CHAPTER 6

The Laplace Transform

6.1 Definition of the Laplace Transform

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals; thus

$$\int_a^{\infty} f(t)\,dt = \lim_{A\to\infty} \int_a^{A} f(t)\,dt, \tag{1}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Let $f(t) = e^{ct},\ t \geq 0$, where c is a real nonzero constant. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\int_0^A e^{ct}\,dt = \lim_{A\to\infty}\left.\frac{e^{ct}}{c}\right|_0^A$$

$$= \lim_{A\to\infty}\frac{1}{c}(e^{cA}-1).$$

It follows that the improper integral converges to the value $-1/c$ if $c < 0$ and diverges if $c > 0$. If $c = 0$, the integrand $f(t)$ is the constant function with value 1. In this case

$$\lim_{A\to\infty}\int_0^A 1\,dt = \lim_{A\to\infty}(A-0) = \infty,$$

so the integral again diverges.

EXAMPLE 2

Let $f(t) = 1/t$, $t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A.$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

Definition:

f is piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$. An example of a piecewise continuous function is shown in Figure 6.1.1.

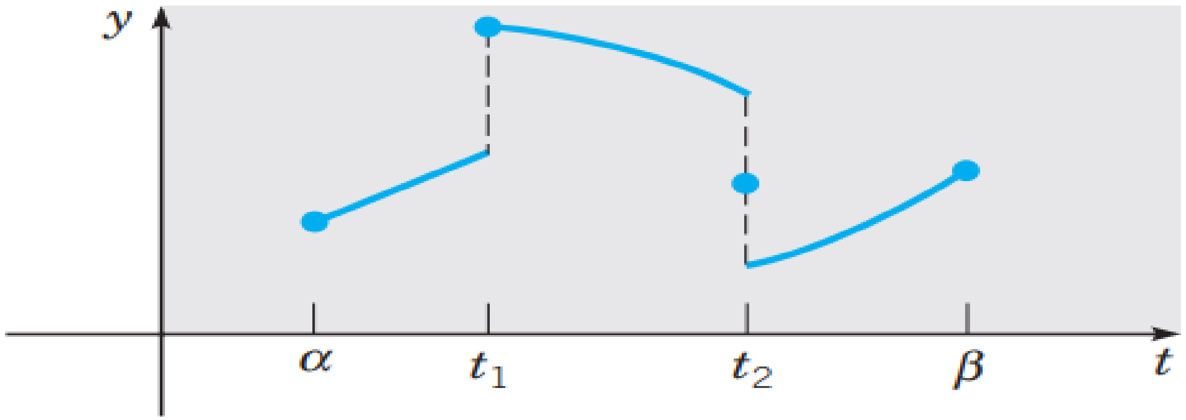

FIGURE 6.1.1 A piecewise continuous function $y = f(t)$.

The integral of a piecewise continuous function on a finite interval is just the sum of the integrals on the subintervals created by the partition points. For instance, for the function $f(t)$ shown in Figure 6.1.1, we have

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^\beta f(t)\,dt. \qquad (2)$$

Laplace Transform

Definition 1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$(1) \qquad F(s) := \int_0^\infty e^{-st} f(t)\, dt\,.$$

The domain of $F(s)$ is all the values of s for which the integral in (1) exists.† The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Theorem 6.1.2 Suppose that

1. f is piecewise continuous on the interval $0 \le t \le A$ for any positive A.
2. $|f(t)| \le Ke^{at}$ when $t \ge M$. In this inequality, K, a, and M are real constants, K and M necessarily positive.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Eq. (4), exists for $s > a$.

Example 1 Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Solution Using the definition of the transform, we compute

$$F(s) = \int_0^{\infty} e^{-st} \cdot 1\, dt = \lim_{N\to\infty} \int_0^N e^{-st}\, dt$$

$$= \lim_{N\to\infty} \frac{-e^{-st}}{s}\Bigg|_{t=0}^{t=N} = \lim_{N\to\infty}\left[\frac{1}{s} - \frac{e^{-sN}}{s}\right].$$

Since $e^{-sN} \to 0$ when $s > 0$ is fixed and $N \to \infty$, we get

$$F(s) = \frac{1}{s} \qquad \text{for} \qquad s > 0\,.$$

When $s \le 0$, the integral $\int_0^{\infty} e^{-st} dt$ diverges. (Why?) Hence $F(s) = 1/s$, with the domain of $F(s)$ being all $s > 0$. ◆

Example 2 Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution Using the definition of the transform,

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt \\
&= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt = \lim_{N\to\infty} \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^N \\
&= \lim_{N\to\infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a} \right] \\
&= \frac{1}{s-a} \quad \text{for} \quad s > a\,.
\end{aligned}
$$

Again, if $s \le a$ the integral diverges, and hence the domain of $F(s)$ is all $s > a$. ◆

Example 3 Find $\mathcal{L}\{\sin bt\}$, where b is a nonzero constant.

Solution We need to compute

$$\mathcal{L}\{\sin bt\}(s) = \int_0^{\infty} e^{-st}\sin bt\,dt = \lim_{N\to\infty}\int_0^{N} e^{-st}\sin bt\,dt\,.$$

Referring to the table of integrals at the back of the book, we see that

$$\begin{aligned}\mathcal{L}\{\sin bt\}(s) &= \lim_{N\to\infty}\left[\frac{e^{-st}}{s^2+b^2}(-s\sin bt - b\cos bt)\Big|_0^N\right]\\ &= \lim_{N\to\infty}\left[\frac{b}{s^2+b^2} - \frac{e^{-sN}}{s^2+b^2}(s\sin bN + b\cos bN)\right]\\ &= \frac{b}{s^2+b^2} \quad \text{for} \quad s>0\end{aligned}$$

(since for such s we have $\lim_{N\to\infty} e^{-sN}(s\sin bN + b\cos bN) = 0$ ◆

Example 4 Determine the Laplace transform of

$$f(t) = \begin{cases} 2\,, & 0 < t < 5\,, \\ 0\,, & 5 < t < 10\,, \\ e^{4t}\,, & 10 < t\,. \end{cases}$$

Solution Since $f(t)$ is defined by a different formula on different intervals, we begin by breaking up the integral in (1) into three separate parts.[†] Thus,

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^5 e^{-st} \cdot 2\, dt + \int_5^{10} e^{-st} \cdot 0\, dt + \int_{10}^\infty e^{-st} e^{4t}\, dt \\
&= 2\int_0^5 e^{-st}\, dt + \lim_{N\to\infty} \int_{10}^N e^{-(s-4)t}\, dt \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \lim_{N\to\infty}\left[\frac{e^{-10(s-4)}}{s-4} - \frac{e^{-(s-4)N}}{s-4}\right] \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4} \qquad \text{for} \quad s > 4\,. \; \blacklozenge
\end{aligned}
$$

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$

Linearity of the Transform

Theorem 1. Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

$$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}\ , \tag{2}$$

$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}\ . \tag{3}$$

Example 5 Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

Solution From the linearity property, we know that the Laplace transform of the sum of any finite number of functions is the sum of their Laplace transforms. Thus,

$$\begin{aligned}\mathscr{L}\{11+5e^{4t}-6\sin 2t\} &= \mathscr{L}\{11\}+\mathscr{L}\{5e^{4t}\}+\mathscr{L}\{-6\sin 2t\}\\ &= 11\mathscr{L}\{1\}+5\mathscr{L}\{e^{4t}\}-6\mathscr{L}\{\sin 2t\}\ .\end{aligned}$$

In Examples 1, 2, and 3, we determined that

$$\mathscr{L}\{1\}(s)=\frac{1}{s},\qquad \mathscr{L}\{e^{4t}\}(s)=\frac{1}{s-4},\qquad \mathscr{L}\{\sin 2t\}(s)=\frac{2}{s^2+2^2}\ .$$

Using these results, we find

$$\begin{aligned}\mathscr{L}\{11+5e^{4t}-6\sin 2t\}(s) &= 11\left(\frac{1}{s}\right)+5\left(\frac{1}{s-4}\right)-6\left(\frac{2}{s^2+4}\right)\\ &= \frac{11}{s}+\frac{5}{s-4}-\frac{12}{s^2+4}\ .\end{aligned}$$

Since $\mathscr{L}\{1\}, \mathscr{L}\{e^{4t}\}$, and $\mathscr{L}\{\sin 2t\}$ are all defined for $s>4$, so is the transform $\mathscr{L}\{11+5e^{4t}-6\sin 2t\}$. ◆

Example 6 Use Table 7.1 to determine $\mathcal{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\}$.

Solution From the table,

$$\mathcal{L}\{t^2e^{-3t}\} = \frac{2!}{[s-(-3)]^{2+1}} = \frac{2}{(s+3)^3} \quad \text{for } s > -3,$$

and

$$\mathcal{L}\{e^{12t}\cos 8t\} = \frac{s-12}{(s-12)^2+8^2} \quad \text{for } s > 12\,.$$

Therefore, by linearity,

$$\mathcal{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\} = \frac{10}{(s+3)^3} - \frac{s-12}{(s-12)^2+64} \quad \text{for } s > 12\,. \ \blacklozenge$$

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

7. $f(t) = \cosh bt$

8. $f(t) = \sinh bt$

9. $f(t) = e^{at} \cosh bt$

10. $f(t) = e^{at} \sinh bt$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$\sinh(at)$	$\dfrac{a}{s^2 - a^2}$
$\cosh(at)$	$\dfrac{s}{s^2 - a^2}$
$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2 - b^2}$
$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2 - b^2}$