Exercises.

Q20: let the Random variables X and y have the joint p.d.f, in each case compute the correlation coefficient of X and y. $\rho = \frac{E(xy) - \mu_x \mu_y}{\sigma_1 \sigma_2}$

a. $f(x,y) = \begin{cases} \frac{1}{3}, & (x,y) = (0,0), (1,1), (2,2) \\ 0, & \text{elsewhere} \end{cases}$

$\rightarrow E(X) = \sum_{x=0}^{x=2} 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) = 1$

$\rightarrow E(y) = \sum_{y=0}^{y=2} 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) = 1$

$\rightarrow E(X^2) = \sum_{x=0}^{x=2} 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) = \frac{5}{3}$

$\rightarrow E(y^2) = \sum_{y=0}^{y=2} 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) = \frac{5}{3}$

$\rightarrow E(xy) = \sum_{(x,y)=0,0}^{(2,2)} 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 2.2\left(\frac{1}{3}\right) = \frac{5}{3}$

$\rightarrow Var(x) = E(x^2) - (E(x))^2 = \frac{5}{3} - 1 = \frac{2}{3} \rightarrow \sigma = \sqrt{\frac{2}{3}}$

$\rightarrow var(y) = E(y^2) - (E(y))^2 = \frac{5}{3} - 1 = \frac{2}{3} \rightarrow \sigma = \sqrt{\frac{2}{3}}$

$\Rightarrow \rho = \frac{\frac{5}{3} - 1}{\sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}} = \frac{2}{3} \div \frac{2}{3} = 1$

$\Rightarrow \boxed{\rho = 1}$

Q21 : let x and y have the joint p.d.f describe as a follows :

(x,y)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
$f(x,y)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

$f(x,y) = 0$ elsewhere .

a. Find the correlation coefficient ρ ?

$\rightarrow E(x) = \sum_{x=1}^{x=2} 1\left(\frac{2}{15}\right) + 1\left(\frac{4}{15}\right) + 1\left(\frac{3}{15}\right) + 2\left(\frac{1}{15}\right) + 2\left(\frac{1}{15}\right) + 2\left(\frac{4}{15}\right) = \frac{21}{15}$

$\rightarrow E(y) = \sum_{y=1}^{y=3} 1\left(\frac{2}{15}\right) + 2\left(\frac{4}{15}\right) + 3\left(\frac{3}{15}\right) + 1\left(\frac{1}{15}\right) + 2\left(\frac{1}{15}\right) + 3\left(\frac{4}{15}\right) = \frac{34}{15}$.

$\rightarrow E(x^2) = \sum_{x=1}^{x=2} 1\left(\frac{2}{15}\right) + 1\left(\frac{4}{15}\right) + 1\left(\frac{3}{15}\right) + 4\left(\frac{1}{15}\right) + 4\left(\frac{1}{15}\right) + 4\left(\frac{4}{15}\right) = \frac{33}{15}$.

$\rightarrow E(y^2) = \sum 1\left(\frac{2}{15}\right) + 4\left(\frac{4}{15}\right) + 9\left(\frac{3}{15}\right) + 1\left(\frac{1}{15}\right) + 4\left(\frac{1}{15}\right) + 9\left(\frac{4}{15}\right) = \frac{86}{15}$

$\rightarrow E(xy) = \sum 1\left(\frac{2}{15}\right) + 2\left(\frac{4}{15}\right) + 3\left(\frac{3}{15}\right) + 2\left(\frac{1}{15}\right) + 4\left(\frac{1}{15}\right) + 6\left(\frac{4}{15}\right) = \frac{49}{15}$

$\rightarrow var(x) = E(x^2) - (E(x))^2 = \frac{22}{15} - \left(\frac{18}{15}\right)^2 = \frac{15 \times 22}{15 \times 15} - \frac{324}{225} = \frac{6}{225}$.

$\rightarrow var(y) = E(y^2) - (E(y))^2 = \frac{86}{15} - \left(\frac{34}{15}\right)^2 = \frac{86}{15} - \frac{1156}{225} = \frac{134}{225}$

$\rightarrow \sigma_x = \frac{\sqrt{6}}{15}$, $\sigma_y = \frac{\sqrt{134}}{15}$

$$\rightarrow \rho = \frac{\frac{49}{15} - \left(\frac{21}{15} \cdot \frac{34}{15}\right)}{\frac{\sqrt{6}}{15} \cdot \frac{\sqrt{134}}{15}} = \frac{\frac{21}{\cancel{225}}}{\frac{\sqrt{804}}{\cancel{225}}} = \frac{21}{\sqrt{804}}$$

جواب الكتاب بطلع $\frac{7}{\sqrt{804}}$ ، في غلط بالحل ؟!

Q22: let $f(x,y) = \begin{cases} 2 & , 0<x<y \ , \ 0<y<1 \\ 0 & , \text{elsewhere} \end{cases}$ be the joint p.d.f of X, Y

Show that the conditional means are $\begin{cases} \frac{1+x}{2} & , 0<x<1 \\ \frac{y}{2} & , 0<y<1 \end{cases}$

and show $\rho = \frac{1}{2}$.

$$\rightarrow E(X|y) = \int_0^y x\, f(x|y)\, dx.$$

$$\rightarrow f(y) = \int_0^y 2\, dx = 2x \Big|_0^y = 2y.$$

$$\rightarrow f(x|y) = \frac{f(x,y)}{f(y)} = \frac{2}{2y} = \frac{1}{y}$$

$$\Rightarrow E(X|y) = \int_0^y x \frac{1}{y}\, dx = \frac{x^2}{2y} \Big|_0^y = \frac{y}{2} \quad , 0<y<1.$$

$$\rightarrow E(y|x) = \int_x^1 y\, f(y|x)\, dy$$

$$\rightarrow f(x) = \int_x^1 2\, dy = 2y \Big|_x^1 = 2 - 2x.$$

$$\rightarrow f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{(1-x)}.$$

$$\Rightarrow E(y|x) = \int_x^1 y \left(\frac{1}{1-x}\right) dy = \frac{1}{2} y^2 \left(\frac{1}{1-x}\right) \Big|_x^1 = \frac{y^2}{2(1-x)} \Big|_x^1$$

$$= \frac{1-x^2}{2(1-x)}$$

$$= \frac{(1-x)(1+x)}{2(1-x)}$$

$$\boxed{E(y|x) = \frac{1+x}{2}}$$

to show $\rho = \frac{1}{2}$

$$\rightsquigarrow E(x) = \int_0^1 \int_0^y x\, f(x,y)\, dx\, dy = \int_0^1 \int_0^y 2x\, dx\, dy = \int_0^1 x^2 \Big|_0^y dy$$

$$= \int_0^1 y^2\, dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\rightsquigarrow E(y) = \int_0^1 \int_0^y 2y\, dx\, dy = \int_0^1 2yx\Big|_0^y dy = \int_0^1 2y^2\, dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$\rightsquigarrow E(xy) = \int_0^1 \int_0^y 2xy\, dx\, dy = \int_0^1 x^2 y\Big|_0^y dy = \int_0^1 y^3\, dy = \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$\rightarrow E(x^2) = \int_0^1 \int_0^y 2x^2\, dx\, dy = \int_0^1 \frac{2x^3}{3}\Big|_0^y dy = \int_0^1 \frac{2}{3} y^3\, dy = \frac{2}{12} y^4\Big|_0^1 = \frac{2}{12} = \frac{1}{6}$$

$$\rightsquigarrow E(y^2) = \int_0^1 \int_0^y 2y^2\, dx\, dy = \int_0^1 2y^2 x\Big|_0^y dy = \int_0^1 2y^3\, dy = \frac{1}{2} y^4\Big|_0^1 = \frac{1}{2}$$

$$\rightarrow Var(x) = E(x^2) - (E(x))^2 = \frac{3\times 1}{3\times 6} - \frac{2\times 1}{2\times 9} = \frac{1}{18} \rightsquigarrow \delta_1 = \frac{1}{\sqrt{18}}$$

$$\rightarrow Var(y) = E(y^2) - (E(y))^2 = \frac{9\times 1}{9\times 2} - \frac{2\times 4}{2\times 9} = \frac{1}{18} \rightsquigarrow \delta_2 = \frac{1}{\sqrt{18}}$$

$$\Rightarrow \rho = \frac{E(xy) - \mu_1 \mu_2}{\delta_1\, \delta_2} = \frac{\frac{1}{4} - \left(\frac{1}{3} \times \frac{2}{3}\right)^{\frac{2}{9}}}{\frac{1}{\sqrt{18}}\, \frac{1}{\sqrt{18}}} = \frac{1}{2}$$