\left(\frac{2^2}{1\times 3}\right)\left(\frac{3^2}{2\times 4}\right)\left(\frac{4^2}{3\times 5}\right)\cdots\left(\frac{n^2}{(n-1)(n+1)}\right) = \frac{2n}{(n+1)} \;:-$$

Proof: using first extended principle of Mathematical Induction :-

a) The statment is true for $n=2$ since $\frac{2^2}{1\times 3} = \frac{2\times 2}{3}$ ✓

b) Suppose that the statment is true for $n=k$, that is

$$\left(\frac{2^2}{1\times 3}\right)\left(\frac{3^2}{2\times 4}\right)\left(\frac{4^2}{3\times 5}\right)\cdots\left(\frac{k^2}{(k-1)(k+1)}\right) = \frac{2k}{(k+1)} \quad --(1)$$

and need to prove that it is true for $n=k+1$, that is

$$\left(\frac{2^2}{1\times 3}\right)\left(\frac{3^2}{2\times 4}\right)\left(\frac{4^2}{3\times 5}\right)\cdots\left(\frac{k^2}{(k-1)(k+1)}\right)\left(\frac{(k+1)^2}{k(k+2)}\right) = \frac{2(k+1)}{k+2} \quad --(2)$$

• Now! $\underbrace{\left(\frac{2^2}{1\times 3}\right)\left(\frac{3^2}{2\times 4}\right)\left(\frac{4^2}{3\times 5}\right)\cdots\left(\frac{k^2}{(k-1)(k+1)}\right)}_{\Downarrow \text{ see (1)}}\left(\frac{(k+1)^2}{k(k+2)}\right) =$ ~~2k~~

$$\frac{2\cancel{k}}{\cancel{(k+1)}} \cdot \left(\frac{(k+1)^{\cancel{2}}}{\cancel{k}(k+2)}\right) = \frac{2(k+1)}{k+2}$$

which is (2) above ✓

Q.19 Prove that for each odd natural number $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^n}{n}\right) = 1 \quad :-$$

• Proof : using first extended principle of Mathematical Induction.

a) The statement is true for $n=3$ since $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = \frac{3}{2}\times\frac{2}{3} = 1$ ✓

b) suppose that the statement is true for $n=k$, that is

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^k}{k}\right) = 1$$

Since k is odd $\Rightarrow \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right) = 1$ --- (1)

and need to prove that it is true for $n=k+2$, that is

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) = 1 \quad \text{--- (2)}$$

• Now : $\underbrace{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1-\frac{1}{k}\right)}_{\Downarrow}\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) = \ ??$

$$(1)\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) =$$

$$\left(\frac{k+2}{k+1}\right)\left(\frac{k+1}{k+2}\right) = 1 \ ✓$$

Q.20 Prove that for each natural number $n \geq 6$, $(n+1)^2 \leq 2^n$

• **Proof**: using first extended principle of Mathematical Induction

a) The statment is true for $x = 6$ since $(6+1)^2 \leq 2^6 \Rightarrow 49 \leq 64$ ✓

b) Suppose the statment is true for $n = k$, that is

$(k+1)^2 \leq 2^k$ ...(1), and need to prove that it is true for

$n = k+1$, that is $(k+2)^2 \leq 2^{k+1}$ ... (2)

• **Now**: $(k+1)^2 = k^2 + 2k + 1$, $(k+2)^2 = k^2 + 4k + 4$

$$k^2 + 2k + 1 \leq 2^k \quad \text{...(1)}$$

adding $2k+3$ for both sides of (1) :-

$$k^2 + 2k + 1 + (2k+1) \leq 2^k + 2k + 3$$

$$k^2 + 4k + 2 \leq 2^k + 2k + 3 \quad , \left(2k+3 \leq 2^k \;\; \forall k \geq 6\right)$$

$$(k+2)^2 \leq 2^k + 2^k$$

$$(k+2)^2 \leq 2 \cdot 2^k$$

$$(k+2)^2 \leq 2^{k+1}$$ which is (2) above ✓

$\boxed{Q.22}$ Prove that for each natural number $n \geq 2$,

$$2^{n+1} \leq 3^n$$

Proof: using first extended principle of Mathematical Induction

a) The statment is true for $n=2$ since $2^3 \leq 3^2$ ✓

b) Suppose that the statment is true for $n=k$, that is

$$2^{k+1} \leq 3^k \quad \text{--(1)}$$

and need to prove that it is true for $n=k+1$, that is

$$2^{k+2} \leq 3^{k+1} \quad \text{--(2)}$$

Now:

$$2^{k+1} \leq 3^k \quad \text{— from (1)}$$

$$2 \leq 3$$

$$2 \times 2^{k+1} \leq 3 \times 3^k$$

$$\boxed{2^{k+2} \leq 3^{k+1}}$$ ✓ which is (2) above