* Impulse Response for rational system function

$$x[n] \longrightarrow \overset{h[n]}{\boxed{LTI}} \longrightarrow y[n]$$

DFE

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \text{if } M \geq N$$

$$= \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k (d_k)^n u[n]$$

Finite Impulse Response (FIR)

- $H(z)$ has no poles except at $z = 0$

$$H(z) = \sum_{k=0}^{M} b_k \delta[n-k]$$

$$= H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

Infinite impulse Response (IIR)

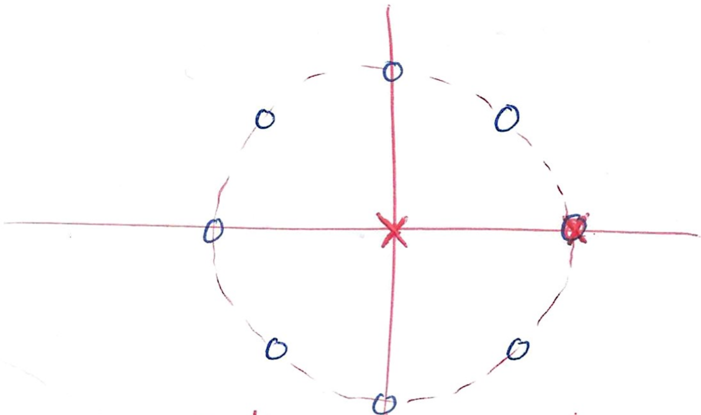
- has at least one pole is not canceld with zero
- feed back

Example:-

$$h[n] = \begin{cases} a^n & 0 \le n \le M \\ 0 & o.w \end{cases}$$

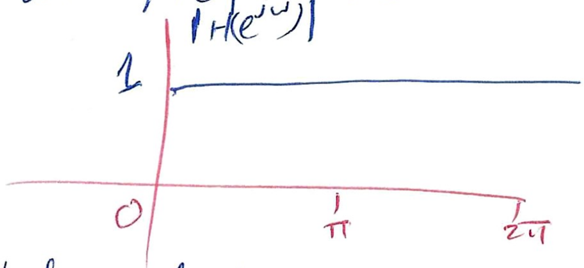
$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - a z^{-1}}$$

$$z_k = a\, e^{j2\pi k/(M+1)} \qquad k = 0, 1, 2, \dots M$$



Types of system
- ① linear phase systems
- ② non linear phase system
- ③ minimum phase systems
- ④ all pass systems

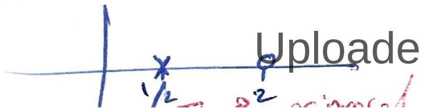
* all pass systems:- pass all frequencies



$$|H_{ap}(e^{jw})| = 1$$

has no effect on the magnitude but has a huge impact on the phase

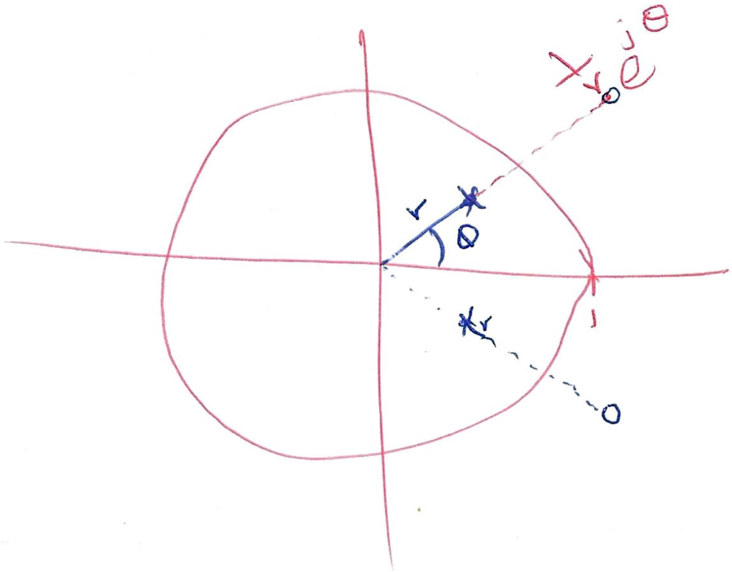
all pass systems:- all poles and zeros are in Conjugate reciprocal



Pole : $c_i = r e^{j\theta}$

$c_i^* = r e^{-j\theta}$

$\frac{1}{c_i^*} = \frac{1}{r} e^{j\theta}$



$$H_{ap}(z) = \prod_{i}^{p} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}}$$

To show $|H_{ap}(e^{jw})| = 1$, consider $p=1$

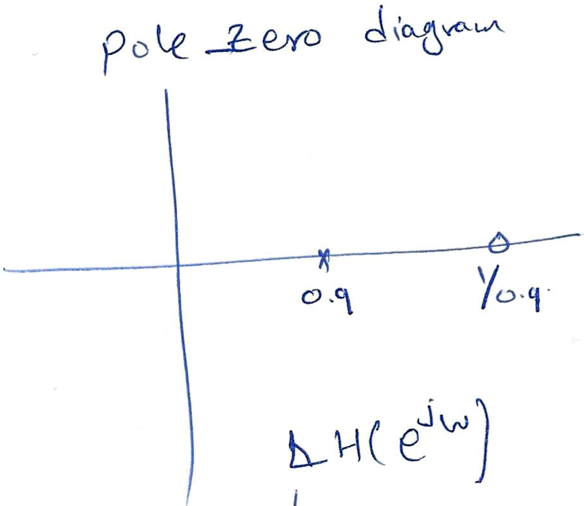
$$|H_{ap}(e^{jw})| = \left| \frac{e^{-jw} - c^*}{1 - c e^{-jw}} \right|$$

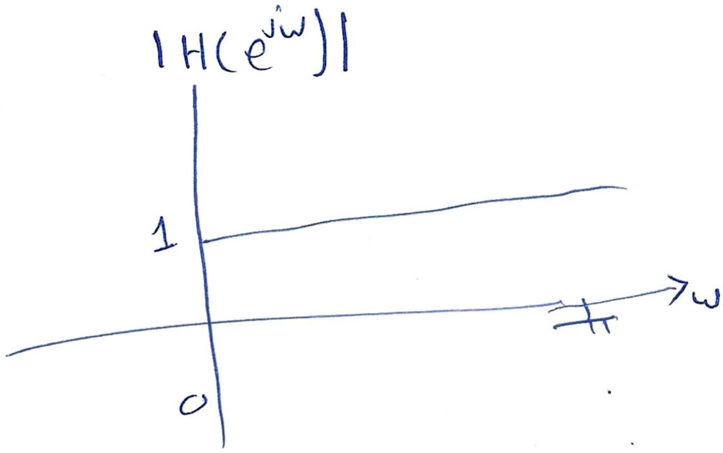
$$= \left| \frac{e^{-jw}(1 - c^* e^{jw})}{1 - c e^{-jw}} \right|$$

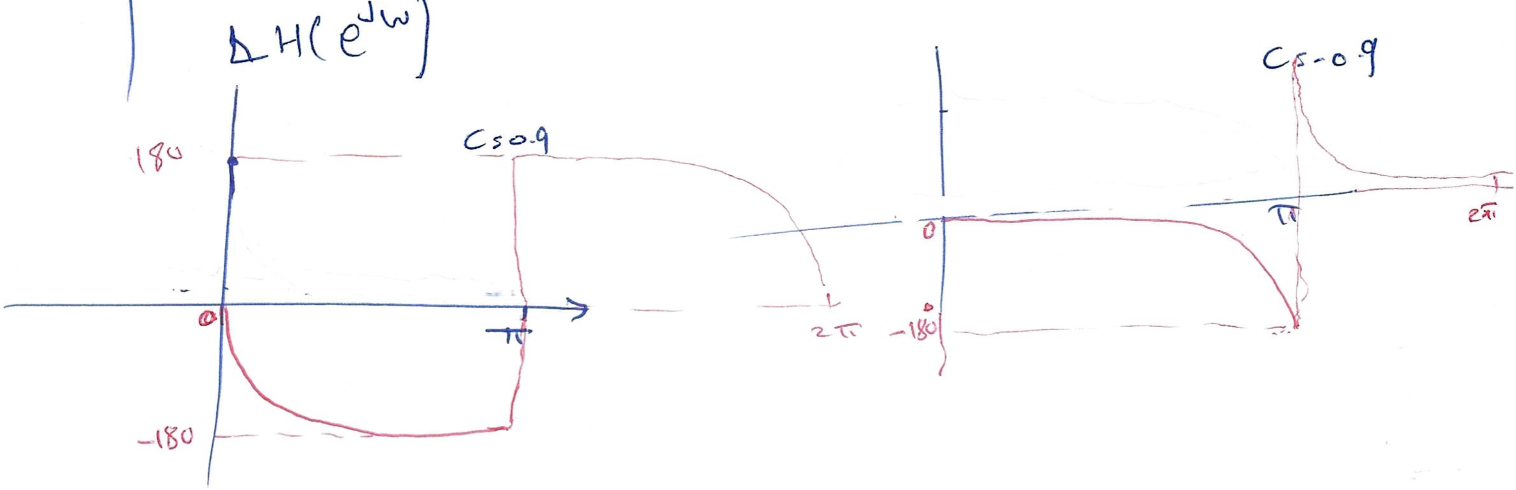
given that $|e^{-jw}| = 1$

$$= \left| \frac{1 - c^* e^{jw}}{1 - c e^{-jw}} \right| = \frac{|b^*|}{|b|} = 1$$

Example: all pass system $H(z) = \frac{z^{-1} - c^*}{1 - cz^{-1}}$, $c = 0.9$







* Minimum phase and all pass decomposition

any arbitrary system can be factorized as a cascade of minimum-phase and all-pass sub systems

H(z) should be written as follow

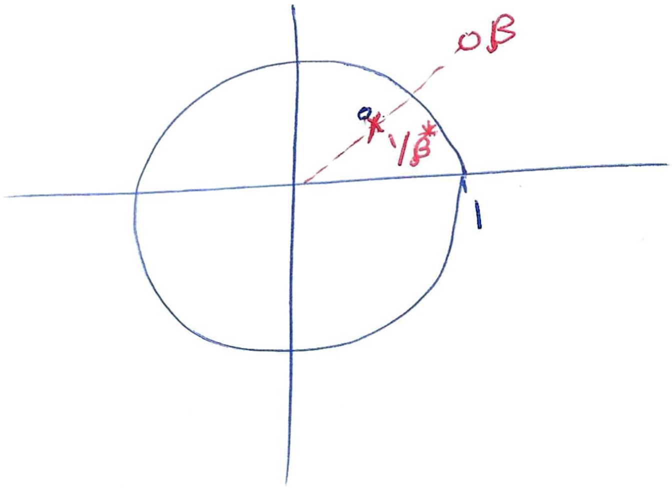
$$H(z) = H_{min}(z) \cdot H_{ap}(z)$$

1. Take zeros that lie outside $|z| = 1$ (unit circle) and Move to $H_{ap}(z)$
2. add poles to $H_{ap}(z)$ in Conjugate reciprocal location of zero
3. put zero $H_{min}(z)$ to Cancel poles added to $H_{ap}(z)$

Example 8- suppose $H(z) = H_1(z)\,(1 - \beta z^{-1})$; $|\beta| > 1$

given that $H_1(z)$ is a minimum phase system

$$\underbrace{H_1(z)}_{min}\,(1 - \beta z^{-1})$$



(1) add pole at $\frac{1}{\beta^*}$

$$\underbrace{H_1(z)}_{min}\;\underbrace{\frac{(1 - \beta z^{-1})}{1 - \frac{1}{\beta^*} z^{-1}}}_{all\ pass}$$

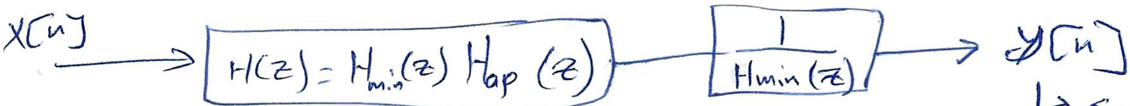
(2) add zero at the same location (To keep the function as it is without any chang in the magnitude or the phase)

$$\underbrace{H_1(z)\left(1 - \frac{1}{\beta^*} z^{-1}\right)}_{min}\;\underbrace{\frac{(1 - \beta z^{-1})}{1 - \frac{1}{\beta^*} z^{-1}}}_{all\ pass}$$

(3) rewrite all pass port as the general formula $\frac{z^{-1} - c^*}{1 - c z^{-1}}$

$$\underbrace{H_1(z)\left(1 - \frac{1}{\beta^*} z^{-1}\right)}_{min} - \beta\;\underbrace{\frac{(z^{-1} - 1/\beta)}{\left(1 - \frac{1}{\beta^*} z^{-1}\right)}}_{all\ pass}$$

later if I want to cancel the phase effect or the magnitud effect, I can add a cascade Inverse system



↳ same as x[n] but with different phase

Example :- decompose H(z) into min-phase and all pass

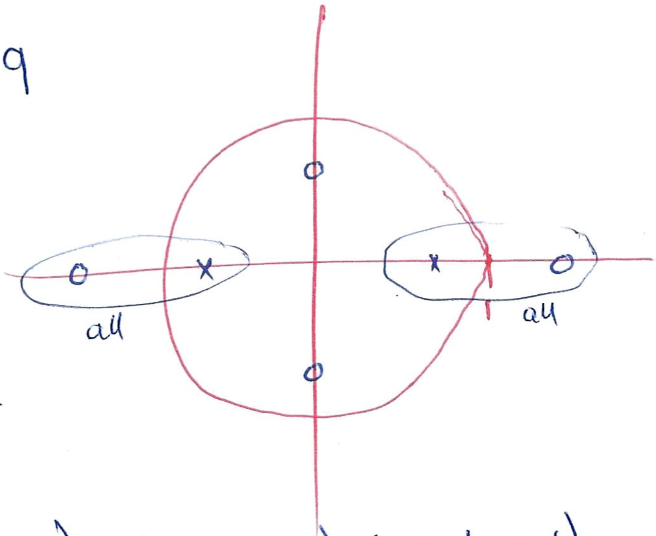
$$H(z) = \left(1 - \frac{1}{0.9}z^{-1}\right)\left(1 + \frac{1}{0.9}z^{-1}\right)\left(1 - j0.7z^{-1}\right)\left(1 + j0.7z^{-1}\right.$$

1) add poles at $\frac{1}{(1/0.9)^*} = 0.9$

2) add zero at the same point

add poles at $+\frac{1}{(\frac{1}{-0.9})^*} = -0.9$

add zero at the same point



$$H(z) = \underbrace{\left(1 - j0.7z^{-1}\right)\left(1 + j0.7z^{-1}\right)}_{\text{minimum phase}}\left(1 - \frac{1}{0.9}z^{-1}\right)\left(1 + \frac{1}{0.9}z^{-1}\right)$$

zero at 0.9 zero at -0.9

$$H(z) = \left(1 - j0.7z^{-1}\right)\left(1 + j0.7z^{-1}\right)\left(1 - \frac{1}{0.9}z^{-1}\right)\left(1 + \frac{1}{0.9}z^{-1}\right)\left(\frac{1-0.9z^{-1}}{1-0.9z^{-1}}\right)\left(\frac{1+0.9z^{-1}}{1+0.9z^{-1}}\right)$$

Pole at 0.9 pole at -0.9

$$H(z) = \underbrace{\left(1 - j0.7z^{-1}\right)\left(1 + j0.7z^{-1}\right)\left(1-0.9z^{-1}\right)\left(1+0.9z^{-1}\right)}_{\text{minimum phase part}}\left(\frac{1 - \frac{1}{0.9}z^{-1}}{1-0.9z^{-1}}\right)\left(\frac{1 + \frac{1}{0.9}z^{-1}}{1+0.9z^{-1}}\right)$$

3) rewrite all pass part as the general formula $\frac{z^{-1} - c^*}{1 - cz^{-1}}$

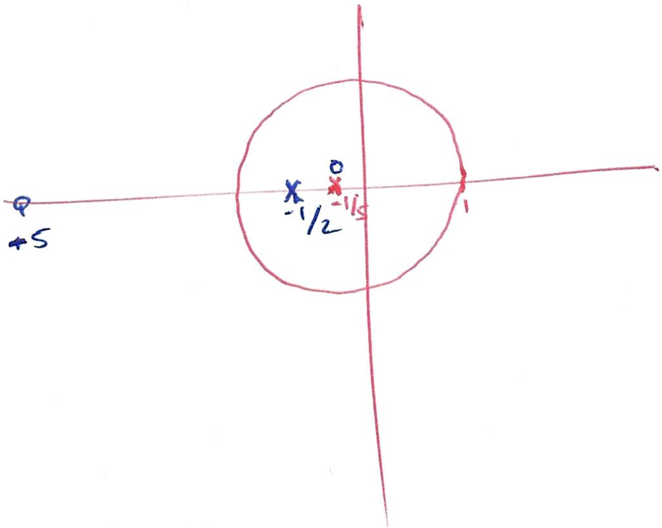
$$\frac{1 - \frac{1}{0.9}z^{-1}}{1 - 0.9z^{-1}} \Longrightarrow \frac{\left(z^{-1} - 0.9\right) - 1/0.9}{1 - 0.9z^{-1}}$$

the new H(z) could be written as:-

$$H(z) = \underbrace{\left(\frac{-1}{0.9}\right)\left(\frac{1}{0.9}\right)\left(1 - j0.7z^{-1}\right)\left(1 + j0.7z^{-1}\right)\left(1-0.9z^{-1}\right)\left(1+0.9z^{-1}\right)}_{\text{minimum phase part}}\underbrace{\left(\frac{z^{-1}-0.9}{1-0.9z^{-1}}\right)\left(\frac{z^{-1}+0.9}{1+0.9z^{-1}}\right)}_{\text{all pass part}}$$

Example: $H(z) = \dfrac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}}$ decompose $H(z)$ into H_{min} and H_{ap} ??

$$H(z) = \frac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}}$$



1) add a pole at $z = -1/5$

2) add a zero at the same point $-1/5$

3) rewrite all pass part as the general formula $\dfrac{z^{-1}-c^*}{1-cz^{-1}}$

1) + 2) : $H(z) = \dfrac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}} \; \dfrac{(1+\frac{1}{5}z^{-1})}{(1+\frac{1}{5}z^{-1})}$

$$= \underbrace{\frac{1+5z^{-1}}{(1+\frac{1}{5}z^{-1})}}_{\text{all pass part}} \; \underbrace{\frac{1+\frac{1}{5}z^{-1}}{1+\frac{1}{2}z^{-1}}}_{\text{minimum phase part}}$$

c

but not in the general form $\dfrac{z^{-1}-c^*}{1-cz^{-1}}$

$c = 1/5 \implies c^* = \frac{1}{5}$

$$\frac{1+5z^{-1}}{1+\frac{1}{5}z^{-1}} \implies 5\left(\frac{z^{-1}+1/5}{1+1/5\,z^{-1}}\right)$$

$\therefore H(z) = \left(\dfrac{z^{-1}+1/5}{1+1/5\,z^{-1}}\right)\left(5 \cdot \dfrac{1+1/5\,z^{-1}}{1+1/2\,z^{-1}}\right)$