Exp Suppose $f(x) = \alpha x^3 - 12x$ has EV at $x=2$
Find α

$f'(x) = 3\alpha x^2 - 12$

since f has EV at $x=2$ and f is diff at $x=2$
$\Rightarrow f'(2) = 0$

$f'(2) = 3\alpha(4) - 12 = 0$
$12\alpha - 12 = 0$
$12\alpha = 12 \quad \Rightarrow \alpha = 1$

Remark The converse of Th* is not true
This means: if $f'(c) = 0$ then f may not have EV at c

Exp $f(x) = x^3$ has no EV's → see page 45

But $f'(x) = 3x^2 = 0 \quad \Rightarrow x = 0 \in D(f)$

This means $f'(0) = 0$ but f has no EV at $x=0$
Note that $(0,0)$ is critical point

Now it is important to classify the critical points: which one of them does f have EV? $\Rightarrow$ we will use

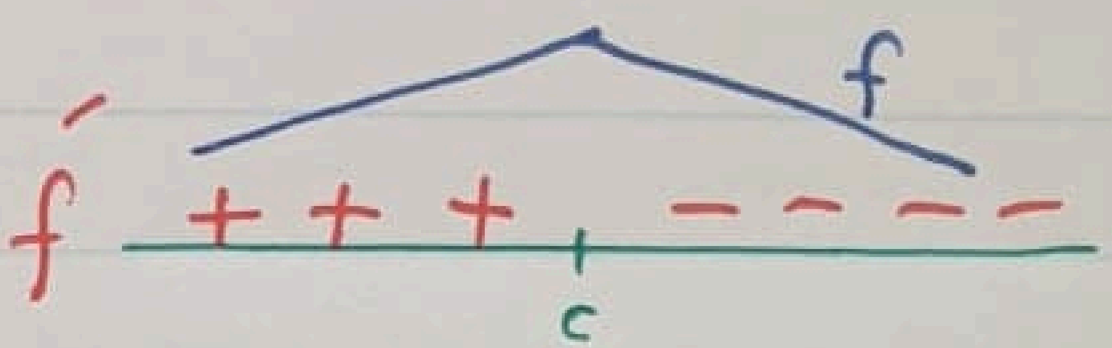
[1] First Derivative Test _ FDT

[2] Second Derivative Test _ SDT

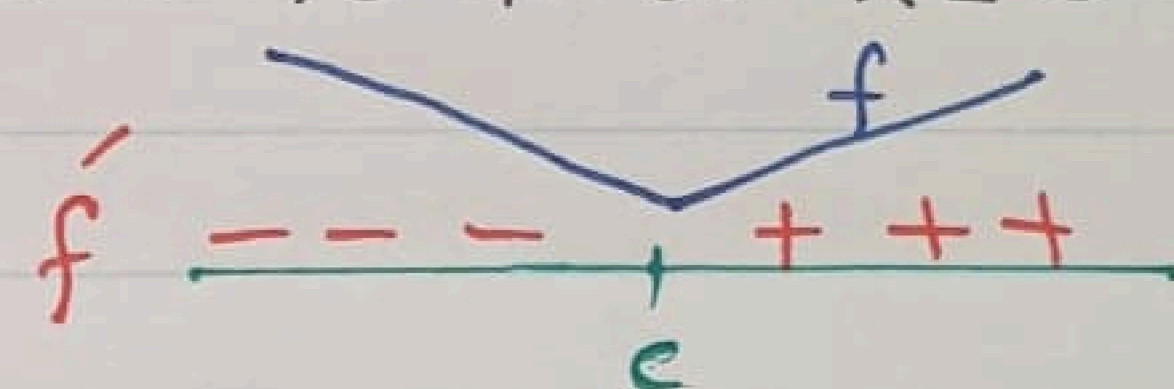
Th (FDT)

Suppose f has critical point at $x=c$ and f' exists in an open interval containing $x=c$. Then

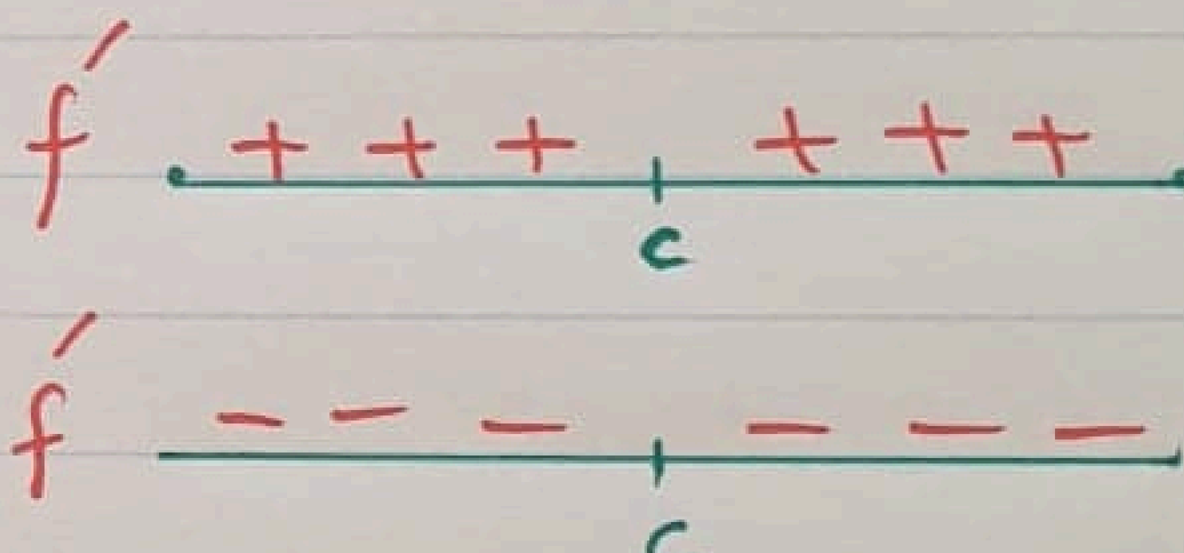
1. if f' changes sign from $+$ to $-$ at $x=c$ then $f(c)$ is local Max



2. if f' changes sign from $-$ to $+$ at $x=c$ then $f(c)$ is local Min



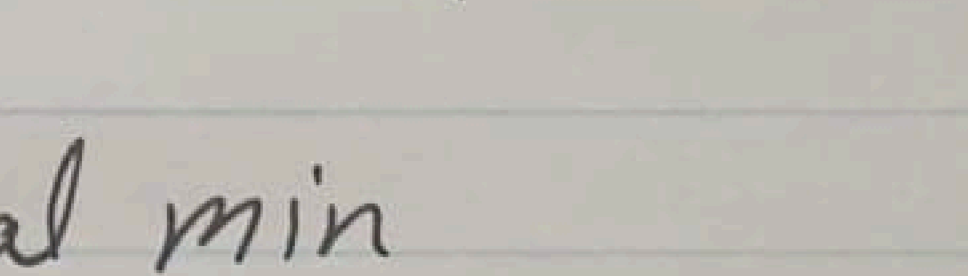
3. if f' does not change sign at $x=c$ then f has no EV's.



Th (SDT) Suppose $f'(c)=0$ and f'' is cont. on an open interval containing c. Then

1. if $f''(c)<0$ then $f(c)$ is local max
2. if $f''(c)>0$ then $f(c)$ is local min
3. if $f''(c)=0$ then the test fails.

Remark 1. If $f''(x)\geq 0 \ \forall x\in I$, then f is concave up on I



2. If $f''(x)\leq 0 \ \forall x\in I$, then f is concave down on I.

Def A point where f has tangent and change concavity is called inflection point.

To find the inflection points we start with the points where $f''(x) = 0$ and check them.

Exp Let $f(x) = x^4 - 4x^3 + 10$. Find the following

① Domain of f $\Rightarrow$ $D = \mathbb{R} = (-\infty, \infty)$ since f is poly.

② Critical points $\Rightarrow$ $f'(x) = 4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

critical points are $\quad x = 0 \quad$ or $\quad x = 3$

$(0, f(0)) = (0, 10)$

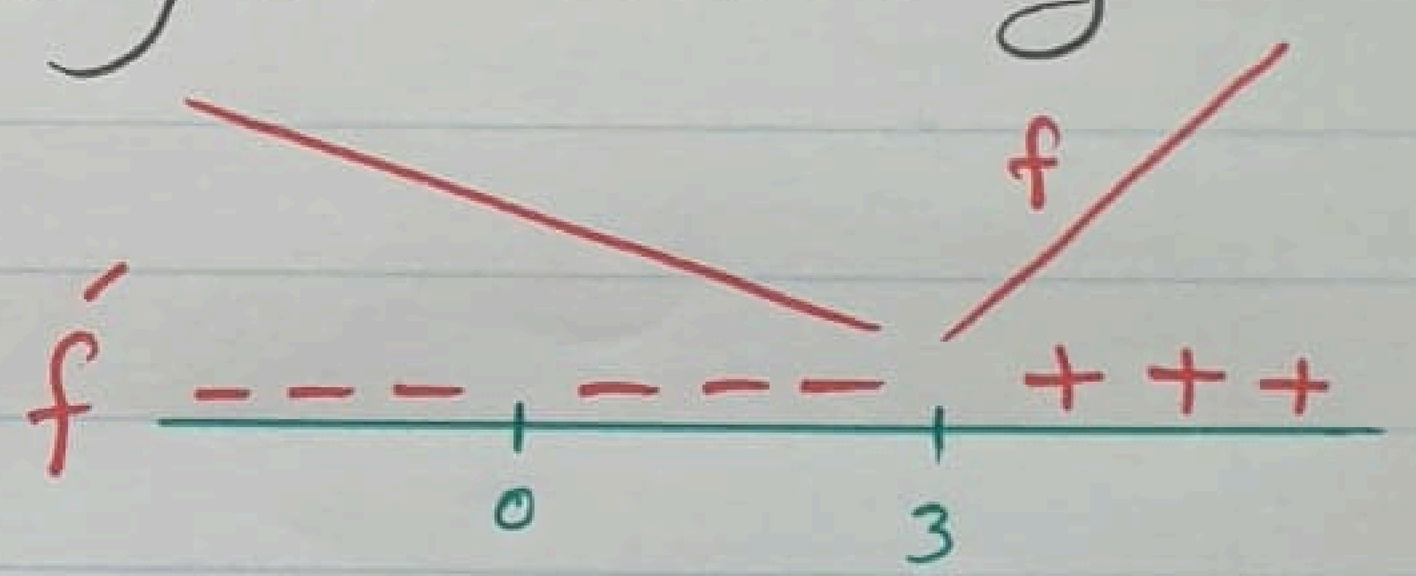
$(3, f(3)) = (3, -17)$ since $f(3) = 81 - 4(27) + 10 = -17$

③ Intervals where f is increasing and decreasing

f is increasing on $[3, \infty)$

f is decreasing on $(-\infty, 3]$

since $0 \in D(f)$

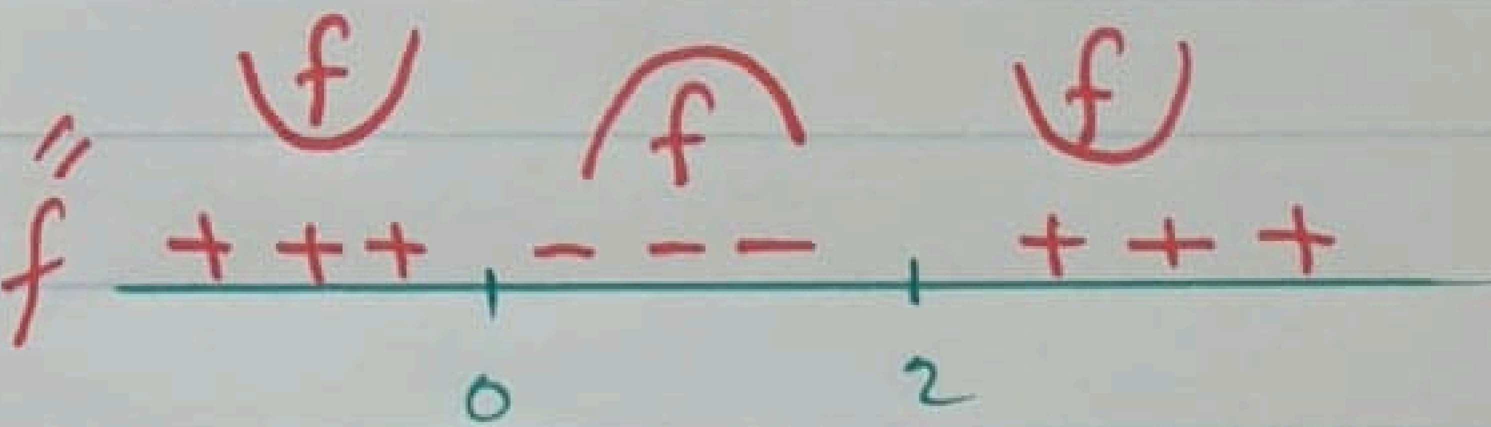


④ Intervals of concavity

$f''(x) = 12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0 \quad$ or $\quad x = 2$



f is concave up on $(-\infty, 0] \cup [2, \infty)$

f is concave down on $[0, 2]$

5 Inflection points

$f''(x) = 12x(x-2) = 0$

$x=0$ or $x=2$

Since f changes concavity around $x=0$ and $x=2$ it follows that the inflection points are

$(0, f(0)) = (0, 10)$

$(2, f(2)) = (2, -6)$ since $f(2) = 16 - 4(8) + 10 = -6$

6 Extreme Values

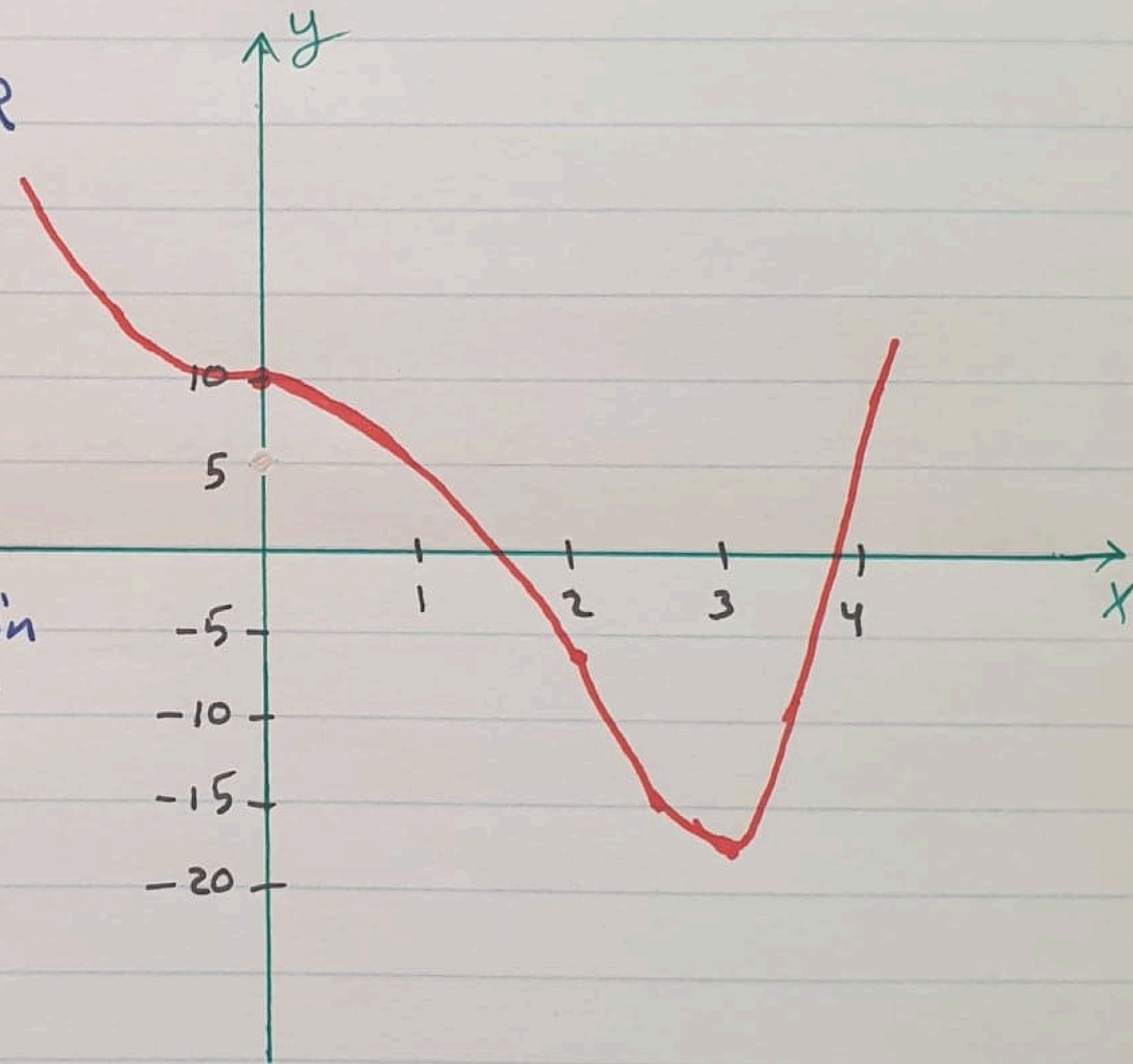
- no endpoint since $D = \mathbb{R}$
- check critical points

$x = 0$, $x = 3$

Using FDT

$(0, f(0))$ is not EV

$(3, f(3)) = (3, -17)$ is Abs. Min (L. Min)



7 Sketch f(x)

8 Range of f(x)

$R(f) = [-17, \infty)$

Exp Let $f(x) = \dfrac{x}{x^2+1}$ with $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$

$f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$ Find $\Rightarrow$

$\boxed{1}$ $D(f) = \mathbb{R} = (-\infty, \infty)$

$\boxed{2}$ Asy. $\Rightarrow$ H. Asy. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{x^2+1} = 0$ $\Rightarrow y=0$ is H Asy.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x}{x^2+1} = 0$ ✓

O. Asy. none

V. Asy. none since $x^2+1 \neq 0$

$\boxed{3}$ critical points

$f' = 0 \Rightarrow \dfrac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1-x^2 = 0$

$(1-x)(1+x) = 0$

$x = 1$ or $x = -1$

$\in D(f)$ $\in D(f)$ ✓

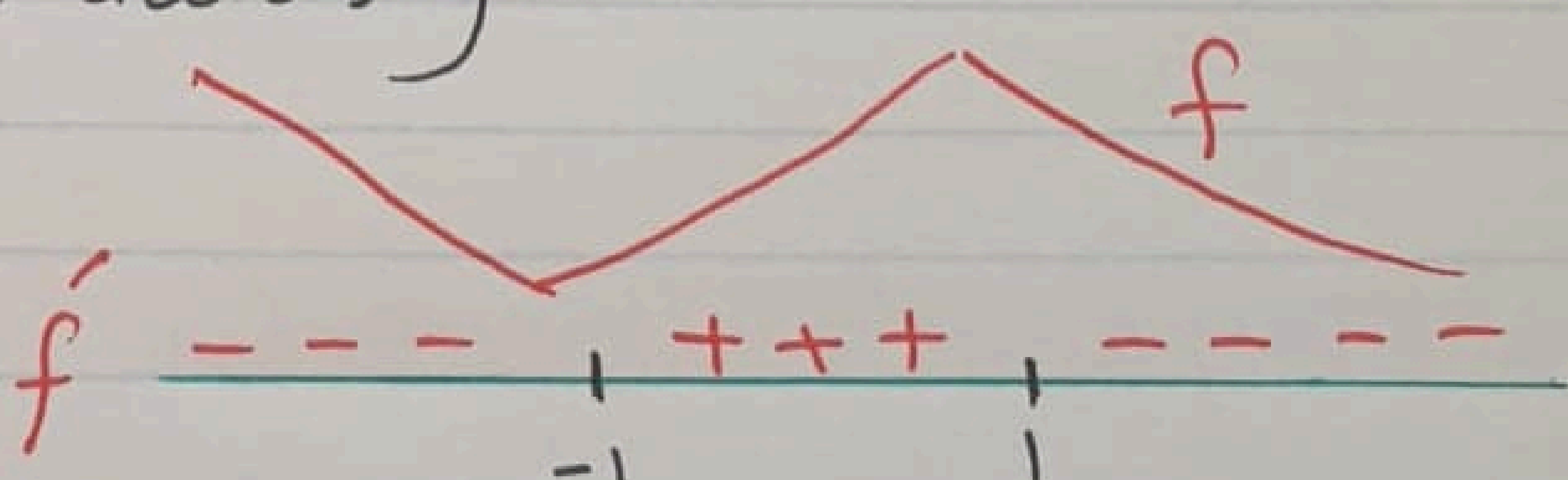
critical points are

$(1, f(1)) = (1, \frac{1}{2})$

$(-1, f(-1)) = (-1, -\frac{1}{2})$

$\boxed{4}$ Interval of increasing and decreasing

$f' = 0 \Rightarrow x = \pm 1$



f is increasing on $[-1, 1]$

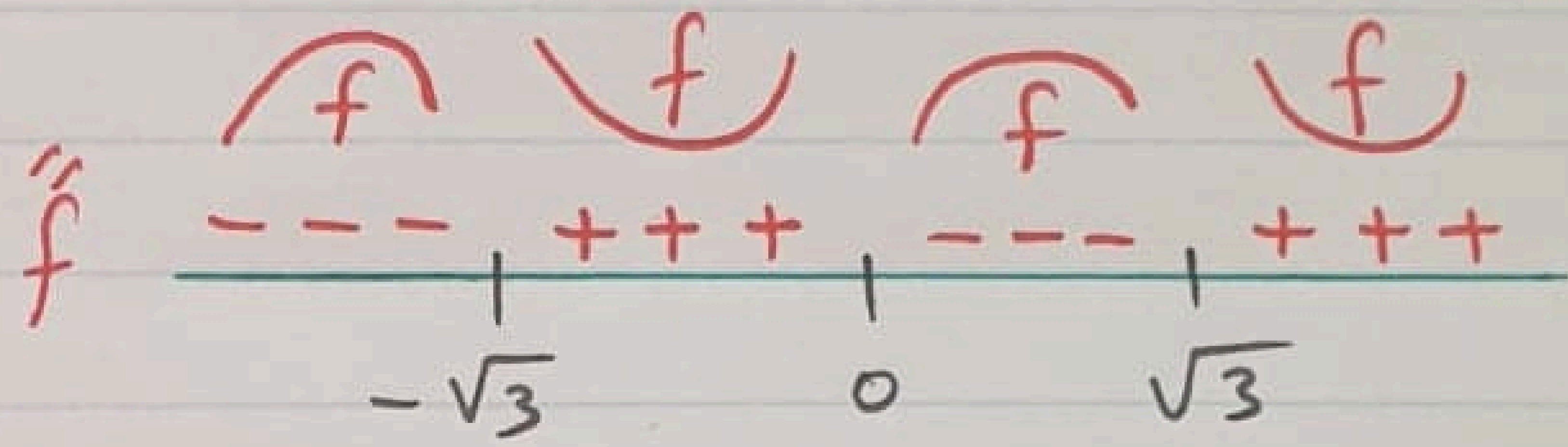
f is decreasing on $(-\infty, -1] \cup [1, \infty)$

5) Interval of concavity

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0 \implies 2x(x^2-3) = 0$$

$x=0$ or $x=\sqrt{3}$ or $x=-\sqrt{3}$

- f is concave up on $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$



- f is concave down on $[-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$

6) Inflection points

$f'' = 0$ at $x = 0, \sqrt{3}, -\sqrt{3}$ and f changes concavity around all of them and all are in $D(f)$

They are

$(0, f(0)) = (0,0)$

$(\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, \frac{\sqrt{3}}{4})$

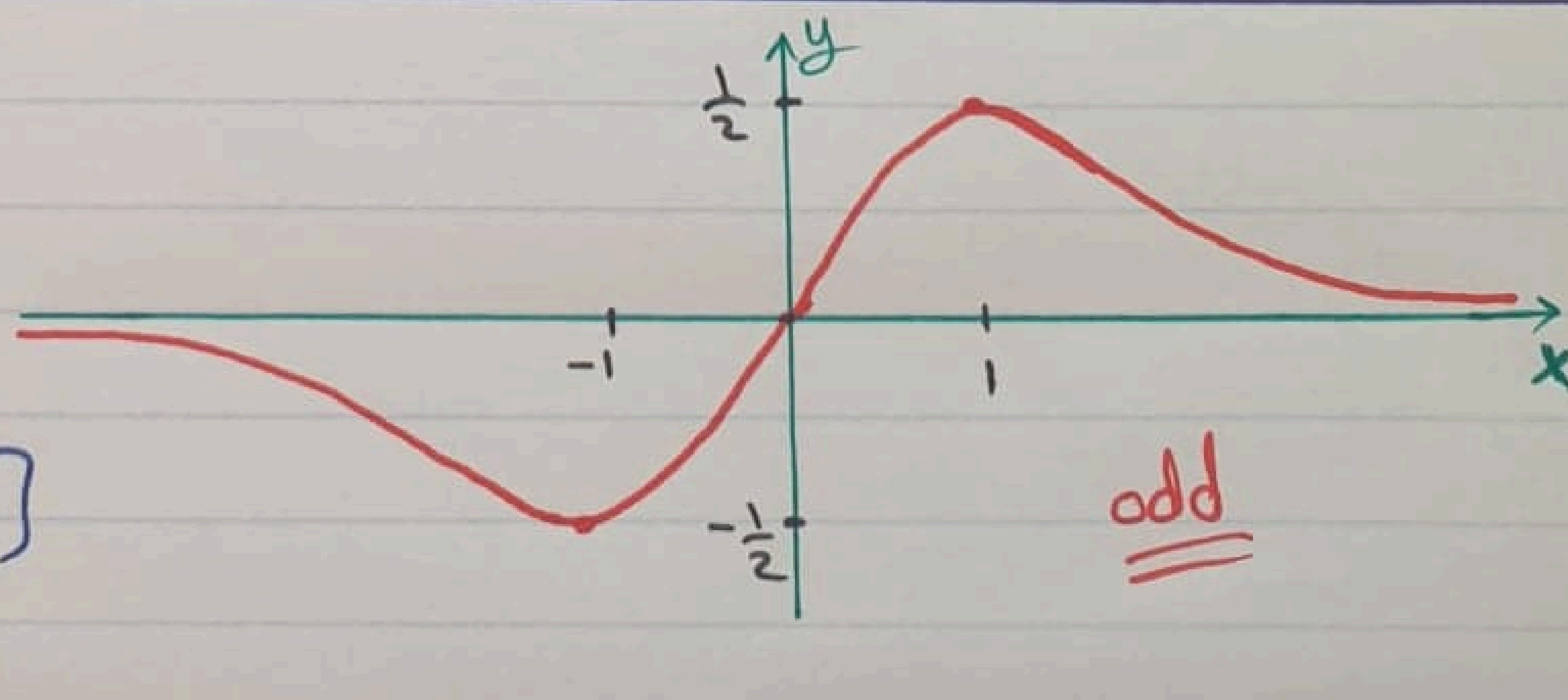
$(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, -\sqrt{3}/4)$

7) Extreme Values: no endpoints $\Rightarrow$ check critical points

f has L. Max of $f(1) = \frac{1}{2}$ at $x=1$ (Abs. Max also)

f has L. Min of $f(-1) = -\frac{1}{2}$ at $x=-1$ (Abs. Min also)

8) Sketch $f(x)$



9) $R(f) = [-\frac{1}{2}, \frac{1}{2}]$