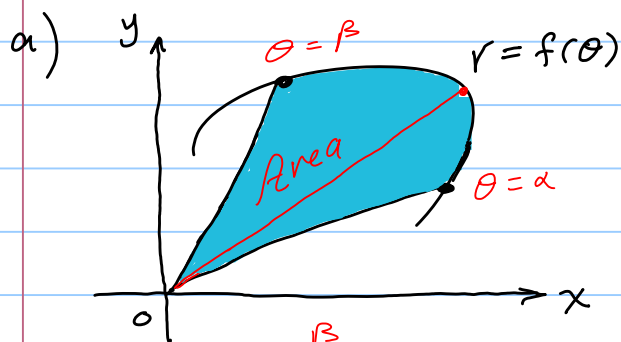


11.5 Areas and Lengths in Polar Coordinates

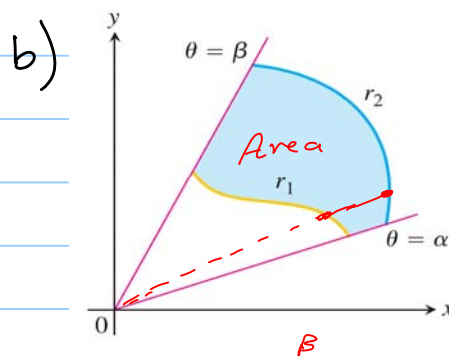
Note Title

33/07/18

Area in the Plane



$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$



$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} (r_2^2 - r_1^2) d\theta$$

- a) **Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta.$$

This is the integral of the **area differential** (Figure 11.31)

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} (f(\theta))^2 d\theta.$$

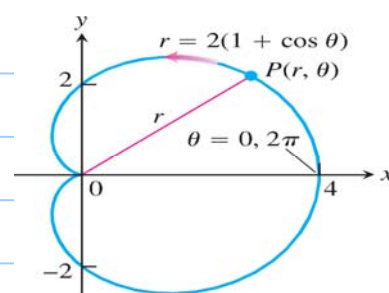
- b) **Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta$$

Examples:

EXAMPLE 1 Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos \theta)$.

Sol: $A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 d\theta$



$$= \frac{1}{2} \cdot 4 \int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta \, d\theta = 2 \int_0^{2\pi} 1 + 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= 2 \left[\theta + 2\sin\theta + \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^{2\pi} = \boxed{6\pi}$$

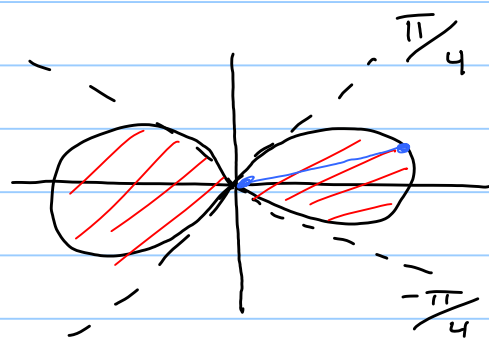
2) Find the area of the region inside
 $r^2 = 4 \cos 2\theta$

sol: (sec 11.4 ex 1,) $\Rightarrow$ $r^2 = 4 \cos 2\theta$

$$A = 2 \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} 4 \cos 2\theta \, d\theta$$

$$= 4 \left[\frac{\sin 2\theta}{2} \right]_{-\pi/4}^{\pi/4} = 2 \left[\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right] = \boxed{4}$$



3) Find the area of:

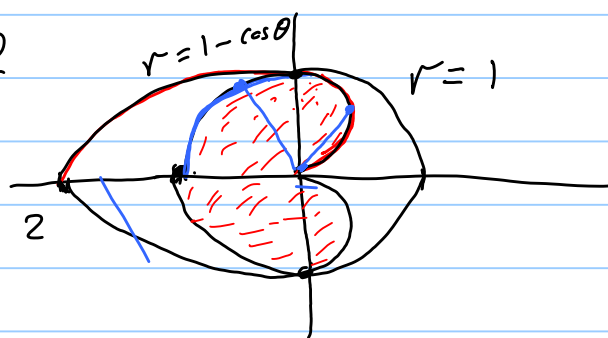
a) the region of intersection of the circle

$r = 1$ and the cardioid $r = 1 - \cos\theta$

sol: Firstly, Find the points of intersection

$$1 = 1 - \cos\theta \Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = -\frac{\pi}{2}, \frac{\pi}{2}$$

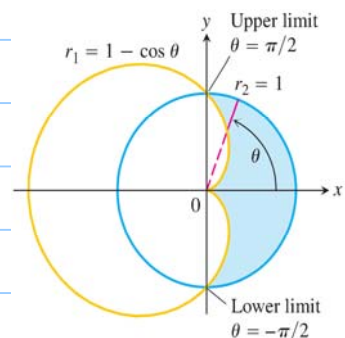


تحليل المسألة: ① لوجود التماثل المحاكى / يمكنه إيجاد مساحة من الأعلى فقط وخرجها من 2
 ② المساحة من الأعلى ليست بيده متخيلة إنما تنقسم إلى جزئين:
 (الجزء الأول: المساحة من الربع الأول وهي محصورة بين $r=0$ و $r=1-\cos\theta$ من $\theta=0$ إلى $\theta=\pi/2$ -
 (الجزء الثاني: المساحة من الربع الثاني وهي محصورة بين $r=0$ و $r=1$ من $\theta=\pi/2$ إلى $\theta=\pi$ -

$$\begin{aligned} \therefore \frac{1}{2} A &= \int_0^{\pi/2} \frac{1}{2} (1-\cos\theta)^2 d\theta + \int_{\pi/2}^{\pi} \frac{1}{2} \cdot 1^2 d\theta = \frac{\pi}{4} \\ &= \frac{1}{2} \int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta d\theta + \frac{\pi}{4} \\ &= \frac{1}{2} \int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2} (1 + \cos 2\theta) d\theta + \frac{\pi}{4} \\ &\quad \text{الآن الحل:} \end{aligned}$$

b) the region inside the circle $r=1$ and outside the cardioid $r=1-\cos\theta$

أيضاً يستخدم التماثل يمكنه إيجاد مساحة من الربع الأول:
 (الأول وخرجها من 2.
 واضح أنه $r_2=1$ أي $r=1$ من الفترة $\theta=0$ إلى $\theta=\pi/2$
 $r_1=1-\cos\theta$



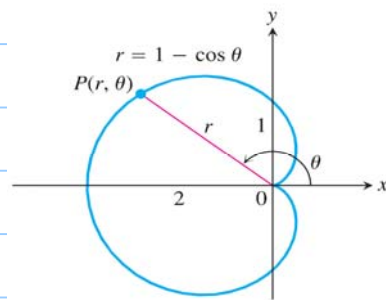
(انظر الشكل (لواصل من المنطقة إلى 0)

$$\begin{aligned} \therefore A &= 2 \int_0^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) d\theta && \text{Symmetry} \\ &= \int_0^{\pi/2} (1 - (1 - 2 \cos \theta + \cos^2 \theta)) d\theta && \text{Square } r_1. \\ &= \int_0^{\pi/2} (2 \cos \theta - \cos^2 \theta) d\theta = \int_0^{\pi/2} \left(2 \cos \theta - \frac{1 + \cos 2\theta}{2} \right) d\theta \\ &= \left[2 \sin \theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$

Length of a Polar Curve

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \quad (3)$$

Examples:



sol: $r = 1 - \cos \theta$, $\frac{dr}{d\theta} = \sin \theta$

$$\begin{aligned}\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + \sin^2\theta \\ &= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta \\ &= 4\sin^2\left(\frac{\theta}{2}\right) \quad (\text{how??})\end{aligned}$$

دایره: نه حول (مختص) مه $\theta = 0$ ای $\theta = \pi$ هو $\frac{1}{2}$ اطوار (مختص)

$$\Rightarrow L = 2 \int_0^{\pi} \sqrt{2 - 2 \cos \theta} \, d\theta = 2 \int_0^{\pi} \sqrt{4 \sin^2 \frac{\theta}{2}} \, d\theta$$

$$= 4 \int_0^{\pi} |\sin \frac{\theta}{2}| d\theta = 4 \int_0^{\pi} \sin \frac{\theta}{2} d\theta \quad (\sin \frac{\theta}{2} > 0)$$

$$= 4 \cdot (-2 \cos \frac{\theta}{2}) \Big|_0^{\pi} = -8 \left(\cos \frac{\pi}{2} - \cos 0 \right) = \boxed{8}$$

b) $r = \cos^3 \frac{\theta}{3}, \quad 0 \leq \theta \leq \frac{\pi}{4}$

sol: $\frac{dr}{d\theta} = 3 \cos^2 \frac{\theta}{3} \cdot -\sin \frac{\theta}{3} \cdot \frac{1}{3} = -\sin \frac{\theta}{3} \cdot \cos^2 \frac{\theta}{3}$

$$\therefore r^2 + \left(\frac{dr}{d\theta} \right)^2 = \cos^6 \frac{\theta}{3} + \sin^2 \frac{\theta}{3} \cos^4 \frac{\theta}{3} = \cos^4 \frac{\theta}{3} \left(\cos^2 \frac{\theta}{3} + \sin^2 \frac{\theta}{3} \right)$$

$$= \cos^4 \frac{\theta}{3}$$

$$\therefore L = \int_0^{\frac{\pi}{4}} \sqrt{\cos^4 \frac{\theta}{3}} d\theta = \int_0^{\frac{\pi}{4}} |\cos^2 \frac{\theta}{3}| d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 + \cos \frac{2\theta}{3} d\theta = \frac{1}{2} \left(\theta + \frac{3}{2} \sin \frac{2\theta}{3} \right) \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{4} + \frac{3}{2} \cdot \frac{1}{2} \right) - \left(0 + \frac{3}{2} \cdot 0 \right) \right] = \boxed{\frac{\pi}{8} + \frac{3}{8}}$$

مثال

Example: Find the length of the curve

$$r = \sqrt{1 + \sin 2\theta}, \quad 0 \leq \theta \leq \sqrt{2}\pi$$

sol: $\frac{dr}{d\theta} = \frac{\cos 2\theta}{\sqrt{1 + \sin 2\theta}} \Rightarrow$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + \sin 2\theta) + \frac{\cos^2 2\theta}{1 + \sin 2\theta}$$

$$= \frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1 + \sin 2\theta}$$

$$= \frac{2 + 2\sin 2\theta}{1 + \sin 2\theta} = 2$$

$$\therefore L = \int_0^{\sqrt{2}\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{2} d\theta = \boxed{2\pi}$$