Birzeit University

Mathematics Department

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Course Code: MATH1321

Title: Calculus II

CH10 Infinite sequences and series

10.1 Sequences.

- A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order.

Each of $a_1, a_2, a_3, \ldots$ represents a number

these are the terms of the sequence

ex. $2, 4, 6, 8, \ldots$

This sequence has first term $a_1 = 2$,

second term $a_2 = 4$, and nth term

$$\boxed{a_n = 2n, \ n \geq 1}.$$

- the integer n is called the index of a_n.

- the order is important, for example

the sequence $2, 4, 6, 8, \ldots$ is NOT

the same as the sequence $4, 2, 6, 8, \ldots$

- the infinite sequence $a_1, a_2, a_3, \dots$ is a function whose domain is the set of positive integers.

Ex. Find a formula of the sequence

$$12, 14, 16, 18, \dots$$

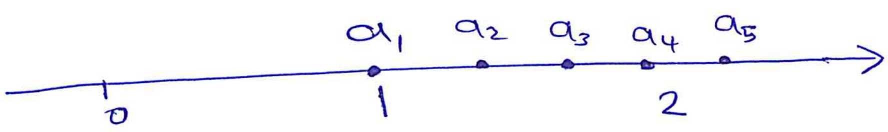
Solution. $a_n = 10 + 2n, \quad n \geq 1$.

Another formula $b_n = 2n, \quad n \geq 6$.

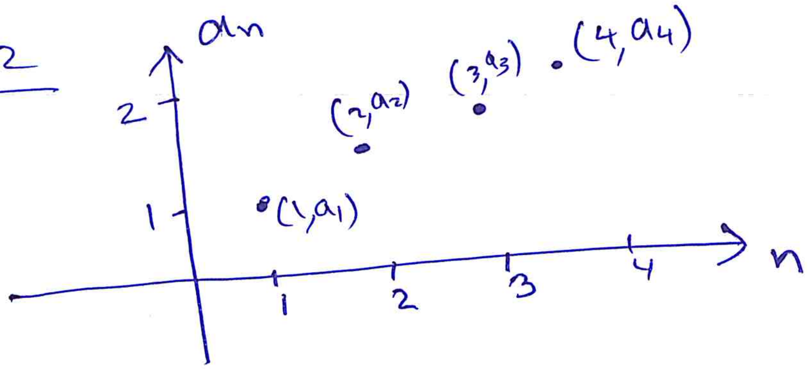
Ex. Graph the sequence

$$\{1, \sqrt{2}, \sqrt{3}, 2, \dots, \sqrt{n}, \dots\}$$

Solution. Method 1.



Method 2



Convergence and divergence

- If $\lim_{n\to\infty} a_n = L$ exists, then we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number L. If no such number L exists, we say that $\{a_n\}_{n=1}^{\infty}$ diverges.

Ex. Which of the sequence $\{a_n\}$ converge and which diverge? Find the limit of each convergent sequence.

① $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \{b_n\}$ converges to zero.

② $a_n = \sqrt{n}$, $n \geq 1$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \infty$ diverges.

③ $c_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, \ldots\}$

$\lim_{n\to\infty} c_n$ DNE (diverges).

4) $a_n = k$, k constant.

$\lim_{n\to\infty} k = k$. converges

5) $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ---, \frac{n-1}{n}, ...\}$

$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} (1-\frac{1}{n}) = 1$

the seq. converges to 1.

Limits Rules

Thm let $\{a_n\}$, $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. If $\boxed{\lim_{n\to\infty} a_n = A}$, and $\boxed{\lim_{n\to\infty} b_n = B}$. Then the following rules hold:

1) Sum Rule $\lim_{n\to\infty} (a_n + b_n) = A + B$

2) Difference Rule $\lim_{n\to\infty} (a_n - b_n) = A - B$

3) Constant. Multiple Rule

$\lim_{n\to\infty} (k \cdot b_n) = k \lim_{n\to\infty} b_n = kB$ (any number k)

4) <u>Product Rule</u> $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

5) <u>Quotient Rule</u> $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, B \neq 0$

<u>Ex.</u> (a) $\lim_{n\to\infty} \frac{2}{n} = 2 \lim_{n\to\infty} \frac{1}{n} = 2 \cdot 0 = 0.$

(b) $\lim_{n\to\infty} \left(\frac{n-4}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{4}{n}\right)$

$= \lim_{n\to\infty} 1 - 4 \lim \frac{1}{n}$

$= 1 - 4 \cdot 0 = 1.$

(c) $\lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n}$

$= 5 \cdot 0 \cdot 0 = 0.$

(d) $\lim_{n\to\infty} \frac{4 - 7n^6}{n^6 + 3}$

$= \lim_{n\to\infty} \frac{\left(\frac{4}{n^6}\right) - 7}{1 + \left(\frac{3}{n^6}\right)} = \frac{0 - 7}{1 + 0} = -7.$

<u>Rmk.</u> The last thm does not say, for example, that each of the sequences $\{a_n\}$ and $\{b_n\}$ have limits if their sum

$\{a_n + b_n\}$ has a limit.

<u>example</u> $\{a_n\} = \{1, 2, 3, \ldots\}$

$\{b_n\} = \{-1, -2, -3, \ldots\}$

both diverges but their sum

$\{a_n + b_n\} = \{0, 0, 0, \ldots\}$ converges to 0

<u>Thm</u> (<u>the Sandwich theorem for sequence</u>)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $\boxed{a_n \le b_n \le c_n \text{ holds}}$ for all n beyond some index N, and if $\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L}$, then

$\lim_{n\to\infty} b_n = L$ also.

<u>Corollary</u>. If $|b_n| \le c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \le b_n \le c_n$.

<u>Examples</u>. (a) $\frac{\cos n}{n} \longrightarrow 0$ because

$\underset{\to 0}{-\frac{1}{n}} \le \frac{\cos n}{n} \le \frac{1}{n} \to 0$

(b) $\frac{(-1)^n}{n^2} \to 0$ because $-\frac{1}{n^2} \le \frac{(-1)^n}{n^2} \le \frac{1}{n^2}$

(c) $\frac{1}{2^n} \to 0$ Since $0 \le \frac{1}{2^n} \le \frac{1}{n}$.

Thm (The Continuous function thm for sequences).

Let $\{a_n\}$ be a sequence of real numbers.

If $\lim_{n\to\infty} a_n = L$ and if f is a continuous function at L and defined at all a_n, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$.

ex. show that $e^{\frac{1}{n}} \to 1$.

proof: We know that $\frac{1}{n} \to 0$. Taking $f(x) = e^x$ and $L = 0$ in the last thm.

$e^{\frac{1}{n}} = f(a_n) = f(\frac{1}{n}) \to f(L) = f(0) = e^0 = 1$

Therefore, the sequence $\{e^{\frac{1}{n}}\}$ Converges to 1.

Using L'Hôpital's Rule

thm. Suppose that f is a function defined for all $x \geq n_0$ and $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$, $n \geq n_0$.

Then $\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$.

ex. Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Solution. let $f(x) = \frac{\ln x}{x}$, $x \geq 1$.

$\therefore \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$

ex. Does the sequence $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim_{n\to\infty} a_n$

Sol. $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n$ $\quad (1)^\infty$.

$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n \ln\left(\frac{n+1}{n-1}\right)$

Then, $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right)$ $\quad (\infty \cdot 0)$

$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}$ $\quad \left(\frac{0}{0}\right)$

$$= \lim_{n\to\infty} \frac{\left(\frac{n-1}{n+1}\right)\left(\frac{(n-1)-(n+1)}{(n-1)^2}\right)}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{-2}{n^2-1}}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

Since $\lim_{n\to\infty} \ln a_n = 2$, then $\lim_{n\to\infty} a_n = e^2$

$\therefore$ The sequence $\{a_n\}$ converges to e^2.

Commonly occurring limits

1) $\lim_{n\to\infty} \frac{\ln n}{n}$

2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

3) $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x>0)$

4) $\lim_{n\to\infty} x^n = 0 \quad (|x|<1)$.

5) $\lim \left(1+\frac{x}{n}\right)^n = e^x \quad$ (any x).

6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad$ (any x).

(10)

ex.

a) $\lim_{n\to\infty} \frac{\ln n^{2020}}{n} = 2020 \lim_{n\to\infty} \frac{\ln n}{n}$

$= (2020)(0) = 0.$

b) $\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^3$

$= \left(\lim_{n\to\infty} \sqrt[n]{n}\right)^3 = (1)^3 = 1.$

c) $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} 3^{\frac{1}{n}} \cdot n^{\frac{1}{n}}$

$= \lim_{n\to\infty} 3^{\frac{1}{n}} \lim_{n\to\infty} n^{\frac{1}{n}}$

$= 1 \cdot 1 = 1$

d) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0.$

e) $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2}$

f) $\lim_{n\to\infty} \frac{n!}{2^n \cdot 3^n} = \lim_{n\to\infty} \frac{n!}{6^n} = \infty.$

g) $a_n = \left(1 - \frac{1}{n^2}\right)^n = \left(1 - \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n \to e^{-1} \cdot e^{1} = 1$

Recursive Definitions

Some sequences are often defined recursively by giving

1) the values of the initial term or terms, and
2) A rule, called a recursion formula, for calculating any later term from terms that precede it.

Ex. consider the sequence $\{a_n\}$, where

$$a_1 = 1, \quad a_n = a_{n-1} + 1, \ n > 1$$

Write out the first fourth terms,

Sol. $a_1 = 1$ (given)

$\boxed{n=2}$ $a_2 = a_1 + 1 = 1+1 = 2$

$n=3$: $a_3 = a_2 + 1 = 2+1 = 3$

$n=4$: $a_4 = a_3 + 1 = 3+1 = 4$.

$\therefore \{a_n\} = \{1, 2, 3, 4, ----$

$a_n = n, \quad n \geq 1.$

ex. $a_1 = 1$, $a_n = n \cdot a_{n-1}$, $n > 1$ define the sequence $1, 2, 6, 24, \ldots, n!, \ldots$

Sol. $a_1 = 1$, $a_2 = 2a_1 = 2(1) = 2!$

$a_3 = 3a_2 = 3 \cdot 2! = 3!$

$\vdots$

$a_n = n!$

ex. $a_1 = 1$, $a_2 = 1$, $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ define the sequence $\{1, 1, 2, 3, 5, \ldots$ of Fibonacci numbers.

Sol. $a_1 = a_2 = 1$ given.

$n=2$: $a_3 = a_2 + a_1 = 1+1 = 2$

$n=3$: $a_4 = a_3 + a_2 = 2+1 = 3$.

$n=4$: $a_5 = a_4 + a_3 = 3+2 = 5$

$\vdots$

and so on.

Q92) Assume that the sequence

$$a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$$ converges,

find its limit.

Solution. Take the limit of both sides:

$$\lim_{n\to\infty} a_{n+1} = \frac{\lim_{n\to\infty} a_n + 6}{\lim_{n\to\infty} a_n + 2}$$

$$L = \frac{L+6}{L+2}$$

$$\Rightarrow \frac{L+6}{L+2} - L = 0$$

$$\Rightarrow \frac{L+6-L^2-2L}{L+2} = 0$$

$$\Rightarrow L^2 + L - 6 = 0$$

$$\Rightarrow (L-2)(L+3) = 0$$

$$\Rightarrow \boxed{L = 2} \checkmark \text{ or } \boxed{L = -3} \text{ reject}$$

$\Rightarrow a_n \to 2$

Since $a_2 = \frac{a_1+6}{a_1+2} = \frac{-1+6}{-1+2} = 5$

$a_3 = \frac{a_2+6}{a_2+2} = \frac{5+6}{5+2} = \frac{11}{7}$

$\{a_n\} = \{-1, 5, \frac{11}{7}, \dots\}$

Bounded monotonic Sequences

Df: (i) A sequence $\{a_n\}$ is bounded from above if there exists a number M such that

$a_n \leq M$, for all n.

The number M is an upper bound for $\{a_n\}$. If M is an upperbound for $\{a_n\}$ but no number less than M, then M is the least upperbound.

(ii) A sequence $\{a_n\}$ is bounded from below if there exists a number m such that $a_n \geq m$ for all n. the number m is a lower bound for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the greatest lower bound.

(iii) If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is bounded. If $\{a_n\}$ is not bounded, then we say that

$\{a_n\}$ is unbounded sequence.

Ex. (a) the sequence $1, 2, 3, \ldots, n, \ldots$ is bounded below by every real number less than or equal 1. The number $m = 1$ is the greatest lower bound.

Notice that this seq. has no upper bound.

(b) the sequence $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by every real number $M \geq 1$. the upper bound $M = 1$ is the least upper bound.

the seq. is also bounded below by every real number $m \leq \frac{1}{2}$. The lower bound $m = \frac{1}{2}$ is the greatest lower bounded.

thus this sequence is bounded.

Rmk: All convergent sequences are bounded but the converse need not be true.

example $\{(-1)^{n+1}\} = \{1, -1, 1, -1, ---$

is bounded but diverges.

Df: (i) A sequence $\{a_n\}$ is nondecreasing if $a_n \leq a_{n+1}$, for all n. that is,

$a_1 \leq a_2 \leq a_3 \leq -----$.

(ii) the sequence is nonincreasing if $a_n \geq a_{n+1}$ for all n, that is,

$a_1 \geq a_2 \geq a_3 \geq -----$

(iii) the sequence $\{a_n\}$ is monotonic if it is either nondecreasing or nonincreasing.

Ex. (a) the seq. $1, 2, 3, ---, n, ----$ is nondecreasing. since $a_n = n < n+1 = a_{n+1}$

(b) the seq. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ---, \frac{n}{n+1}, ---$ is nondecreasing

Since, if $f(x) = \frac{x}{x+1}$, $x \geq 1$, then

$$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

$\Rightarrow f$ is nondecreasing

$\Rightarrow a_n = \frac{n}{n+1}$ is nondecreasing

(c) The seq. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$

is nonincreasing since

$$a_{n+1} = \frac{1}{2^{n+1}} \leq \frac{1}{2^n} = a_n.$$

(d) the constant sequence $3, 3, 3, \dots, 3, \dots$

is both nonincreasing and nondecreasing.

(e) the sequence $1, -1, 1, -1, \dots$ is not monotonic.

<u>Thm.</u> (the monotonic Sequence thm).

If a sequence is both bounded and monotonic, then the sequence converges.

Remarks

1) The last thm does not say that convergent sequences are monotonic. For example, the sequence $\left\{ \frac{(-1)^{n+1}}{n} \right\}$ converges to zero since $\underset{\to 0}{-\frac{1}{n}} \le \frac{(-1)^{n+1}}{n} \le \underset{\to 0}{\frac{1}{n}}$ and bounded but it is NOT montonic (why?).

2) The last thm says that a nondecreasing sequence converges when it is bounded from above, but it diverges to ∞ otherwise.

Summary (Bounded Monotonic Sequences)

(1) $\{a_n\}$ is bounded $\Rightarrow$ $\{a_n\}$ conv. (False)

(2) $\{a_n\}$ conv. $\Rightarrow$ $\{a_n\}$ bounded (True)

(3) $\{a_n\}$ Monotonic $\Rightarrow$ $\{a_n\}$ conv. (False)

(4) $\{a_n\}$ conv. $\Rightarrow$ $\{a_n\}$ Monotonic (False)

(5) $\{a_n\}$ bdd and Monotonic $\Rightarrow$ $\{a_n\}$ conv. (True).

Lecture problems

10.1 46, 59, 68, 84

$\boxed{46}$ $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$ since $0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$

$\Rightarrow a_n = \frac{\sin^2 n}{2^n}$ converges by the Sandwich theorem for sequences.

$\boxed{59}$ $\lim_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{\infty}{1} = \infty$ diverge

$\boxed{68}$ $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n$

$= \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$ since $y = \ln x$ is continuous on $x > 0$.

$= \ln e = 1$ converges.

$\boxed{84}$ $a_n = \sqrt[n]{n^2+n}$

$\lim_{n\to\infty} e^{\ln\left(\sqrt[n]{n^2+n}\right)}$ Notice that $x = e^{\ln x}$, $x > 0$

$= \lim_{n\to\infty} e^{\frac{\ln(n^2+n)}{n}} = e^{\lim_{n\to\infty} \frac{\ln(n^2+n)}{n}}$

$= e^{\lim_{n\to\infty}\left(\frac{2n+1}{n^2+n}\right)} = e^0 = 1$

10.2 Infinite series.

Df: Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1+a_2+a_3+\cdots+a_n+\cdots$ is an infinite series. The number a_n is the nth term of the series.

· Sequence of partial sums

The sequence $\{S_n\} = \{S_1, S_2, S_3, \cdots, S_n, \cdots\}$ defined by

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the sequence of partial sums of the series $a_1+a_2+a_3+\cdots+a_n+\cdots$

S_n is the nth partial sum.

If $\lim_{n\to\infty} S_n = L$ converges, we say that the series converges and its sum is L.

that is $a_1+a_2+\cdots+a_n+\cdots = \sum_{n=1}^{\infty} a_n = L.$

If the sequence of partial sums $\{S_n\}$ does not converge, we say that the series diverge.

Notation. $\sum_{n=1}^{\infty} a_n$, $\sum_{k=1}^{\infty} a_k$, or $\sum a_n$

example. Test for convergence.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$

Sol. $S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

$\vdots$

$S_n = \frac{2^n - 1}{2^{n-1}} = 2 - \left(\frac{1}{2}\right)^{n-1}$

Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2 - \left(\frac{1}{2}\right)^{n-1}\right) = 2 - 0 = 2$ conv

$\therefore \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ conv. and its sum is 2.

that is, $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2$.

(b) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)$

$+\cdots+\frac{1}{n}-\frac{1}{n+1}+\cdots$

Sol. $S_1 = a_1 = 1-\frac{1}{2}$

$S_2 = a_1+a_2 = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) = 1-\frac{1}{3}$

$S_3 = a_1+a_2+a_3 = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) = 1-\frac{1}{4}$

$\vdots$

$S_n = 1-\frac{1}{n+1}$

Since $\lim_{n\to\infty} S_n = 1-0 = 1$ conv., then

$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ conv. and its sum is 1

Notice that the series in (b) is called Telescoping series. other examples are follows.

(c) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty}\left[\ln(n)-\ln(n+1)\right]$

$= (\ln 1-\ln 2)+(\ln 2-\ln 3)$

$+(\ln 3-\ln 4)+\cdots+\ln(n)-\ln(n+1)$

$+\cdots$

Sol. $S_1 = a_1 = \ln 1-\ln 2 = -\ln 2$

$S_2 = a_1+a_2 = \ln 1-\ln 2+\ln 2-\ln 3$

$= -\ln 3$

$\vdots$

$$S_n = -\ln(n+1)$$

$$\lim_{n\to\infty} S_n = -\lim_{n\to\infty} \ln(n+1) = -\infty \text{ div.}$$

$$\Rightarrow \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) \text{ div.}$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Sol. $\displaystyle\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$$\Rightarrow 1 = A(n+1) + B(n) = (A+B)n + A$$

$$\Rightarrow A = 1,\quad A+B=0 \Rightarrow \boxed{B=-1}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \text{ exactly (b).}$$

(e) $\displaystyle\sum_{n=1}^{\infty} \left[\tan^{-1} n - \tan^{-1}(n+1)\right]$

$$= (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3) + \cdots$$

$$S_1 = a_1 = \tan^{-1} 1 - \tan^{-1} 2 = \frac{\pi}{4} - \tan^{-1} 2$$

$$S_2 = a_1 + a_2 = \frac{\pi}{4} - \tan^{-1} 2 + \tan^{-1} 2 - \tan^{-1} 3$$

$$= \frac{\pi}{4} - \tan^{-1} 3$$

$\vdots$

$$S_n = \frac{\pi}{4} - \tan^{-1}(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{\pi}{4} - \tan^{-1}(n+1)$$

$$= \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} \quad \text{conv.}$$

$\Rightarrow$ the series conv. and its sum is $-\frac{\pi}{4}$.

Geometric Series (G.S)

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

$a \neq 0$: The first term

r : the ratio can be positive or negative.

ex. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots$ is a G.S with $a = 1$, $r = \frac{1}{2}$.

ex. $1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots$ is a G.S with $a = 1$, $r = -\frac{1}{3}$.

The nth partial sum of the G.S

$S_1 = a_1 = a$

$S_2 = a_1 + a_2 = a + ar$

$\vdots$

$$\boxed{S_n = a_1 + a_2 + \cdots + a_n = a + ar + \cdots + ar^{n-1}}$$

multiply by r:

$$\boxed{rS_n = ar + ar^2 + \cdots + ar^n}$$

$S_n - rS_n = a - ar^n$

$(1-r)S_n = a(1-r^n)$

$$\Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

and $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

Summary. The G.S $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

converges to $\frac{a}{1-r}$ if $|r| < 1$

That is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r|<1$

If $|r| \geq 1$, the series diverges.

Ex. (a) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is

a G.S with $a=1$, $r = \frac{1}{2}$.

since $|r| = |\frac{1}{2}| = \frac{1}{2} < 1$, then the series

conv. and its sum is $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

That is $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

is a G.S with $a=5$, $r = -\frac{1}{4}$.

since $|-\frac{1}{4}| = \frac{1}{4} < 1$, then the series

conv. to $\frac{a}{1-r} = \frac{5}{1+\frac{1}{4}} = 4$.

(c) $\sum_{n=1}^{\infty} 3^n = 3+9+27+\cdots$ is a G.S with $r=3$

Since $|r|=|3|=3>1$, the series diverges.

(d) $\sum_{n=1}^{\infty} 4 = 4+4+4+\cdots+4+\cdots$ G.S with $r=1$ which diverges.

<u>Ex.</u> Express the repeating decimal $5.\overline{23}$ as the ratio of two integers.

<u>sol.</u> $5.\overline{23} = 5.232323\cdots$

$$= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\Big[\underbrace{1 + \frac{1}{100} + \frac{1}{(100)^2} + \cdots}_{G.S\ a=1,\ r=\frac{1}{100}}\Big]$$

$$= 5 + \frac{23}{100}\left[\frac{1}{1-\frac{1}{100}}\right]$$

$$= 5 + \frac{23}{100}\cdot\frac{100}{99} = 5 + \frac{23}{99}$$

$$= \frac{518}{99}$$

The nth term Test for a divergent series

Thm. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

The nth-term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist

or $\lim_{n\to\infty} a_n \neq 0$

Ex: (a) $\sum_{n=1}^{\infty} n^2$ div. since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ div. since $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

(c) $\sum_{n=1}^{\infty} \left(1 - \frac{2020}{n}\right)^n$ div. since

$\lim_{n\to\infty} \left(1 - \frac{2020}{n}\right)^n = e^{-2020} \neq 0$

(d) $\sum_{n=1}^{\infty} (-1)^{n+1}$ div. since $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

(e) the test fails for $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ since $\lim \frac{1}{2^{n-1}} = 0$ but it is conv. by G.S. Test.

Combining Series

Thm: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1) $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$.

2) $\sum(k a_n) = k \sum a_n = kA$ (any number k).

Corollary. 1) Every nonzero constant multiple of a divergent series diverges.

2) If $\sum a_n$ conv. and $\sum b_n$ div., then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both div.

تنبيه Caution $\sum(a_n + b_n)$ can conv. if $\sum a_n$ and $\sum b_n$ both diverges. For example

$\sum a_n = 1 + 1 + 1 + \cdots$ and
$\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverge

but $\sum_{n=1}^{\infty}(a_n + b_n) = 0 + 0 + \cdots + 0 + \cdots$ converg to zero.

Ex. Find the sums.

(a) $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}} = 2 - \frac{6}{5} = \frac{4}{5}.$$

(b) $\sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \frac{1}{2^n}$

$$= 4 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right)$$

$$= 4 \frac{1}{1-\frac{1}{2}} = 8.$$

Adding or Deleting terms

If $\sum_{n=1}^{\infty} a_n$ conv., then $\sum_{n=k}^{\infty} a_n$ conv. for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ conv. for any $k>1$, then $\sum_{n=1}^{\infty} a_n$ converges.

ex. $$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and $$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}.$$

Reindexing

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots$$

ex. $$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

and $$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

or $$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=5} \frac{1}{2^{n-5}}$$

or even $$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

(32)

10.2 37, 51, 64, 75 (Lecture problems) (13)

$\boxed{37}$ $\sum_{n=1}^{\infty}\left[\ln\sqrt{n+1}-\ln\sqrt{n}\right]$

$$= \underbrace{(\ln\sqrt{2}-\ln 1)}_{a_1} + \underbrace{(\ln\sqrt{3}-\ln\sqrt{2})}_{a_2} + \underbrace{(\ln\sqrt{4}-\ln\sqrt{3})}_{a_3} + \cdots$$

$$S_1 = a_1 = \ln\sqrt{2} - \cancel{\ln\sqrt{1}}^{\,0} = \ln\sqrt{2}$$

$$S_2 = a_1 + a_2 = \cancel{\ln\sqrt{2}} - \cancel{\ln 1}^{\,0} + \ln\sqrt{3} - \cancel{\ln\sqrt{2}}$$

$$= \ln\sqrt{3}$$

$\vdots$

$$S_n = \ln(\sqrt{n+1})$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\sqrt{n+1} = +\infty \text{ diverges.}$$

$\Rightarrow$ Series diverges.

$\boxed{51}$ $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$

$$= \frac{3}{2} - \frac{3}{2^2} + \frac{3}{2^3} - \cdots$$

is Geometric series with $a=\frac{3}{2}$, $r=-\frac{1}{2}$

since $|r| = |-\frac{1}{2}| = \frac{1}{2} < 1$, then the series converges and its sum $= \frac{a}{1-r} = \frac{\frac{3}{2}}{1+\frac{1}{2}} = 1.$

$\boxed{64}$ $\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+4^n}$

$$\lim_{n\to\infty} \frac{2^n+4^n}{3^n+4^n} = \lim_{n\to\infty} \frac{4^n\left[\frac{2^n}{4^n}+1\right]}{4^n\left[\frac{3^n}{4^n}+1\right]}$$

$$= \lim_{n\to\infty} \frac{(\frac{1}{2})^n+1}{(\frac{3}{4})^n+1} = \frac{0+1}{0+1} = 1 \neq 0$$

$\Rightarrow$ Series diverges by n^{th} term test for divergence.

$\boxed{75}$ $\sum_{n=0}^{\infty} (-1)^n (x+1)^n = \sum_{n=0}^{\infty} (-x-1)^n$

$$= 1 - (x+1) + (x+1)^2 - \dots$$

is a Geometric series with $a=1$, $r=-x-1$

Converges to $\frac{1}{1+(x+1)} = \frac{1}{2+x}$ for

$|-x+1| < 1$

or $|x+1| < 1$

$-2 < x < 0$.

10.3 The Integral Test

Thm. (The integral test)

let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is continuous, positive, decreasing function of x for all $x \geq N$ (N is a positive integer). then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

ex. Does the following series converge? diverge? Justify.

① $\sum_{n=1}^{\infty} \frac{1}{n}$ (is called the harmonic series)

let $f(x) = \frac{1}{x}$, $x \geq 1$

f is positive, continuous and decreasing

for $x \geq 1$ (since $f'(x) = -\frac{1}{x^2} < 0$)

and $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{A\to\infty}\int_1^A \frac{1}{x}\,dx$

$$= \lim_{A\to\infty} \operatorname{Ln}|x|\Big|_1^A$$

$$= \lim_{A\to\infty}(\operatorname{Ln}A - \ln 1) = \infty$$

div

$\therefore$ the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges by integral test.

② $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$

$f(x) = \dfrac{1}{x^2+1}$, $x \geq 1$.

(i) f is positive for $x \geq 1$. (even for every x)

(ii) f is continuous $\forall x$.

(iii) $f'(x) = \dfrac{-2x}{(x^2+1)^2} < 0$

$\therefore f$ is decreasing

Now, $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{A\to\infty}\int_1^A \frac{1}{x^2+1}\,dx$

(36)

$$= \lim_{A\to\infty} \tan^{-1}x \Big|_1^A = \lim_{A\to\infty} (\tan^{-1}A - \tan^{-1}1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ Conv}$$

$\therefore$ the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Conv. by integral test but we do not know the value of its sum. that is, we can not emphasize that $\frac{\pi}{4}$ is the sum.

③ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$f(x) = \frac{1}{x^2}$, $x \geq 1$ is positive, continuous and decreasing on $x \geq 1$

أفحص ذلك !

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{A\to\infty} \int_1^A x^{-2} dx$$

$$= \lim_{A\to\infty} \frac{-1}{x} \Big]_1^A$$

$$= \lim_{A\to\infty} \left(-\frac{1}{A} + 1\right) = 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by integral test

④ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p - series)

show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

Solution. If $p > 1$, then $f(x) = \frac{1}{x^p}$ is a positive, continuous, decreasing on $x \geq 1$

Since $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{A \to \infty} \int_1^A x^{-p} dx$

$$= \lim_{A \to \infty} \left. \frac{x^{1-p}}{1-p} \right|_1^A$$

$$= \lim_{A \to \infty} \frac{1}{1-p} \left[\frac{1}{A^{p-1}} - 1 \right]$$

$$= \frac{1}{1-p} [0 - 1] = \frac{1}{p-1}$$

the series Converges by the integral test.

We emphasize that the sum of the p-series is NOT $\frac{1}{p-1}$. that is, the series conv. but we don't know the value it converges to.

- If $p<1$, then $1-p>0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{A\to\infty}\left(A^{1-p}-1\right) = \underset{\text{div.}}{\infty}$$

- If $p=1$, we have the (divergent) harmonic series (example 1).

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^p} = \begin{cases} \text{conv.} & \text{if } p>1 \\ \text{div.} & \text{if } p\le 1. \end{cases}$$

ex. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ conv. (p-series $p=\frac{3}{2}>1$).

ex. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\pi-e}}$ div. (p-series with $p=\pi-e<1$.

Lecture Problems 8, 26, 34.

(8) $\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$.

$f(x) = \frac{\ln(x^2)}{x}$, $x \geq 2$

(i) f is positive for $x \geq 2$

(ii) f is continuous for $x \geq 2$

(iii) $f'(x) = \frac{x \cdot \frac{2x}{x^2} - \ln(x^2) \cdot 1}{x^2}$

$= \frac{2 - \ln(x^2)}{x^2} < 0$

< 0 if $2 - \ln(x^2) < 0$

if $\ln x^2 > 2$

if $x^2 > e^2$

if $|x| > e$

if $\boxed{x > e}$ since $x \geq 2$

thus f is decreasing for $x \geq 3$.

$$\int_3^\infty \frac{\ln(x^2)}{x}\,dx = \lim_{B\to\infty} \int_3^B \frac{\ln x^2}{x}\,dx$$

$$= \lim_{B\to\infty} (\ln x)^2 \Big|_3^B$$

$$= \lim_{B\to\infty} \left((\ln B)^2 - (\ln 3)^2 \right) = \infty$$

$$\Rightarrow \int_3^\infty \frac{\ln x^2}{x}\,dx \text{ diverges}$$

$$\Rightarrow \sum_{n=3}^\infty \frac{\ln(n^2)}{n} \text{ div.}$$

$$\Rightarrow \sum_{n=2}^\infty \frac{\ln n^2}{n} = \frac{\ln 4}{2} + \sum_{n=3}^\infty \frac{\ln(n^2)}{n} \text{ div.}$$

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(26) $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

$f(x) = \dfrac{1}{\sqrt{x}(\sqrt{x}+1)}$ is positive, continuous, decreasing for $x \geq 1$

الاختبار

$$\int_1^\infty \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx = \lim_{A\to\infty}\int_1^A \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx$$

$$= \lim_{A\to\infty} 2\ln(\sqrt{x}+1)\Big|_1^A$$

$$= \lim_{A\to\infty}\left(2\ln(\sqrt{A}+1) - 2\ln 2\right)$$

$$= \infty \text{ diverges}$$

$\therefore$ the series diverges by the Integral Test.

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(34) $\sum_{n=1}^{\infty} n\tan\left(\frac{1}{n}\right)$ diverges by the <u>nth term test for divergence</u> since

$$\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) \quad (\infty\cdot 0)$$

$$= \lim_{n\to\infty}\frac{\tan\left(\frac{1}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\sec^2\left(\frac{1}{n}\right)\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}}$$

$$= 1 \neq 0.$$

10.4 Comparison Test

Theorem [the Comparison Test].

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} c_n$, and $\sum_{n=1}^{\infty} d_n$ be series with nonnegative terms. Suppose that for some integer N

$$\boxed{d_n \leq a_n \leq c_n, \quad \forall n > N.}$$

(a) If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also conv.

(b) If $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges

Ex. Determine if each series converges or diverges.

$\boxed{1}$ $\sum_{n=1}^{\infty} \frac{5}{5n-1}$, Notice that $5n > 5n-1$

$$\Rightarrow \frac{5}{5n} < \frac{5}{5n-1}$$

$$\Rightarrow \underbrace{\frac{1}{n}}_{d_n} < \underbrace{\frac{5}{5n-1}}_{a_n}, \forall n \geq 1$$

Since $\sum_{n\geq 1}^{\infty} \frac{1}{n}$ div. (p-test) and $\frac{\overset{+ve}{1}}{n} < \frac{\overset{+ve}{5}}{5n-1}$,

then $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ div. by Comparison Test. (C-T)

② $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by (C-T)

since its terms are all positive and

$$\underbrace{\frac{1}{n^2+1}}_{a_n} < \underbrace{\frac{1}{n^2}}_{c_n} \quad \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ conv. p-test} \right)$$

③ $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{2020}}$ Notince that $\tan^{-1} n < \frac{\pi}{2}$

$$\Rightarrow \underbrace{\frac{\tan^{-1} n}{n^{2020}}}_{a_n} < \underbrace{\frac{\pi}{2n^{2020}}}_{b_n}$$

Since $\sum_{n=1}^{\infty} \frac{\pi}{2n^{2020}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{2020}}$ conv. (p-test)

then, by C.T, $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{2020}}$ converges.

④ $$\sum_{n=1}^{\infty} \left(\frac{1}{2^n + \sqrt{n}} \right)$$

Notice that $\underbrace{\frac{1}{2^n+\sqrt{n}}}_{a_n} < \underbrace{\frac{1}{2^n}}_{c_n}$

a_n & c_n are with nonnegative terms.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$ converges (Geometric series with $r = \frac{1}{2} < 1$)

by C.T, $$\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$$ converges.

Thm [The Limit Comparison Test]

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

(1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \boxed{c} > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \boxed{0}$ and $\boxed{\sum b_n \text{ conv.}}$, then $\boxed{\sum a_n \text{ conv.}}$

(3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \boxed{\infty}$ and $\boxed{\sum b_n \text{ div}}$, then $\boxed{\sum a_n \text{ div.}}$

ex. Which of the series converge & which diverge?

① $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ let $\boxed{a_n = \dfrac{2n+1}{n^2+2n+1}}$

For large n, we expect a_n to behave like $\dfrac{2n}{n^2} = \dfrac{2}{n}$, so we let $\boxed{b_n = \dfrac{1}{n}}$

Since • $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges and

• $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2 > 0$,

then $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ div. by L.C.T.

② $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{2019} e^{-n}$

let $a_n = \left(1+\dfrac{1}{n}\right)^{2019} e^{-n}$

Take $b_n = e^{-n}$

Since $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ conv. (Geometric series) and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{2019} \cancel{e^{-n}}}{\cancel{e^{-n}}} = 1 > 0$,

then $\sum_{n=1}^{\infty} a_n$ conv. by L.C.T.

③ $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. Let $\boxed{a_n = \frac{1}{2^n-1}}$

For large n, we expect a_n like $\frac{1}{2^n}$,

so, we let $\boxed{b_n = \frac{1}{2^n}}$. ~~Not~~. Using L.C.T,

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{2^n \ln 2}{2^n \ln 2} = 1 >$

- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ Conv. (Geometric series)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ Conv. by L.C.T.

④ $\sum_{n=1}^{\infty} \frac{1 + n\ln n}{n^2+5}$ a_n

Using L.C.T with $\sum_{n=1}^{\infty} b_n$ $\sum_{n=1}^{\infty} \frac{1}{n}$

$\cdot \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n + n^2 \ln n}{n^2+5} \quad \left(\dfrac{\infty}{\infty}\right)$

L'Hôpital

$= \lim\limits_{n\to\infty} \dfrac{1 + n^2 \cdot \frac{1}{n} + 2n \ln n}{2n}$

$= \lim\limits_{n\to\infty} \dfrac{n + 2n \ln n}{2n} \quad \dfrac{\infty}{\infty}$

$= \lim\limits_{n\to\infty} \dfrac{1 + 2\ln n + 2}{2} = \infty$

$\cdot \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ div. (p-test).

$\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ div. by L.C.T (part 3).

⑤ $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^{\frac{3}{2}}}$ a_n · Using L.C.T with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{5/4}}$.

$\cdot \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{n^{\frac{1}{4}}} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}}{\frac{1}{4} n^{-3/4}}$

$= \lim\limits_{n\to\infty} \dfrac{4}{n^{\frac{1}{4}}} = 0$

$\cdot \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\frac{5}{4}}}$ conv. (p-test).

$\Rightarrow \sum\limits_{n=1} a_n$ conv. by L.C.T (part 2).

10.5 The Ratio and Root Tests

Thm. [The Ratio Test].

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$.

Spse that $\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.}$ then

(a) the series converges if $\rho < 1$.

(b) " " diverges if $\rho > 1$ or ρ is infinite.

(c) the test is inconclusive if $\rho = 1$.

Ex. Investigate the convergence of the following series.

(a) $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$. We apply the Ratio test

Let $a_n = \frac{2^n + 5}{3^n} \Rightarrow a_{n+1} = \frac{2^{n+1} + 5}{3^{n+1}}$.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2^{n+1}+5}{3^{n+1}}\right)\left(\frac{3^n}{2^n+5}\right)$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{2^{n+1}\ln 2}{2^n \ln 2} = \frac{1}{3} \lim_{n\to\infty} 2 = \frac{2}{3} < 1$$

$\therefore$ The series converges by Ratio Test since

$\rho = \frac{2}{3} < 1.$

Warning, this does not mean that $\frac{2}{3}$ is the sum.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

If $a_n = \dfrac{(2n)!}{n!\,n!}$, then $a_{n+1} = \dfrac{[2(n+1)]!}{(n+1)!(n+1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)\cancel{(2n)!}\ \cancel{n!}\ \cancel{n!}}{(n+1)\cancel{n!}\,(n+1)\,\cancel{n!}\quad \cancel{(2n)!}}$$

$$= \frac{2(n+1)(2n+1)}{(n+1)^2} = \frac{4n+2}{n+1}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4n+2}{n+1} = 4 > 1$$

